AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Calculus and Linear Algebra

2:00 pm Friday, January 30, 2009

Seeley Mudd 205

There are 13 problems (totalling 150 points) on this portion of the examination. Record your answers in your blue book(s). SHOW ALL WORK.

1. **(15 points.)** Evaluate the following limits.

   (a) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

   (b) $\lim_{x \to 0} (\sec x)^{1/x^2}$

   (c) $\lim_{x \to -1} \frac{2x + |x| + 1}{x^2 - 1}$

2. **(15 points.)** Evaluate the following integrals.

   (a) $\int x^2 \sqrt{x - 1}\, dx$

   (b) $\int x^2 \tan^{-1} x\, dx$

   (c) $\int_3^4 \frac{25\, dx}{(25 - x^2)^{3/2}}$

3. **(15 points.)** Determine whether each of the following series converges or diverges. Justify your answers.

   (a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - 1}}$

   (b) $\sum_{n=1}^{\infty} \frac{n!}{e^{(n^2)}}$

   (c) $\sum_{n=1}^{\infty} \frac{1}{2 + \sin(\ln n)}$

4. **(10 points.)** Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n 3^n}$.

5. **(15 points.)**

   (a) Find the absolute maximum and minimum values of the function $g(x) = \frac{x}{x^2 + 1}$ on $(-\infty, \infty)$. (*Suggestion: It may help to break the domain into pieces.*)

   (b) State the Mean Value Theorem.

   (c) Prove that $|\ln(b^2 + 1) - \ln(a^2 + 1)| \leq b - a$ for any real numbers $a < b$.

6. **(15 points.)** Let $f(x, y) = 2x + 5y^2$. Find the maximum and minimum values of $f(x, y)$ on the curve $x^2 + 5y^4 = 9$.

7. **(10 points.)** Find the mass of a ball of radius 1 centered at the origin in 3-space if the density at the point (x, y, z) of the ball is equal to $3 - (x^2 + y^2 + z^2)$.

8. **(10 points.)** Let C be the triangle with vertices $(0,0)$, $(2,0)$, and $(0,2)$, traversed counterclockwise. Compute

$$\int_C x \cos y \sin y \, dx - x^2 \sin^2 y \, dy.$$

9. **(10 points.)** Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^3 + xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Compute $f_x(0,0)$ and $f_y(0,0)$.

   (b) Prove that f is not continuous at $(0,0)$.

10. **(10 points.)** Let A be an $n \times n$ matrix with real entries such that $A^2 = 0$.

   (a) Prove that the column space of A is contained in the nullspace of A. (Recall that the nullspace, or kernel, is the set of vectors $v \in \mathbb{R}^n$ satisfying $Av = 0$.)

   (b) Prove that $\text{rank}(A) \leq \text{nullity}(A)$. (Recall that the rank is the dimension of the column space and the nullity is the dimension of the nullspace.)

   (c) Prove that $\text{nullity}(A) \geq \frac{1}{2}n$.

11. **(5 points.)** Let V and W be vector spaces, let $v_1 \ldots, v_n \in V$, and let $f : V \to W$ be a linear map that is onto. Assume that $v_1, \ldots, v_n$ span V. Prove that $f(v_1), \ldots, f(v_n)$ span W.

12. **(10 points.)** Let M be the matrix

$$M = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

   (a) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors, or else prove that there is no such basis.

   (b) Is M diagonalizable? Why or why not?

13. **(10 points.)** Let $M_{2\times 2}$ be the vector space of 2×2 matrices with real entries. It is a fact that $\dim M_{2\times 2} = 4$, and that $\{e_1, e_2, e_3, e_4\}$ is a basis for $M_{2\times 2}$, where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the function $T : M_{2\times 2} \to M_{2\times 2}$ given by $T(A) = A \begin{pmatrix} -1 & 7 \\ 3 & 4 \end{pmatrix}$.

   (a) Prove that T is linear.

   (b) Find the matrix for T with respect to the basis $\{e_1, e_2, e_3, e_4\}$.

