Core March 2009

There are 13 problems (totalling 150 points) on this portion of the examination. Record your answers in your blue book(s). SHOW ALL WORK.

1. **(15 points.)** Evaluate the following limits:

   (a) $\lim_{x\to 3}\left[\dfrac{x}{x-3}-\dfrac{9}{x^2-3x}\right]$

   (b) $\lim_{x\to -\infty} x^2 e^x$

   (c) $\lim_{x\to 0}\left(\tan\left(x+\dfrac{\pi}{4}\right)\right)^{1/x}$

2. **(15 points.)** Evaluate the following integrals:

   (a) $\int x^9 \ln x\, dx$

   (b) $\int \dfrac{x^2-2}{x^2-x-2}\, dx$

   (c) $\int \dfrac{dx}{x^2\sqrt{x^2+4}}$

3. **(15 points.)** Determine whether each of the following series converges or diverges. Justify your answers.

   (a) $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^3}$

   (b) $\sum_{n=1}^{\infty} \dfrac{(-1)^n n}{n^2+10}$

   (c) $\sum_{n=1}^{\infty} \dfrac{n!}{e^{3n+1}}$

4. **(10 points.)** Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{n^2(x+2)^n}{2^n}$.

5. **(10 points.)** Find the maximum and minimum values of the function $f(x,y)=xy$ on the ellipse $x^2+9y^2=18$.

6. **(10 points.)** Let C be the circle $x^2+y^2=4$, oriented counterclockwise. Compute

$$\int_C (xy^2-y^3)dx+(x^3+x^2y)dy.$$

7. **(15 points.)** Let R be the solid in 3-dimensional space above the xy-plane, below the cone $z=\sqrt{x^2+y^2}$, and inside the cylinder $x^2+y^2=4$.

   (a) Express $\iiint_R (x^2+y^2+z^2)\,dV$ in rectangular, cylindrical, and spherical coordinates.

   (b) Evaluate one of the integrals in part (a).

8. **(10 points.)** The temperature at the point (x, y, z) is

$$T(x, y, z) = \frac{1}{\pi}\sin(\pi xy) + \ln(z^2 + 1) + 60.$$

(a) Find a vector pointing in the direction in which the temperature increases most rapidly at the point $(2, -1, 1)$.

(b) Let $\vec{v} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. (Notice that $\vec{v}$ is **not** a unit vector). What is the rate of change of the temperature at the point $(2, -1, 1)$ in the direction of $\vec{v}$?

9. **(10 points.)** Consider the function

$$f(x, y) = \begin{cases} \dfrac{2xy^2 + x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Compute $f_x(0, 0)$ and $f_y(0, 0)$.

(b) Compute the directional derivative $D_{\vec{u}}f(0, 0)$, where $\vec{u} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$.

(c) Explain why f is not differentiable at $(0, 0)$.

10. **(10 points.)** Let M be a 4×6 matrix with real entries.

(a) Explain why the columns of M cannot be linearly independent.

(b) Assume that the rows of M are linearly independent. Determine the dimension of the null space $N(M) = \{v \in \mathbb{R}^6 \mid Mv = 0\}$. Justify your reasoning.

11. **(10 points.)** Let V be the vector space of polynomials with real coefficients and of degree at most 3, and let $W = \{f \in V \mid f(0) = f''(0) \text{ and } f'(1) = 0\}$.

(a) Prove that W is a subspace of V.

(b) Find a basis for W.

12. **(10 points.)** Let W be a finite dimensional vector space, and let $U \subseteq W$ and $V \subseteq W$ be subspaces. Define $U + V = \{u + v \mid u \in U, v \in V\}$.

(a) Prove that $U + V$ is a subspace of W.

(b) Suppose $\{u_1, \ldots, u_m\}$ is a basis for U, and $\{v_1, \ldots, v_n\}$ is a basis for V. Prove that $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ spans $U + V$.

(c) Prove that $\dim(U + V) \leq \dim(U) + \dim(V)$, where $\dim(\ldots)$ denotes dimension.

13. **(10 points.)** Let M be the matrix

$$M = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of M.

(b) Is M diagonalizable? Why or why not?

Amherst College
Department of Mathematics and Computer Science

Comprehensive Examination: Mathematics 26

Friday, March 27, 2009

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. **(25 points).** Let G be a group, and let $s \in G$. The *centralizer* of s in G is defined to be

$$C(s) = \{g \in G : gs = sg\}.$$

(a). **(10 points).** Prove that $C(s)$ is a subgroup of G.

(b). **(5 points).** Prove that $\langle s \rangle$ is contained in $C(s)$.
(Recall that $\langle s \rangle$ is the subgroup of G generated by s.)

