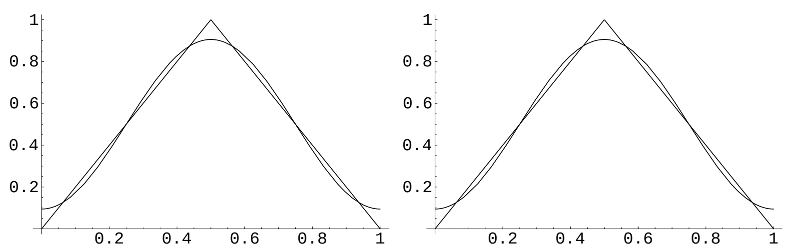
# Norms and Functions (Exercises due Fri 9/14)

Given that the goal of Fourier series is to approximate a given function $f(x)$ on $[0, a]$, we need a means of determining how well we have approximated the function. There are a couple of different points of view here. One point of view is that we want to know how many terms are needed so that the partial sum of the Fourier series is within some error tolerance of the function at each point of $[0, a]$. A broader point of view is to ask whether the series itself converges and, if so, whether it converges to the function. This convergence may be in a pointwise sense or in some average sense over the interval.

**Definition** $L^\infty$ is the set of all bounded functions on $[0, a]$. The $L^\infty$ norm (also called the sup norm) is $||f||_\infty = \sup\{|f(x)| : x \in [0, a]\}$.

If we want a series to converge pointwise uniformly on $[0, a]$, then we want the sup norm of the difference between the function and the partial sums to go to zero as we take more and more terms (that is, the maximum error on the interval is going to zero):

$$\lim_{N\to\infty} ||f - \sum_{n=-N}^{N} c_n e^{2\pi i n x/a}||_\infty = 0.$$




**Definition** $L^1$ is the set of all integrable functions on $[0, a]$. The $L^1$ norm is simply the integral of the absolute value of the function, measuring the area between its graph and the $x$-axis: $||f||_1 = \int_0^a |f(x)|dx$.

Sometimes we want to know whether a series converges to a function in some average sense, for example, by measuring the area between the graphs of the partial sums and the function and checking that it goes to zero. This is equivalent to the $L^1$-norm of the difference between $f$ and the partial sums going to zero. On occasion it turns out that the integral of the difference squared is most pertinent (reminiscent of seeking the least squared error in regression), leading us to consider a third function space:

**Definition** $L^2$ is the set of all square-integrable functions on $[0, a]$ (meaning $f^2 \in L^1$). The $L^2$ norm is the square root of the integral of the square of the function: $||f||_2 = \sqrt{\int_0^a |f(x)|^2 dx}$.

**Exercise 1** Section 1.1 proves that $L^\infty \subset L^1$ and $L^2 \subset L^1$. Show that these sets are not identical by finding an example of a function that is $L^1$ but not $L^\infty$ or $L^2$ on [0,1]. (Note that we use these labels both as names of the spaces and as adjectives for the functions that belong in them, and we are assuming a finite length interval.)

**Exercise 2** Calculate $||f||_\infty$, $||f||_1$, and $||f||_2$ for $f(x) = x$ on $[0, 1]$.

**Exercise 3** Calculate $||f - g||_\infty$ and $||f - g||_2$ if $f(x) = \chi_{[0,1/2]}$ on $[0, 1]$ and $g$ is the partial sum of the Fourier series of $f$ with (a) n=0; (b) n=-1..1. Use the fact that $\int_0^1 \sin^2(bx)dx = \frac{1}{2} - \frac{\sin(2b)}{4b}$.