Research Statement - Benjamin Hutz

I am interested in arithmetic properties of periodic and pre-periodic points arising from iterating morphisms of projective varieties and associated computational problems. The major guiding problem of my research is the uniform boundedness conjecture of Morton and Silverman. Note, for example, that this conjecture implies Merel's theorem on the uniform boundedness of torsion on elliptic curves.

Conjecture 1. (Morton-Silverman [8]). For all positive integers D, N, d with $d > 2$, there exists an integer $\kappa(D, N, d)$ such that for each number field K of degree D over $\mathbb{Q}$, and each morphism $\phi : \mathbb{P}^N(K) \to \mathbb{P}^N(K)$ defined by homogenous polynomials of degree d over K, the number of pre-periodic points of ϕ in $\mathbb{P}^N(K)$ is less than or equal to $\kappa(D, N, d)$.

Local Information

In Morton and Silverman [8], the authors give an upper bound on the possible primitive periods of periodic points for rational maps on $\mathbb{P}^1$ (and also *auto*morphisms of $\mathbb{P}^N$). I generalize their main theorem to non-degenerate morphisms on smooth, irreducible projective varieties and provide a similar upper bound on primitive periods based on good reduction information for these more general dynamical systems.

Theorem 1. *Let K be the field of fractions of a discrete valuation ring, k be the associated residue field, and $\overline{}$ denote reduction modulo a uniformizer. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. Assume that X is non-singular and has good reduction and $\phi : X \to X$ is a morphism with good reduction.*

$n =$ primitive period of P for ϕ

$m =$ primitive period of $\overline{P}$ for $\overline{\phi}$.

$p =$ characteristic of k.

$r =$ order of $\overline{d\phi}$ (the map induced on the tangent space by $\overline{\phi}$) in $k^\times$.

Then $n = m, n = mr$, or $n = mrp^e$ for some $e \geq 1$.

Corollary 2. *Let K be a number field. Let $\mathfrak{p}$ and $\mathfrak{q}$ be primes of K such that their residue characteristics are distinct. Let X be an irreducible, non-singular projective variety of dimension d with good reduction at $\mathfrak{p}$ and $\mathfrak{q}$. Let $\phi : X \to X$ be a map over K with good reduction at $\mathfrak{p}$ and $\mathfrak{q}$. Then the primitive period n of any periodic point of ϕ in $X(K)$ satisfies*

$$n \leq (N\mathfrak{p}^{2d+2} - 1)(N\mathfrak{q}^{2d+2} - 1)$$

where $N\mathfrak{p}$ and $N\mathfrak{q}$ denote the norms of $\mathfrak{p}$ and $\mathfrak{q}$ respectively.

The idea of the proof, which is to iterate a local power series representation of the morphism, is similar to the one dimensional case. However, there is significant non-trivial work necessary to carry out the details in higher dimensions. The main tools are a p-adic version of the implicit function theorem, information about the local ring of a variety at a point, and good reduction information on both the variety and the morphism.

Dynatomic Cycles

In the case of iteration of single variable polynomial maps, $\phi(z) \in K[z]$, we can define $\Phi_n(\phi) = \phi^n(z) - z$. This polynomial is often called the *n-th division polynomial* and its roots are periodic points of order n.

Going a step further we can define $\Phi_n^*(\phi) = \prod_{d|n} \Phi_d^{\mu(n/d)}$ the *n-dynatomic polynomial.* As shown by Morton [13], this is in fact a polynomial and in the case where it has no multiple roots, its roots are the points of primitive period n. In the more general setting of morphisms on projective varieties, these definitions are no longer suitable, so we turn to and intersection theory for an appropriate generalization. Let X be a projective variety and $\phi : X \to X$ a morphism. Consider the cycles in $X \times X$, the graph of ϕ^n defined as $\Gamma_n = \sum_{x \in X}(x, \phi^n(x))$ and the diagonal $\Delta = \sum_{x \in X}(x, x)$ and define $\Phi_n(\phi)$ as the intersection of Γ_n and Δ. Then $\Phi_n(\phi)$ is a zero-cycle whose point of positive multiplicity are the points of period n. We can then proceed with the inclusion-exclusion sum as above, obtaining