(c). **(10 points).** Prove that $\langle s \rangle$ is a **normal** subgroup of $C(s)$.

2. **(25 points).** Let S_n denote the symmetric group of degree n, also known as the group of permutations of the set $\{1, 2, \ldots, n\}$.
Let $Y \subseteq \{1, 2, \ldots, n\}$. Define $H_Y = \{f \in S_n : f(y) \in Y \text{ for all } y \in Y\}$.

(a). **(15 points).** Prove that H_Y is a subgroup of S_n.

(b). **(10 points).** In the case $n = 4$ and $Y = \{1, 2\}$, list all the elements of H_Y.

3. **(25 points).** Let R be a ring.

(a). **(10 points).** Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
If you use other terms like "closed" or "coset" or "subgroup" or "subring" or "maximal" in your definition, you must define those terms as well.

(b). **(15 points).** Let $I \subseteq R$ be an ideal of R, let S be another ring, and let $\phi : R \to S$ be an **onto** homomorphism. Prove that $\phi(I) = \{\phi(x) : x \in I\}$ is an ideal of S.

4. **(25 points).** Let $\mathbb{F}$ be a field, let $R = \mathbb{F}[x]$ be the ring of polynomials in one variable with coefficients in $\mathbb{F}$, and let $f(x) \in R$ be a polynomial of degree 2009. Let

$$I = \{g(x)f(x) : g \in R\}$$

be the set of all polynomials which are multiples of $f(x)$. It is a fact, which you may assume, that I is an ideal of R.

(a). **(10 points).** For any $g, h \in R$, prove that $I + g = I + h$ if and only if $(g - h)$ is divisible by f.

(b). **(15 points).** Prove that for any $g \in R$, there is a unique polynomial $h \in R$ with $\deg h < 2009$ such that $I + g = I + h$.

AMHERST COLLEGE
Department of Mathematics
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
March 27, 2009

Work the following five problems. Record your answers in the blue book provided. The number of points each problem is worth is indicated in parentheses.
PLEASE SHOW ALL OF YOUR WORK.

1. (10 points) State the Heine-Borel Theorem.

2. (25 points) Show that the sequence of functions $\{f_n\}$ with $f_n(x) = x^n(1-x)$ for $n \in \mathbb{N}$ converges uniformly on $[0, 1]$ to the zero function.

3. (25 points) Let $s_n = \sum_{k=1}^{n} \frac{1}{k}$.

   (a). Use induction to prove that $s_{2^n} \geq \frac{1}{2}(n+2)$ for every $n \geq 0$.

   (b). Use part (a) to prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

4. (20 points) Prove that a continuous real-valued function on a compact set must be bounded.

5. (20 points) The boundary ∂A of a set A of real numbers is the set of points $x \in \mathbb{R}$ such that for every $\epsilon > 0$, the ball of radius ϵ centered at x contains a point in A and a point not in A.

   Prove that A is closed (i.e., contains all of its limit points) if and only if it contains its boundary.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Calculus and Linear Algebra

2:00 pm Friday, January 30, 2009

Seeley Mudd 205

There are 13 problems (totalling 150 points) on this portion of the examination. Record your answers in your blue book(s). SHOW ALL WORK.

1. **(15 points.)** Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

   (b) $\displaystyle\lim_{x\to 0}(\sec x)^{1/x^2}$

   (c) $\displaystyle\lim_{x\to -1}\frac{2x+|x|+1}{x^2-1}$

2. **(15 points.)** Evaluate the following integrals.

   (a) $\displaystyle\int x^2\sqrt{x-1}\,dx$

   (b) $\displaystyle\int x^2\tan^{-1}x\,dx$

   (c) $\displaystyle\int_3^4\frac{25\,dx}{(25-x^2)^{3/2}}$

3. **(15 points.)** Determine whether each of the following series converges or diverges. Justify your answers.

   (a) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^3-1}}$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{e^{(n^2)}}$

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2+\sin(\ln n)}$

4. **(10 points.)** Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(2x+1)^n}{n3^n}$.

5. **(15 points.)**

   (a) Find the absolute maximum and minimum values of the function $g(x)=\dfrac{x}{x^2+1}$ on $(-\infty,\infty)$. (*Suggestion: It may help to break the domain into pieces.*)

   (b) State the Mean Value Theorem.

   (c) Prove that $|\ln(b^2+1)-\ln(a^2+1)|\le b-a$ for any real numbers $a<b$.

6. **(15 points.)** Let $f(x,y)=2x+5y^2$. Find the maximum and minimum values of $f(x,y)$ on the curve $x^2+5y^4=9$.

7. **(10 points.)** Find the mass of a ball of radius 1 centered at the origin in 3-space if the density at the point (x, y, z) of the ball is equal to $3 - (x^2 + y^2 + z^2)$.

