Introductory Exercises: Computing Fourier Series (Due Wed 9/12)

Always keep in mind that the goal of using Fourier series is to *approximate* a given function $f(x)$ on a particular interval. Fourier series are superpositions of oscillations and so can only represent periodic functions. Many other types of approximations have been developed for various purposes and kinds of functions, e.g., Taylor series, Legendre polynomials, splines, step functions, and wavelets.

At the moment we wish to rewrite the function $f(x)$ as a sum of oscillations so that we may see what frequencies dominate. The great advantage of Fourier analysis is that it reveals *frequency* information. The period a of $f(x)$ limits the possible frequencies to be of form n/a for integers n (meaning it repeats n times on $[0, a]$).

If $f(x)$ has period a, then the graphs looks the same on $[0, a]$ as it does on $[-a, 0]$ and $[98a, 99a]$. Hence we focus on the interval $[0, a]$, but remember that the function is assumed to be periodic with period a when doing Fourier series.

Fourier series can be written in two different forms. The more intuitive form consists of sines and cosines, the prototypical oscillating functions, with frequencies n/a:

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\Big(\frac{2\pi n x}{a}\Big) + B_n \sin\Big(\frac{2\pi n x}{a}\Big) \quad \text{on } [0, a]$$

$$A_0 = \frac{1}{a}\int_0^a f(x)dx, \quad A_n = \frac{2}{a}\int_0^a f(x)\cos\Big(\frac{2\pi n x}{a}\Big)dx, \quad B_n = \frac{2}{a}\int_0^a f(x)\sin\Big(\frac{2\pi n x}{a}\Big)dx.$$

The use of $\sim$ is to remind us that this is a representation of the function that may or may not actually agree with the function at each point. For example, we will see that, if f is piecewise differentiable, this series in fact converges to exactly the value $f(x)$ if f is continuous at x, but does not do so at discontinuities.

The more mathematically convenient form involves exponential functions $e^{2\pi i n x/a}$. To interpret this expression, use Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$. This yields the Fourier series formulae:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/a} \quad \text{on } [0, a], \quad c_n = \frac{1}{a}\int_0^a f(x)e^{-2\pi i n x/a}dx.$$

For today, let's build some intuition of Fourier series via examples using the sine/cosine version. A bit of notation before we begin: the characteristic function $\chi_{[a,b]}(x)$ equals 1 if $x \in [a, b]$ and 0 otherwise.

Exercise 1 Consider the function $f(x) = \chi_{[0,1/2]}(x)$ for $0 \le x < 1$, extended to the real line with period 1. Sketch the function $f(x)$ on [-2,2]. Find the Fourier series for $f(x)$ in terms of sines and cosines. What happens to the coefficients as n increases?

Exercise 2 Consider the function $f(x) = \chi_{[1/2,2/3]}(x)$ for $0 \le x < 1$, extended to the real line with period 1. Sketch the function $f(x)$ on [-2,2]. Find the Fourier series for $f(x)$ in terms of exponentials. What happens to the coefficients as n increases?

Exercise 3 Show that the two forms of Fourier series are indeed equivalent. Find c_n in terms of A_0, A_n, B_n.