Amherst College
Department of Mathematics and Computer Science

Comprehensive Examination: Mathematics 26

Friday, January 30, 2009

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. **(25 points).** Let G be a group, and let $H \subseteq G$ be a subgroup. Suppose that for every $x, y \in G$ satisfying $xy \in H$, we have $yx \in H$. Prove that H is a **normal** subgroup of G.

2. **(25 points).** Recall that S_7 denotes the group of permutations of the set $\{1, 2, \ldots, 7\}$. Define the permutations $\sigma, \tau \in S_7$ by

$$\sigma = (1,2)(4,6,7,5) \quad \text{and} \quad \tau = (1,3,5)(2,7,4,6).$$

   (a) **(10 points).** Write $\sigma\tau$ as a product of disjoint cycles.

   (b) **(10 points).** Find the order of each of σ, τ, and $\sigma\tau$.

   (c) **(5 points).** Determine whether each of σ, τ, and $\sigma\tau$ is even or odd.

3. **(25 points).** Let R be a ring.

   (a) **(10 points).** Define what it means for a subset $I \subseteq R$ to be an **ideal** of R. If you use other terms like "closed" or "coset" or "subgroup" or "subring" or "maximal" in your definition, you must define those terms as well.

   (b) **(15 points).** Let $I \subseteq R$ be an ideal of R, and suppose that $xy - yx \in I$ for every $x, y \in R$. Prove that the quotient ring R/I is commutative.

4. **(25 points).** Let $\mathbb{F}_2 = \{0, 1\}$ denote the field of two elements, and let $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ denote the field of five elements.

   (a) **(15 points).** Prove that $f(X) = X^4 + X^2 + 1$ is **reducible** in the polynomial ring $\mathbb{F}_2[X]$.

   (b) **(10 points).** Prove that $g(X) = X^3 + 2X^2 + 2X + 3$ is **irreducible** in the polynomial ring $\mathbb{F}_5[X]$.

AMHERST COLLEGE
Department of Mathematics
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
January 30, 2009

Work the following five problems. Record your answers in the blue book provided.
The number of points each problem is worth is indicated in parentheses.
PLEASE SHOW ALL OF YOUR WORK.

1. (10 points) State the Completeness Axiom (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

2. (20 points) Consider the sequence $\{a_n\}$ defined recursively as follows:

$$a_1 = 2 \quad \text{and} \quad a_{n+1} = 5 - \frac{4}{a_n} \text{ for } n \geq 1.$$

   (a) Prove that for $n \geq 1$, $a_{n+1} \geq a_n$.

   (b) Prove that the sequence $\{a_n\}$ is bounded above.

   (c) Prove that the sequence $\{a_n\}$ converges and find $\lim_{n\to\infty} a_n$.

3. (a) (10 points) State Taylor's Theorem, including the definition of a Taylor polynomial and an expression for the remainder.

   (b) (14 points) Find a polynomial P such that $\left|P(x) - \frac{1}{x}\right| < 10^{-5}$ for all x in $(100, 102)$.

4. (20 points) Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on an interval $[a, b]$. Use the definitions of continuity and uniform convergence to prove that f is continuous.

5. (a) (10 points) State the Bolzano-Weierstrass Theorem.

   (b) (16 points) Prove that every bounded sequence has a monotone subsequence.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Calculus and Linear Algebra

2:00 pm Friday, February 1, 2008

Seeley Mudd 206

There are 13 problems on this examination. Record your answers in the blue book provided. SHOW ALL WORK.

1. Evaluate the following limits:

   (a) $\lim_{x \to 0} (\cos x)^{1/x^2}$.

   (b) $\lim_{x \to 2} \left[\frac{1}{x-2} - \frac{2}{x^2 - 2x} \right]$.

2. Evaluate the following integrals:

   (a) $\int \frac{x}{\sqrt{4-x}}\, dx$.

   (b) $\int \frac{x^2}{\sqrt{4-x^2}}\, dx$.

   (c) $\int \ln(x^2+1)\, dx$.

3. Determine whether the following series converge or diverge. Justify your answers.

   (a) $\sum_{n=0}^{\infty} \frac{\sin n}{\sqrt{n^3+1}}$.

   (b) $\sum_{n=1}^{\infty} \frac{\ln(e^n - n)}{n}$.

