Amherst College
Economics 58

Substitution and Income Effects with the Cobb-Douglas

If $U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ we know that

$$x_1 = \frac{\alpha I}{p_1}$$
$$x_1^c = \alpha A^{-1} p_1^{\alpha-1} p_2^{1-\alpha} U$$
$$U = A p_1^{-\alpha} p_2^{-(1-\alpha)} I$$

1. Own price Slutsky Equation: $\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^c}{\partial p_1} - x_1 \frac{\partial x_1}{\partial I}$. Using the Marshallian demand function directly gives $\frac{\partial x_1}{\partial p_1} = \frac{-\alpha I}{p_1^2}$. Calculating the components of the Slutsky Equation gives:

$$\frac{\partial x_1^c}{\partial p_1} = \alpha(\alpha-1) A^{-1} p_1^{\alpha-2} p_2^{1-\alpha} U = \alpha(\alpha-1) p_1^{-2} I$$
$$-x_1 \frac{\partial x_1}{\partial I} = -\frac{\alpha I}{p_1} \cdot \frac{\alpha}{p_1} = -\alpha^2 p_1^{-2} I$$

Summing these two effects gives $\frac{\partial x_1}{\partial p_1} =$ Sub + Income Effect= $-\alpha p_1^{-2} I$ just as was derived from direct differentiation of the Marshallian demand function. Note, if $\alpha = 0.5$, we have

$$\frac{\partial x_1}{\partial p_1} = -0.5 p_1^{-2} I$$
$$\frac{\partial x_1^c}{\partial p_1} = -0.25 p_1^{-2} I$$
$$-x_1 \frac{\partial x_1}{\partial I} = -0.25 p_1^{-2} I$$

That is, the total price effect in the Marshallian demand function is half income effect and half substitution effect. If, say, $\alpha = 0.3$, the substitution effect would be 70 percent of the total effect ($0.7 = \frac{0.3 \cdot 0.7}{0.3}$) and the income effect would be only 30 percent of the total. These proportions would be reversed if $\alpha = 0.7$.

2. Cross-price Slutsky equation : $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_1^c}{\partial p_2} - x_2 \frac{\partial x_1}{\partial I}$.

Now direct differentiation gives: $\frac{\partial x_1}{\partial p_2} = 0$ and we wish to know why. To calculate the Slutsky Equation we have to know the Marshallian demand for good 2 which is $x_2 = \frac{(1-\alpha)I}{p_2}$. So the Slutsky components are:

$$\frac{\partial x_1^c}{\partial p_2} = \alpha(1-\alpha)A^{-1} p_1^{\alpha-1} p_2^{-\alpha} U = \alpha(1-\alpha) p_1^{-1} p_2^{-1} I$$

$$-x_2 \frac{\partial x_1}{\partial I} = -(1-\alpha) I p_2^{-1} \cdot \alpha\, p_1^{-1} = -\alpha(1-\alpha) p_1^{-1} p_2^{-1} I$$

Which shows that the substitution and income effects always precisely cancel out regardless of the value of α . That makes the Cobb-Douglas a very special case.