8. **(10 points.)** Let C be the triangle with vertices $(0,0)$, $(2,0)$, and $(0,2)$, traversed counterclockwise. Compute

$$\int_C x \cos y \sin y \, dx - x^2 \sin^2 y \, dy.$$

9. **(10 points.)** Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3 + xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Compute $f_x(0,0)$ and $f_y(0,0)$.

   (b) Prove that f is not continuous at $(0,0)$.

10. **(10 points.)** Let A be an $n \times n$ matrix with real entries such that $A^2 = 0$.

   (a) Prove that the column space of A is contained in the nullspace of A. (Recall that the nullspace, or kernel, is the set of vectors $v \in \mathbb{R}^n$ satisfying $Av = 0$.)

   (b) Prove that $\operatorname{rank}(A) \leq \operatorname{nullity}(A)$. (Recall that the rank is the dimension of the column space and the nullity is the dimension of the nullspace.)

   (c) Prove that $\operatorname{nullity}(A) \geq \frac{1}{2}n$.

11. **(5 points.)** Let V and W be vector spaces, let $v_1 \ldots, v_n \in V$, and let $f : V \to W$ be a linear map that is onto. Assume that $v_1, \ldots, v_n$ span V. Prove that $f(v_1), \ldots, f(v_n)$ span W.

12. **(10 points.)** Let M be the matrix

$$M = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

   (a) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors, or else prove that there is no such basis.

   (b) Is M diagonalizable? Why or why not?

13. **(10 points.)** Let $M_{2\times2}$ be the vector space of 2×2 matrices with real entries. It is a fact that $\dim M_{2\times2} = 4$, and that $\{e_1, e_2, e_3, e_4\}$ is a basis for $M_{2\times2}$, where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Consider the function $T : M_{2\times2} \to M_{2\times2}$ given by $T(A) = A \begin{pmatrix} -1 & 7 \\ 3 & 4 \end{pmatrix}$.

   (a) Prove that T is linear.

   (b) Find the matrix for T with respect to the basis $\{e_1, e_2, e_3, e_4\}$.

Amherst College
Department of Mathematics and Computer Science

Comprehensive Examination: Mathematics 26

Friday, January 30, 2009

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. **(25 points).** Let G be a group, and let $H \subseteq G$ be a subgroup. Suppose that for every $x, y \in G$ satisfying $xy \in H$, we have $yx \in H$. Prove that H is a **normal** subgroup of G.

2. **(25 points).** Recall that S_7 denotes the group of permutations of the set $\{1, 2, \ldots, 7\}$. Define the permutations $\sigma, \tau \in S_7$ by

$$\sigma = (1,2)(4,6,7,5) \quad \text{and} \quad \tau = (1,3,5)(2,7,4,6).$$

   (a) **(10 points).** Write $\sigma\tau$ as a product of disjoint cycles.

   (b) **(10 points).** Find the order of each of σ, τ, and $\sigma\tau$.

   (c) **(5 points).** Determine whether each of σ, τ, and $\sigma\tau$ is even or odd.

3. **(25 points).** Let R be a ring.

   (a) **(10 points).** Define what it means for a subset $I \subseteq R$ to be an **ideal** of R. If you use other terms like "closed" or "coset" or "subgroup" or "subring" or "maximal" in your definition, you must define those terms as well.

   (b) **(15 points).** Let $I \subseteq R$ be an ideal of R, and suppose that $xy - yx \in I$ for every $x, y \in R$. Prove that the quotient ring R/I is commutative.

4. **(25 points).** Let $\mathbb{F}_2 = \{0, 1\}$ denote the field of two elements, and let $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ denote the field of five elements.

   (a) **(15 points).** Prove that $f(X) = X^4 + X^2 + 1$ is **reducible** in the polynomial ring $\mathbb{F}_2[X]$.

   (b) **(10 points).** Prove that $g(X) = X^3 + 2X^2 + 2X + 3$ is **irreducible** in the polynomial ring $\mathbb{F}_5[X]$.

AMHERST COLLEGE
Department of Mathematics
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
January 30, 2009

Work the following five problems. Record your answers in the blue book provided.
The number of points each problem is worth is indicated in parentheses.
PLEASE SHOW ALL OF YOUR WORK.

1. (10 points) State the Completeness Axiom (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

2. (20 points) Consider the sequence $\{a_n\}$ defined recursively as follows:

$$a_1 = 2 \quad \text{and} \quad a_{n+1} = 5 - \frac{4}{a_n} \quad \text{for} \quad n \geq 1.$$

   (a) Prove that for $n \geq 1$, $a_{n+1} \geq a_n$.