   (c) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n+1)!}$.

4. Determine the values of x for which the following series converges absolutely, converges conditionally, and diverges. Justify your answer.

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}(n+1)}$$

5. Find the value of the following infinite series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n)!}$$

6. Evaluate the following integrals:

   (a) $\int_0^1 \int_{2x}^2 e^{(y^2)}\, dy\, dx.$

   (b) $\int_C (e^x + y^3)\, dx + (6y^2x + x^3)\, dy$, where C is the circle $x^2 + y^2 = 1$, oriented counterclockwise.

7. Find the volume of the region that is inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$.

8. Consider the function:

$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

   (a) Find $f_x(0, 0)$ and $f_y(0, 0)$.

   (b) Use the definition of directional derivative to find the directional derivative of f at $(0, 0)$ in the direction $\vec{u} = (\sqrt{2}/2, \sqrt{2}/2)$.

   (c) Is f differentiable at $(0, 0)$? Justify your answer.

9. Find the maximum value of the function $f(x, y) = x^2y$ on the ellipse $x^2 + 2y^2 = 24$.

10. (a) State the Mean Value Theorem.

    (b) Prove that for every $x > 1$, $\ln x < x - 1$.

11. Let

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 0 & -5 \\ 1 & 0 & 0 \end{pmatrix}.$$

    (a) Find all eigenvalues of A.

    (b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix, or show that there is no such matrix P.

12. Suppose that u, v, and w are distinct vectors in a vector space V, and $\{u, v, w\}$ is linearly independent. Prove that $\{u + 2v, v + 2w, u + 2w\}$ is linearly independent.

13. Suppose V, W, and Z are finite-dimensional vector spaces, $T : V \to W$ and $U : W \to Z$ are linear transformations, and T is onto.

    (a) Prove that $\text{range}(UT) = \text{range}(U)$.

    (b) Prove that $\text{nullity}(UT) = \text{nullity}(U) + \text{nullity}(T)$.

Amherst College

Department of Mathematics and Computer Science

Comprehensive Examination: Mathematics 26

Friday, February 1, 2008

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. Let G be a group, let $H \subseteq G$ be a subgroup, and define the **normalizer** of H to be

$$N(H) = \{x \in G : x^{-1}Hx = H\}.$$

   (a) Prove that $N(H)$ is a subgroup of G.

   (b) Prove that H is a subgroup of $N(H)$.

   (c) Prove that H is a **normal** subgroup of $N(H)$.

2. Recall that S_n denotes the group of permutations on n symbols.

   (a) Find an element of S_{10} of order 21.

   (b) Prove that no element of S_{10} has order 11.

3. Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   (b) Let S be another ring, and let $\phi : R \to S$ be a ring homomorphism. Let $I \subseteq R$ be an ideal of R, and set

$$J = \{x \in I : \phi(x) = 0_S\},$$

   where 0_S denotes the zero element of S. Prove that J is an ideal of R.

4. Let $R = \mathbb{R}[x]$ be the ring of polynomials with coefficients in the field $\mathbb{R}$ of real numbers. Let $I \subseteq R$ be the subset

$$I = \{f \in \mathbb{R}[x] : f(1) = f(2) = 0\}.$$

   (a) Prove that I is an ideal of R.

   (b) Prove that R/I has zero-divisors. That is, show that there are two nonzero elements of R/I whose product is zero in R/I.

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
February 1, 2008

Work the following four problems. Record your answers in the blue book provided.
PLEASE SHOW ALL OF YOUR WORK.

1. (a) State the Completeness Axiom (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

   (b) State the Bolzano-Weierstrass Theorem.

2. Use induction to prove the binomial theorem: If a and b are real numbers and n is a positive integer, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

3. (a) Give a condition that is both necessary and sufficient for a set of real numbers to be *compact* in $\mathbb{R}$.

   (b) Use the condition in part (a) to determine whether the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$ is or is not a compact set of real numbers. Be sure to justify your answer completely.

4. (a) Define what it means for a sequence $\{x_n\}_{n=1}^{\infty}$ to be a Cauchy sequence.