$$\Phi_n^*(\phi) = \sum_{d|n} \mu(n/d)\Phi_d(\phi) = \sum_P a_P^*(n)(P),$$

where $a_P^*(n)$ is the multiplicity of the point P and the points of primitive period n have $a_P^*(n) > 0$. Define the *n-th dynatomic cycle* to be $\Phi_n^*(\phi)$.

In Morton and Silverman [9], the authors construct the dynamic analogue of cyclotomic (and elliptical) units. In the course of doing so they prove that $\Phi_n^*(\phi)$ is an effective zero-cycle for rational maps on $\mathbb{P}^1$ (and also *auto*morphisms of $\mathbb{P}^N$), ie. $a_P^*(n) \geq 0$ for all points P on the variety and all integers $n \geq 1$. They also conjecture that $\Phi_n^*(\phi)$ is effective for non-degenerate morphisms of non-singular projective varieties. I resolve this conjecture in the affirmative.

Theorem 3. *Let $X \subset \mathbb{P}^N$ be an irreducible, non-singular projective variety defined over a field K. Let $\phi : X \to X$ be a non-degenerate morphism defined over K and let $d\phi$ be the map on the tangent space induced by ϕ. Let $\lambda_1, \ldots, \lambda_l$ be the distinct eigenvalues of $d\phi$. Fix a point $P \in X$ and define integers*

$m =$ *the exact ϕ-period of P (set $m = \infty$ if $P \notin Per(\phi)$).*

$p =$ *the characteristic of K*

$r_i =$ *the multiplicative period of λ_i^m in $\overline{K}^*$ (set $r_i = \infty$ if λ_i is not a root of unity).*

Then

- *(a) $a_P^*(n) \geq 0$ for all $n \geq 1$.*
- *(b) Let $n \geq 1$. Then $a_P^*(n) \geq 1$ only if n has one of the following forms*
  - *(i) $n = m$*
  - *(ii) $n = m \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})$ for $1 \leq k \leq l$.*
  - *(iii) $n = m \operatorname{lcm}(r_{i_1}, \ldots, r_{i_k})p^e$ for $1 \leq k \leq l$ and some $e \geq 1$.*

As in the one dimensional case, the proof is carried out by carefully examining when the multiplicity of a point P in $\Phi_n(\phi)$ is greater than the multiplicity of P in $\Phi_1(\phi)$. However, several new ideas and a lot of additional work are needed in the higher dimensional case. Some of the difficulties encountered are taking into account the higher Tor modules in the intersection theory, which turn out to all be identically 0, using the theory of standard bases to get information about the multiplicity of a point in $\Phi_n(\phi)$, and iteration of local power series representations of the morphism.

$\Phi_n(\phi)$ and $\Phi_n^*(\phi)$ occur with great frequency in the literature, under a variety of notations, with a number of results stemming from the fact they are effective, for example see [8], [9], [10], [11], [12], [13], [14], [17]. In particular, [13] and [17] contain Galois theoretic results in the single variable polynomial case where $\Phi_n^*(\phi)$ has no multiple roots. Along with proving some general properties of $\Phi_n^*(\phi)$, I also generalize several of the Galois theoretic results under similar hypotheses.

Wehler's K3 Surfaces

Let $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the intersection of a (1,1) and a (2,2) effective divisor. Wehler [18] showed that this always is a K3 surface with an infinite automorphism group. Silverman [16] constructed a height function on these surfaces and showed the set of preperiodic points is a set of bounded height. Call and Silverman [3] gave explicit algorithms for computing the automorphisms and the height function. Combining their work with the results above allows for an explicit investigation of the rational points on these surfaces.