   (b) Prove that the sequence $\{a_n\}$ is bounded above.

   (c) Prove that the sequence $\{a_n\}$ converges and find $\lim_{n \to \infty} a_n$.

3. (a) (10 points) State Taylor's Theorem, including the definition of a Taylor polynomial and an expression for the remainder.

   (b) (14 points) Find a polynomial P such that $\left|P(x) - \frac{1}{x}\right| < 10^{-5}$ for all x in $(100, 102)$.

4. (20 points) Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on an interval $[a, b]$. Use the definitions of continuity and uniform convergence to prove that f is continuous.

5. (a) (10 points) State the Bolzano-Weierstrass Theorem.

   (b) (16 points) Prove that every bounded sequence has a monotone subsequence.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Calculus and Linear Algebra

2:00 pm Friday, March 28, 2008

Seeley Mudd 206

There are 13 problems on this examination. Record your answers in the blue book provided. SHOW ALL WORK.

1. Evaluate the following limits:

   (a) $\lim_{x \to 4} \frac{x-4}{\sqrt{x}-2}$.

   (b) $\lim_{x \to 0} \frac{\int_0^x \sqrt{2+\cos^2 t}\, dt}{x}$.

2. Evaluate the following integrals:

   (a) $\int_1^\infty \frac{\ln x}{x^2}\, dx$.

   (b) $\int \frac{x^3}{\sqrt{4+x^2}}\, dx$.

3. Determine whether the following series converge or diverge. Justify your answers.

   (a) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2-1}$.

   (b) $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$.

   (c) $\sum_{n=1}^{\infty} n \sin(1/n)$.

4. Determine the values of x for which the following series converges absolutely, converges conditionally, and diverges. Justify your answer.

$$\sum_{n=0}^{\infty} \frac{(2x-1)^n}{\sqrt{n+1}}.$$

5. (a) State the Intermediate Value Theorem.

   (b) State the Mean Value Theorem.

   (c) Use the Intermediate Value Theorem and the Mean Value Theorem to prove that the equation $e^x = -x$ has exactly one solution.

6. (a) Find the Maclaurin series for the function $f(x) = \sin(x^2)$. Express your answer in Σ-notation.

   (b) Use part (a) to express $\int_0^1 \sin(x^2)\, dx$ as an infinite series. Write the series in Σ-notation, and also write out the first three nonzero terms.

   (c) Use part (b) to approximate $\int_0^1 \sin(x^2)\, dx$, with an error less than 0.001.

7. Let C be the triangle with vertices $(0,0)$, $(2,0)$, and $(2,1)$, oriented counterclockwise. Find $\int_C (\ln(x^2+1) + 4x^3y)\, dx + (x^2+y)^2\, dy$.

8. Find the maximum value of the function $f(x,y) = x^2 + 2y$ subject to the constraint $x^4 + y^2 = 5$.

9. Consider the integral

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} e^{x^2+y^2}\, dy dx.$$

   (a) Reexpress the integral in polar coordinates.

   (b) What is the value of the integral?

10. Let $f(x,y) = \tan^{-1}(xy)$.

    (a) Find a unit vector $\vec{u}$ pointing in the direction in which $f(x,y)$ increases most rapidly at the point $(2,-1)$.

    (b) With $\vec{u}$ as in part (a), find the directional derivative of $f(x,y)$ at the point $(2,-1)$ in the direction $\vec{u}$.

11. Let

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

    (a) Find all eigenvalues of A.

    (b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, or show that there is no such matrix P.

12. Suppose that V is a vector space, $T : V \to V$ is a linear transformation, and $\text{range}(T) \cap \text{nullspace}(T) = \{0\}$.

    (a) Prove that $\text{nullspace}(T^2) = \text{nullspace}(T)$.

    (b) Prove that if V is finite dimensional then $\text{range}(T^2) = \text{range}(T)$.

13. Suppose that $T : V \to W$ is a linear transformation and $v_1, v_2, \ldots, v_n \in V$. Prove that if $\text{Span}(\{v_1, v_2, \ldots, v_n\}) = V$ and T is onto then $\text{Span}(\{T(v_1), T(v_2), \ldots, T(v_n)\}) = W$.

Amherst College
Department of Mathematics and Computer Science

Comprehensive Examination: Mathematics 26

Friday, March 28, 2008

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. Let G be a group, let $H \subseteq G$ be a subgroup, and let $N \subseteq G$ be a **normal** subgroup. Define
$$NH = \{nh : n \in N, h \in H\}.$$
Prove that NH is a subgroup of G.