   (b) Let f be a real-valued function defined on a subset S of $\mathbb{R}$. Define what it means for f to be uniformly continuous on S.

   (c) Let f be uniformly continuous on S and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence of elements in S. Prove that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Calculus and Linear Algebra

2:00 pm Friday, February 2, 2007

Seeley Mudd 206

There are 12 problems on this examination. Record your answers in the blue book provided. SHOW ALL WORK.

1. Evaluate the following limits.

   (a) $\lim_{x \to 0} \dfrac{x \sin(\sin x)}{\sin^2 x}$

   (b) $\lim_{x \to \infty} \left(1 + \dfrac{1}{3x}\right)^{2x}$

2. Evaluate the following integrals.

   (a) $\displaystyle\int_e^{\infty} \frac{1}{x\sqrt{(1 + \ln x)^3}}\, dx$

   (b) $\displaystyle\int_0^{3} \frac{1}{x^2 + 3}\, dx$

3. Determine whether each series converges or diverges. Justify your reasoning.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)!(n-1)!}{(2n+1)!}$

4. Determine the values of $x \in \mathbf{R}$ for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n+2}$ converges absolutely, converges conditionally, or diverges. Justify your reasoning

5. Let $F(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

   (a) Prove that $F(x, y)$ is continuous at $(0, 0)$.

   (b) Compute $F_x(0, 0)$ and $F_y(0, 0)$.

   (c) Prove that $F(x, y)$ is not differentiable at $(0, 0)$.

6. Find the maximum and minimum values of $F(x, y) = 3x^2 + 2y^3$ on the unit circle $x^2 + y^2 = 1$.

7. Evaluate the following integrals

(a) $\int_C (e^x - y^3)\,dx + (\cos y + x^3)\,dy$, where C is the circle $x^2 + y^2 = 2$ oriented counter clockwise.

(b) $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx$

8. Let $F(x, y, z) = \frac{x+z}{z-y}$ and let $P = (2, 3, 4)$.

(a) Compute the gradient of F at P.

(b) Compute the directional derivative of F at P in the direction of most rapid increase of F.

9. Consider the region in 3-dimensional space that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$.

(a) Express the volume of the region as a triple integral in cartesian, cylindrical and spherical coordinates.

(b) Compute one of the integrals of part (a).

10. Consider the matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, a 3×3 matrix with entries in $\mathbf{R}$.

(a) Compute a basis of solutions of the equation $AX = 0$.

(b) Compute the characteristic polynomial and eigenvectors of A.

(c) Find a basis of $\mathbf{R}^3$ consisting of eigenvectors of A.

11. Let V and W be vector spaces over $\mathbf{R}$ and let $T : V \to W$ be a linear map. Assume that V has a basis $v_1, \ldots, v_n$ such that $T(v_1), \ldots, T(v_n)$ span W.

(a) Prove that T is onto.

(b) If $\dim V = \dim W$, what else can you conclude about T? Explain your reasoning.

12. Let V be a finite-dimensional vector space over $\mathbf{R}$ and let $T : V \to V$ be a linear map such that the composition $T \circ T$ is the zero linear map. Let

$N(T) = \{v \in V : T(v) = 0\}$

$R(T) = \{v \in V : v = T(w) \text{ for some } w \in V\}$.

You may assume that these are subspaces of V.

(a) Prove that $R(T) \subseteq N(T)$.

(b) Prove that $\dim N(T) \geq \frac{1}{2} \dim V$.

Amherst College
Department of Mathematics and Computer Science

Comprehensive Examination: Mathematics 26

Friday, February 2, 2007

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. Let G be a group, and let $H \subseteq G$ be a subgroup. Let L be the set of left cosets of H in G, and let R be the set of right cosets of H in G. That is,

$$L = \{aH : a \in G\}, \qquad \text{and} \qquad R = \{Ha : a \in G\}.$$

Define the function $f : L \to R$ by $f(aH) = Ha^{-1}$.

   (a) Prove that f is well-defined.

   (b) Prove that f is onto.

