The Uncertainty Principle (Exercises due Fri 10/12)

(Adapted from *A First Course in Wavelets*, Boggess and Narcowich and *A Wavelet Tour of Signal Processing*, Mallat)

The uncertainty principle states that a function cannot simultaneously have restricted support in time as well as in frequency. This is essentially the same principle (with a different interpretation) as the famous Heisenberg uncertainty principle of quantum physics. For a nice discussion, see p. 48–51 of *The World According to Wavelets*, Hubbard or p. 30–33 of *A Wavelet Tour of Signal Processing*, Mallat.

Define the *dispersion* of $f \in L^2$ about the point $a \in R$ to be

$$\triangle_a f = \frac{\int_{-\infty}^{\infty} (t-a)^2 |f(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt}.$$

The dispersion measures the spread of the graph of f about a point a. The more spread out the graph is near that point, the larger the dispersion will be. In terms of probability, $|f|^2/\int |f|^2$ is a probability distribution, and if a is set equal to the mean then $\triangle_a f$ is the variance. If f is a wave function describing the state of a particle, a is the average location of the particle, and α is its average momentum, then $\triangle_a f$ is the variance of the position and $\triangle_\alpha \hat{f}$ is the variance of the momentum. The uncertainty principle says that the smaller the variance (hence the uncertainty) of the position, the greater the variance (uncertainty) of the momentum, and vice versa:

Theorem (Uncertainty Principle) Suppose f is a function in $L^2(R)$ that vanishes at $\pm\infty$. Then $\triangle_a f \cdot \triangle_\alpha \hat{f} \geq \frac{1}{16\pi^2}$.

In Fourier analysis terms, the uncertainty principle says that if we try to localize a signal in time, then we end up with a greater spread of frequencies. If we try to narrow the bandwidth, then we get a signal widely dispersed in time. There's just no way around it.

For example, consider a Gaussian $f(t) = e^{-bt^2}$ for some $b > 0$. Its dispersion about $a = 0$ is $\frac{1}{4b}$, while the dispersion of its transform $\hat{f}(\gamma) = (\sqrt{\pi/b})e^{-\pi^2\gamma^2/b}$ about $\alpha = 0$ is $\frac{b}{4\pi^2}$. Observe that we have equality in this special case: $\triangle_0 f \cdot \triangle_0 \hat{f} = \frac{1}{16\pi^2}$. The more f spreads out (b small), the more concentrated $\hat{f}$ becomes, and vice versa. For a point like $a = \alpha = 1$, we have $\triangle_1 f = \frac{1+4b}{4b}$ and $\triangle_1 \hat{f} = 1 + \frac{b}{4\pi^2}$, leading to $\triangle_1 f \cdot \triangle_1 \hat{f} > \frac{1}{16\pi^2}$.

Despite the uncertainty principle, it was hoped for a time that a function f with compact support ((essentially, f nonzero on a bounded interval) whose Fourier transform also had compact support could be constructed. However, the following theorem dashed those hopes permanently:

Theorem (Compact Support) If $f \neq 0$ (meaning that f is not identically zero everywhere) has compact support, then $\hat{f}(\gamma)$ cannot be zero on a whole interval. Similarly, if $\hat{f} \neq 0$ has compact support, then $f(t)$ cannot be zero on a whole interval.

The proof will be done in steps in the exercises on the back of this page.

Exercise 1 Suppose $\hat{f}$ is zero outside the interval $[-b, b]$. Write down $f(t)$ as the inverse transform of $\hat{f}$.

Exercise 2 Suppose that $f(t)$ is zero on the interval $[c, d]$. Let $t_0 = (c + d)/2$ (midpoint of this interval), and find the nth derivative of $f(t)$ at $t = t_0$, using the expression in Exercise 1. This derivative must be zero, since $f(t)$ is zero on $[c, d]$.

Exercise 3 Rewrite the expression for $f(t)$ in Exercise 1 by replacing $e^{2\pi i\gamma t}$ with $e^{2\pi i\gamma(t-t_0)}e^{2\pi i\gamma t_0}$. Then rewrite $e^{2\pi i\gamma(t-t_0)}$ as an infinite series $\sum_{n=0}^{\infty} \frac{[2\pi i\gamma(t-t_0)]^n}{n!}$. Rearrange a bit to find an integral that matches the nth derivative found in Exercise 2, which we said equals zero. But this means that the entire series equals zero and that $f(t) = 0$ for all t. This is a contradiction, which proves that if $f \neq 0$ then it can't be zero on a whole interval.

Exercise 4 How can you quickly prove the second statement of the theorem?