2. Recall that S_8 denotes the group of permutations on 8 symbols.
Let $\sigma = (1,3,2)(4,7)(5,6) \in S_8$ and $\tau = (1,4)(3,7,8,2,5) \in S_8$.
   (a) Write $\sigma\tau$ as a product of disjoint cycles.
   (b) Find the orders of σ, τ, and $\sigma\tau$.
   (c) Determine whether each of σ, τ, and $\sigma\tau$ is even or odd.

3. Let R be a ring.
   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.
   (b) Let R be a commutative ring with unity, and let $I \subsetneq R$ be a proper ideal. Prove that I is a prime ideal if and only if R/I is an integral domain.
   [Recall that a proper ideal $I \subsetneq R$ is said to be a **prime ideal** if for any $x, y \in R$ for which $xy \in I$, either $x \in I$ or $y \in I$. Recall also that an **integral domain** is a commutative ring S with unity with no zero divisors, i.e., no nonzero $a, b \in S$ for which $ab = 0$.]

4. Let $\mathbb{F}_5 = \{0,1,2,3,4\}$ denote the field of five elements, and let $\mathbb{F}_7 = \{0,1,2,3,4,5,6\}$ denote the field of seven elements. Write
$$f(x) = x^2 + 3 \in \mathbb{F}_5[x] \qquad \text{and} \qquad g(x) = x^2 + 3 \in \mathbb{F}_7[x].$$
(They might appear to be the same polynomial, but they lie in different polynomial rings.)
   (a) Prove that f is irreducible in $\mathbb{F}_5[x]$.
   (b) Prove that g is reducible in $\mathbb{F}_7[x]$.
   (c) Let $\langle g \rangle \subseteq \mathbb{F}_7[x]$ denote the principal ideal $\{gh : h \in \mathbb{F}_7[x]\}$. Find an ideal $I \subseteq \mathbb{F}_7[x]$ such that $\langle g \rangle \subsetneq I \subsetneq \mathbb{F}_7[x]$.
   (*Hint: use part b.*)

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
March 28, 2008

Work the following four problems. Record your answers in the blue book provided.
PLEASE SHOW ALL OF YOUR WORK.

1. Complete the following statement in a non-trivial manner: A bounded sequence of real numbers...

2. Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ using induction.

3. (a) Define the notion of uniform convergence on a set A of a sequence $\{f_n\}_{n=1}^{\infty}$ of real valued functions of a real variable.

   (b) Let $f_n(x) = \frac{1}{1+n^2x^2}$. Prove that $\{f_n\}_{n=1}^{\infty}$ converges pointwise but not uniformly on [0,1].

4. Suppose that E is a bounded subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is uniformly continuous on E. Prove that f is bounded on E.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Calculus and Linear Algebra

2:00 pm Friday, March 30, 2007

Seeley Mudd 206

There are 12 problems on this examination. Record your answers in the blue book provided. SHOW ALL WORK.

1. Evaluate the following limits.

   (a) $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

   (b) $\lim_{x\to 0} (1+\tan x)^{\frac{1}{x+x^2}}$

2. Decide whether the following series converge absolutely, converge conditionally, or diverge. Give reasons.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

   (b) $\sum_{n=1}^{\infty} \frac{1+3\sin n}{n\sqrt{n}}$

   (c) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$

3. Evaluate the following integrals.

   (a) $\int_0^{\ln 6} \frac{e^x}{\sqrt{3+e^x}}\,dx$

   (b) $\int \sin^5 x \cos^2 x \, dx$

   (c) $\int_C e^{-x^2}\,dx + x^2 y \sin y\,dy$ where C is the boundary (oriented counterclockwise) of the rectangle with vertices $(0,0)$, $(1,0)$, $(1,\pi)$ and $(0,\pi)$.

4. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} x^n$.

5. Find the first 3 terms of the Taylor series of $f(x) = x\ln x$ about $x = e$.

6. Let $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

   (a) Compute $f_x(0,0)$ and $f_y(0,0)$.

   (b) Let $\vec{u} = (a,b)$ be a unit vector (i.e. $|\vec{u}| = \sqrt{a^2+b^2} = 1$). Compute $D_{\vec{u}}f(0,0)$ using the definition of directional derivative.

   (c) If f is differentiable at $(0,0)$, then $D_{\vec{u}}f(0,0) = \vec{u}\cdot\nabla f(0,0)$ for all unit vectors $\vec{u}$. Use this plus parts (a) and (b) to show that f is not differentiable at $(0,0)$.

7. Let $F(x, y, z) = xy^2z^3$.

   (a) Find the equation of the tangent plane to the level surface $F(x, y, z) = 1$ at the point $(1, 1, 1)$

   (b) Compute $\nabla F(1, 1, 1) \times \vec{v}$, where $\vec{v} = (2, -1, 3)$

8. Consider $f(x, y) = 2x^2 + 3y^2$ on the closed disk $x^2 + y^2 \leq 1$

   (a) Find the critical points of f in the interior of the disk and classify them using the 2nd derivative test.