In particular, I develop an algorithm to search for periodic points over $\mathbb{Q}$ using Theorem 1. The key fact is if there is a periodic point then the local period for a prime of good reduction must divide the primitive period, i.e. $m \mid n$ in the notation of the theorem. While this may result in false positives due to failure of the Hasse principle, it provides an effective way to search quickly. I use it successfully in Pari/gp to find surfaces with $\mathbb{Q}$-rational points of primitive period $1, \ldots, 11$, and 13.

This algorithm can also be used to obtain information on the number of points, number of cycles, and cycles lengths for S non-degenerate over $\mathbb{F}_p$. Experimentally, S has $p^2 + 2p + 1$ $\mathbb{F}_p$-rational points on average divided into p cycles of average length p. However, for a random permutation of p^2 points we would expect to have $\ln(p^2)$ cycles on average. A suitable explanation for this is still open.

Going a step further, I optimize the algorithm for point counting using the fact that projection to each component is a double cover and checking whether there are 0, 1, or 2 points over each point in $\mathbb{P}^2(\mathbb{F}_p)$. I then show how this can be used to determine the Zeta function of such a surface using the Weil conjectures. However, for this to be successful, I must know when such a surface is non-singular. In this direction, I use methods from GKZ [5], as in Grier et al. [6], to determine information about the homogeneous polynomial which vanishes if and only if the surface is singular, ie. the A-discriminant. In particular, I determine the bi-degree and illustrate a method for determining the leading monomial for a given monomial ordering.

Note that, in computing explicitly with these surfaces, there were several side problems that I did not pursue that would be excellent advanced undergraduate or graduate student projects. For example, examining the distribution of cycle lengths over finite fields as opposed to examining the average cycle length or applying similar computational methods to a similarly constructed class of surfaces such as those in Baragar [1], Billard [2], or McMullen [7]

Future Directions

In broad scope, I am interested in K3 (or projective variety) analogues of the Uniform Boundedness Conjecture, Zhang's conjectures on preperiodic subvarieties and preperiodic points, and Serre's Image of Galois theorem, and algorithmic and computational aspects of these problems.

The following three areas extend the work of my dissertation.

(1) Bound the number of rational preperiodic points on Wehler's K3 surfaces. Two possible approaches to this are to examine conditions for primes of good reduction or to examine the moduli space of pairs (S, P) with S a K3 surface as above and P a fixed point, similar to the approach of Morton [12] for rational 4-cycles.

(2) Examine a similarly constructed class of surfaces such as those in Baragar [1], Billard [2], or McMullen [7], and apply similar methods as in my dissertation and examine the results for parallels. Two possible directions for generalizations are to general K3 surfaces and other compact complex manifolds in the Enriques-Kodiara classification or towards projective varieties with an ample line bundle (allowing for a canonical height function) as described in Fakhruddin [4].

(3) On the algorithmic side I would like to improve the point counting algorithm by using the existence of an automorphism and also look at more sophisticated point counting algorithms along the lines of those described in Schoof [15]. One possible direction is to look for an analogue of the following property of twists of elliptic curves. Given an elliptic curve E there exists a twist E' such that P is not a point of $E(\mathbb{F}_q)$ if and only if P is a point of $E'(\mathbb{F}_q)$.

I am also interested in broadening my research interests beyond the work of my dissertation. An appealing direction is Zhang's general conjectures in [19]. Zhang states several conjectures in broad scope including a dynamical Manin-Mumford conjecture, a dynamical Bogomolov conjecture, and an equidistribution of dynamically generic preperiodic points conjecture. Examining the dynamical Manin-Mumford conjecture, which states that a subvariety is preperiodic if and only if it contains many preperiodic points, for the special case of dynamical systems on projective varieties with an ample line bundle would provide a familiar basis for starting work on Zhang's conjectures on general Kähler manifolds.

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