Linear Filters (Due Wednesday 10/10)

(Adapted from *A First Course in Wavelets*, Boggess and Narcowich)

A filter takes a signal, processes it in some way (e.g., removes noise), then returns an output signal. Mathematically, a *filter* is a transformation L that maps a function (signal) $f : \mathbf{R} \to \mathbf{C}$ into another signal $\tilde{f}$.

A *linear filter* satisfies $L[cf + g] = cL[f] + L[g]$ for all real numbers c and signals f and g.

We also typically want a filter to be *time invariant*, meaning that any time-shifted signal $f(t-a)$ is transformed by L into the time-shifted output signal $(Lf)(t-a)$. That is, you get the same output if you filter then time-shift or if you time-shift then filter.

Exercise 1 Let $\ell(t) = \chi_{[0,1]}(t)$. Define a filter L via $(Lf)(t) = (\ell * f)(t) = \int_{-\infty}^{\infty} \ell(t-x)f(x)dx$. Prove that this filter is both linear and time-invariant.

Exercise 2 Define a filter L via $(Lf)(t) = \int_0^t f(x)dx$. Prove that this filter is linear but not time-invariant.

Theorem 1 Let L be a linear, time-invariant transformation on the space of signals that are piecewise continuous functions. Then there exists an integrable function h such that $Lf = f * h$ for all signals.

The function $h(t)$ is called the *impulse response function* since if the signal is a single pulse at time $t = a$ (modeled as a Dirac delta function: $f(t) = \delta(t-a)$), then $Lf = h * f = h(a)$. Theorem 1 tells us that designing a filter is equivalent to construction of the impulse response function h. Let's design a filter (construct an h) that reduces high frequencies, which in many signals tends to be noise. Such a filter is called a "low-pass filter." Let's apply a Fourier transform to work in the frequency domain: $\hat{Lf}[\gamma] = \hat{h}(\gamma)\hat{f}(\gamma)$.

Exercise 3 As a naive first attempt at designing a filter, let's do the obvious thing and cutoff all frequencies above γ_{cutoff}. What function $\hat{h}$ will do this? Once you have a choice, find its inverse transform to obtain $h(t)$. Apply this new filter to the signal $f(t) = \chi_{[0,1]}(t)$ and plot the result, which should strike you as very unsatisfactory. For instance, the output signal Lf is nonzero for $t < 0$, so that the output signal starts long before the original signal does!

The result of this exercise leads us to a further condition for a sensible filter. A *causal filter* is one for which the output signal begins after the input signal has started to arrive.

Theorem 2 Let L be a time invariant filter with response function h. L is a causal filter if and only if $h(t) = 0$ for all $t < 0$.

Exercise 4 An old noise-reducing filter is the Butterworth filter, with $h(t) = Ae^{-\alpha t}$ if $t \geq 0$ and $h(t) = 0$ if $t < 0$. Find the Fourier transform of h and explain how this filter reduces noise (that is, what happens to high frequencies).

Exercise 5 Let $f(t) = e^{-t/3}(\sin 2t + 2\sin 4t + 0.4\sin 2t \sin 40t)$ for $0 \leq t \leq \pi$, and zero otherwise. Plot this signal and the output signal from the Butterworth filter with $A = \alpha = 10$. How does the filter affect the low ($\gamma = 2, 4$) and high frequency ($\gamma = 40$) components of this signal?