   (b) Find the minimum and maximum values of $f(x, y)$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange multipliers.

   (c) What are the minimum and maximum values of f on $x^2 + y^2 \leq 1$?

9. Consider the region in 3-dimensional space that is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above by the paraboloid of revolution $z = 2 - x^2 - y^2$

   (a) Show that the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 2 - x^2 - y^2$ intersect where $z = 1$ and $x^2 + y^2 = 1$. You may wish to illustrate your answer with a drawing.

   (c) Compute the volume of the region.

10. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

   (a) Compute the inverse of A.

   (b) Compute the characteristic polynomial of A.

11. Let $M_{2\times 3}$ be the vector space of 2×3 matrices with real entries, and let

$$W = \left\{ B \in M_{2\times 3} : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

   (a) Prove that W is a subspace of $M_{2\times 3}$.

   (b) Find a basis of W.

12. Let V_1 and V_2 be subspaces of a vector space V and assume that $V_1 \cap V_2 = \{0\}$. Let $v_1 \in V_1$ and $v_2 \in V_2$ and assume $v_1 \neq 0$, $v_2 \neq 0$. Prove that $\{v_1, v_2\}$ is linearly independent.

Amherst College
Department of Mathematics and Computer Science

Comprehensive Examination: Mathematics 26

Friday, March 30, 2007

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. Let G and G' be groups, let $H \subseteq G$ be a subgroup, and let $\phi : G \to G'$ be a homomorphism.

   Let $\phi(H)$ denote the set $\{\phi(h) : h \in G\}$, often called the image of H under ϕ.

   (a) Prove that $\phi(H)$ is a subgroup of G'.

   (b) Prove that if ϕ is onto and if H is a normal subgroup of G, then $\phi(H)$ is a normal subgroup of G'.

2. Let $p, q \geq 2$ be prime numbers, and let G be a group of order $|G| = pq$. Let $H \subsetneq G$ be a *proper* subgroup (i.e., H is a subgroup but is not the full group G).

   Prove that H is cyclic.

3. Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   (b) If $I, J \subseteq R$ are ideals, prove that $I + J$ is an ideal of R, where

$$I + J = \{x + y : x \in I \text{ and } y \in J\}.$$

4. Let $d, n \geq 2$ be integers. Let R be a commutative ring with exactly n elements. Let $S = R[x]$ be the ring of polynomials with coefficients in R.

   Let $I = \langle x^d \rangle = \{x^d \cdot f(x) : f \in S\}$ be the principal ideal generated by $x^d \in S$.

   (a) Prove that

$$S/I = \{a_0 + a_1 x + \cdots + a_{d-1} x^{d-1} + I \quad : \quad a_0, \ldots, a_{d-1} \in R\}.$$

   (b) Prove that the quotient ring S/I contains exactly n^d elements.

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
March 30, 2007

Work the following four problems. Record your answers in the blue book provided.
PLEASE SHOW ALL OF YOUR WORK.

1. Complete the statement in a non-trivial manner: A continuous real-valued function on a compact set will ...

2. Show that the set $\{1/n : n \in \mathbb{N}\}$ is not compact by showing that there is an open cover with no finite subcover.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(1) = 5$, $f(2) = 13$ and for $n \geq 2$, $f(n) = 2f(n-2) + f(n-1)$. Prove that $f(n) = 3(2^n) + (-1)^n$ for all $n \in \mathbb{N}$.

4. (a) Let A be a subset of $\mathbb{R}$ and $M > 0$ such that $x < M$ for all $x \in A$. Show that $\sup A \leq M$. (For a nonempty set A, $\sup A$ = least upper bound of A.)

   (b) Suppose the sequence $\{f_n\}$ of real-valued functions on the set E converges pointwise to f on E. For each $n \in \mathbb{N}$ set

   $$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

   and assume $M_n < \infty$ for all $n \in \mathbb{N}$. Show that if $\{f_n\}$ converges uniformly on E, then $\lim_{n\to\infty} M_n = 0$.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, March 31, 2006

Seeley Mudd 206

There are 12 problems on this examination. Record your answers in the blue book provided. SHOW ALL WORK.

1. Find each of the following limits.

   (a) $\lim_{x \to 0} \dfrac{x^2}{\ln(1+x) - x}$

   (b) $\lim_{x \to 0} (1 + \sin x)^{\cot(2x)}$

2. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{1}{x^3 + x}\, dx$

   (b) $\displaystyle\int_0^{\pi/3} x \sin x \, dx$

   (c) $\displaystyle\int_C (\tan^{-1} x - y)\, dx + (x + e^{\sin y})\, dy$, where C is the circle $(x-1)^2 + (y+1)^2 = 3$ oriented counter-clockwise.

3. The sum of integrals $\displaystyle\int_0^1 \int_0^x \sqrt{x^2+y^2}\, dy\, dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ can be expressed as $\displaystyle\int\int_R \sqrt{x^2+y^2}\, dx dy$ for a certain region R in the plane.

   (a) Draw a picture of R

   (b) Evaluate $\displaystyle\int\int_R \sqrt{x^2+y^2}\, dx\, dy$

4. Determine in each case whether the given series converges absolutely, converges conditionally, or diverges. Justify your answers.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n \cos n}{n^3 + 1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^n n!}$

5. Find all values of x for which $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$ converges. Justify your answer.

6. (a) State the Mean Value Theorem.

   (b) Use (a) to show that $\frac{1}{61} < \tan^{-1} 11 - \tan^{1} 9 < \frac{1}{41}$

7. Define

$$f(x,y) = \begin{cases} \dfrac{3x^2+4y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   (a) Is f continuous at $(0,0)$? Justify your answer.

   (b) Compute $\dfrac{\partial f}{\partial y}$ at $(1,1)$ and $(0,0)$.

8. Locate the critical points of $f(x,y) = (x+y)^3 + 6(x^2+y^2)$ and determine the type (local maximum, local minimum, saddle point) of each critical point.

9. Let $F(x,y,z) = x^2 + y^2 + z^2$

   (a) Compute the gradient vector ∇F.

   (b) Compute $\displaystyle\iiint_R ||\nabla F||\, dV$, where R is the region $\{(x,y,z) \in \mathbf{R}^3 : x^2+y^2+z^2 \leq 1\}$ and $||v||$ denotes the length of the vector v.

10. Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ whose matrix relative to the standard basis of $\mathbf{R}^2$ is $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

11. Let U and V be subspaces of a vector space W.

    (a) Prove that $U \cap V$ is a subspace of W.

    (b) Prove that $U + V = \{u + v : u \in U, v \in V\}$ is a subspace of W.

    (c) Given an example to show that $U \cup V$ need not be a subspace of W.

12. Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be a linear map.

    (a) Could T be one-to-one? Explain your reasoning.

    (b) Could it happen that the nullspace (also called the kernel) of T had dimension 2 and the range (also called the image) of T had dimension 3? Explain your reasoning.

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 26
March 31, 2006

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. Let G be group and let g be an element of G. The centralizer $Z(g)$ of g is defined to be the set $\{x \in G : xg = gx\}$.

   (a) Show that $Z(g)$ is a subgroup of G.

   (b) Suppose that $Z(g)$ is normal in G. Show that if $a \in G$, then $aga^{-1}g = gaga^{-1}$.

   (c) Suppose that $Z(g)$ has finite order. Show that $o(g)$ divides $o(Z(g))$. (Here, $o(g)$ denotes the order of g and $o(Z(g))$ denotes the number of elements in the set $Z(g)$.)

2. Let H be a subgroup of a group G. Show that if the index of H in G is 2, then $x^2 \in H$ for each x in G.

3. Let R be a ring and let I be a subset of R.

   (a) Define what it means for I to be an **ideal** of R.

   (b) Suppose that I is an ideal in R, and let r_1, r_2, s_1 and s_2 be in R. Show that if $r_1 - r_2$ and $s_1 - s_2$ are both in I, then $r_1s_1 - r_2s_2$ is also in I.

4. Let R be a ring.

   (a) Define what it means for R to be a **field**.

   (b) Suppose that R is a field and R' is a ring. Show that if ϕ is a nontrivial homomorphism from R to R', then ϕ is one-to-one. (The trivial homomorphism maps all elements of R to the zero element of R'.)

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
March 31, 2006

Work the following four problems. Record your answers in the blue book provided.
PLEASE SHOW ALL OF YOUR WORK.

1. State the Completeness Axiom (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

2. Consider the sequence $\{a_n\}$ defined recursively as follows:

$$a_1 = 3 \quad \text{and} \quad a_{n+1} = 8 - \frac{12}{a_n} \quad \text{for} \quad n \geq 1.$$

   (a) Prove that for $n \geq 1$, $a_{n+1} \geq a_n$.

   (b) Prove that the sequence $\{a_n\}$ is bounded above.

   (c) Prove that the sequence $\{a_n\}$ converges and find $\lim_{n\to\infty} a_n$.

3. Suppose f is a real-valued function defined on the entire real line. Use the Mean Value Theorem to prove that if $f'(x)$ exists and is bounded on all of $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.

4. (a) Let $\{f_n\}$ be a sequence of bounded functions on $[a, b]$. Prove that if $\{f_n\}$ converges *uniformly* to f on $[a, b]$, then f is bounded.

   (b) Give an example to show that this statement is false if uniform convergence is replaced by pointwise convergence.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, March 25, 2005

Seeley Mudd 205

1. Evaluate the following limits.

(a) $\lim_{x \to 0+} (\cos \sqrt{x})^{1/x}$

(b) $\lim_{x \to \infty} x \ln \left(\frac{x+1}{x-1} \right)$

(c) $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{(-1)^k}{3^k}$

2. Evaluate the following integrals.

(a) $\int x \tan^{-1} x \, dx$

(b) $\int_4^{\infty} \frac{1}{x^2 - 2x + 10} \, dx$

(c) $\int_0^3 \sqrt{9 - x^2} \, dx$

(d) $\int_C y(3x^2 - 2)\, dx + (x^3 + e^y)\, dy$, where C is the boundary of the curve $x^2 + y^2 + x = 1$ oriented counterclockwise.

3. The curve $y = x^2$ from $x = 0$ to $x = 1$ is revolved about the y-axis. Find the surface area of the resulting figure.

4. In each case determine whether the given series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n^2)}{n^2 + 1}$

(c) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n^2)}$

5. Let $f(x, y) = x^2 + 2y^2 - x^2 y$. Find all critical points of f and determine in each case whether the point gives a relative maximum, a relative minimum, or is a saddle point.

6. (a) Use the method of Lagrange multipliers to find the minimum value of $x^2 + y^2 + z^2 + 2z + 1$ subject to the constraint $x + 2y + 3z = 11$.

(b) Using (a) or otherwise, find the minimum distance between the point $(0, 0, -1)$ and the plane $x + 2y + 3z = 11$

7. Let Π be the plane tangent to the ellipsoid $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$.

   (a) Find an equation for Π.

   (b) Find the acute angle between Π and the xy-plane.

   (c) The part of Π in the first octant is a triangular region. Find its area.

8. Let $f(x, y) = \begin{cases} \frac{y^2(2x-y)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

   (a) Prove or disprove: f is continuous at $(0, 0)$.

   (b) Does $f_x(0, 0)$ exist? If so, what is it? Answer the same question for $f_y(0, 0)$.

   (c) Prove or disprove: f is differentiable at $(0, 0)$.

9. Let R be the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 8$. Suppose that R has density $\delta(x, y, z) = z$. Set up three triple integrals giving the mass of R, one in Cartesian, one is cylindrical and one in spherical coordinates. Then evaluate one of your three integrals.

10. Let $W = \{(x, y, z) \in \mathbf{R}^3 : \ x + 2y + 3z = 0\}$.

    (a) Prove that W is a subspace of the vector space $\mathbf{R}^3$.

    (b) Exhibit a basis of W.

11. Let V and W be vector spaces over $\mathbf{R}$ and let T be a linear transformation from V to W. Suppose that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for V over $\mathbf{R}$. Show that if T is one-one, then $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is linearly independent in W.

12. Let T be a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$. Suppose that $T(1, 2) = (3, 4)$ and $T(2, 3) = (1, 1)$.

    (a) Find $T(x, y)$ for any (x, y) in $\mathbf{R}^2$.

    (b) Show that T is invertible and find $T^{-1}(x, y)$ for any (x, y) in $\mathbf{R}^2$.

13. Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $T(x, y, z) = (2x + y - 4z, 4y - 5z, -z)$.

    (a) Prove or disprove: T is invertible.

    (b) Find a basis of $\mathbf{R}^3$ consisting of eigenvectors of T.

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
March 25, 2005

Work the following three problems. Record your answers in the blue book provided.
PLEASE SHOW ALL YOUR WORK.

1. (a) State the Completeness Axiom for the Real Numbers (also known as Axiom C or the Axiom of Continuity).

   (b) State the Bolzano-Weierstrass Theorem.

2. For $n \geq 1$, define the function g_n on $[0, 1]$ by

$$g_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{2n} \\ n - n^2 x & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$$

   (a) Draw the graph of g_n for $n = 1, 2, 3$.

   (b) Compute $\lim_{n\to\infty} g_n(x)$ for all $x \in [0, 1]$

   (c) Compute $\lim_{n\to\infty} \int_0^1 g_n(x)\, dx$.

   (d) Show that $\{g_n\}_{n=1}^{\infty}$ does not converge uniformly on $[0, 1]$. Justify your answer using either definitions or standard theorems.

3. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers and suppose that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Prove that $\lim_{n\to\infty} (a_n + b_n) = A + B$.