   (*Note: you may* **not** *assume that L or R is finite, and you may* **not** *assume that H is normal in G.*)

2. Fix an integer $n \geq 2$, and write S_n for the permutation group on n letters. Let

$$\phi : S_n \to G$$

be a homomorphism, where G is a group of odd order. (I.e., G is a finite group with an odd number of elements.)

   (a) Prove that every **transposition** (i.e., 2-cycle) $\tau \in S_n$ is in $\ker \phi$. That is, prove that $\phi(\tau) = e$.

   (b) Prove that ϕ is the trivial homomorphism; i.e., prove that for all $\sigma \in S_n$, we have $\phi(\sigma) = e$.

3. Let R be a ring with unity, and let $I \subseteq R$ be a subset.

   (a) Define what it means for I to be an **ideal** of R.

   (b) Recall that a **unit** is an element $u \in R$ that has a multiplicative inverse $v \in R$. If I is an ideal and contains a unit, prove that $I = R$.

4. Let $R = \mathbb{Z}[x]$ be the ring of polynomials (in one variable) with integer coefficients. Note that the constant polynomial 2 and the degree one polynomial x are both elements of R. Define

$$I = \{2f + xg : f, g \in R\}.$$

   (a) Prove that I is an ideal of R.

   (b) Prove that

$$I = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{Z} \text{ and } a_0 \text{ is even}\}.$$

   That is, prove that I consists of exactly those polynomials in R with even constant term.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION: MATHEMATICS 28

February 2, 2007

Work the following four problems. Record your answers in the blue book provided.
PLEASE SHOW ALL OF YOUR WORK.

1. (a) State the Completeness Axiom (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

   (b) Prove that a decreasing sequence of real numbers which is bounded below must converge.

2. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

   (a) Find a simple formula (not involving summation notation) for the kth partial sum,
$$\sum_{n=1}^{k} \frac{1}{n(n+1)},$$
   and prove that it is correct.

   (b) Prove that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges (and clearly state what it converges to).

3. Suppose $f : [-1,1] \to \mathbb{R}$ is continuous and satisfies $f(-1) = f(1)$. Use the Intermediate Value Theorem to prove that there exists $\gamma \in [0,1]$ such that $f(\gamma) = f(\gamma - 1)$.

4. Define a sequence of functions $\{f_n\}$, where $f_n : [0,1] \to \mathbb{R}$, via
$$f_n(x) = \frac{x^n}{1+x^n}.$$

   (a) Show that $\{f_n\}$ converges uniformly to 0 on $[0,a]$ for any a satisfying $0 < a < 1$.

   (b) Does $\{f_n\}$ converge uniformly on [0,1]? Explain your reasoning.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, February 3, 2006

Seeley Mudd 206

There are 12 problems on this examination. Record your answers in the blue book provided. SHOW ALL WORK.

1. Evaluate the following limits.

   (a) $\lim_{x \to 0} \frac{xe^x - x}{\sin^2 x}$

   (b) $\lim_{x \to \infty} \sin\left(\frac{1}{x}\right) \ln(1 + e^x)$

2. Evaluate the following integrals.

   (a) $\int \frac{(1+x)^2}{\sqrt{x}}\, dx$

   (b) $\int \sqrt{4 - x^2}\, dx$

   (c) $\int_C (2xy^2 - y^3)\, dx + (2x^2y + x^3)\, dy$, where C is the circle $x^2 + y^2 = 2$ oriented counterclockwise.

3. For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges.

   (a) $\sum_{n=1}^{\infty} \frac{1 + \sin(n)}{2^{2n}}$

   (b) $\sum_{n=1}^{\infty} \frac{(2n)!}{2^n (n!)^2}$

   (c) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{1 + n^3}$

4. Find all real numbers x for which for series $\sum_{n=1}^{\infty} (-1)^n \frac{e^{nx}}{n^2}$ converges.

5. Find the volume of the region in 3-dimensional space inside the cylinder $x^2 + y^2 = 1$, above the xy plane, and below the plane $x + z = 1$.

6. Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Compute $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

(b) Prove that $f(x,y)$ is not differentiable at $(0,0)$.

7. Let $f(x,y) = xy + \displaystyle\int_0^y \sin(t^2)\,dt$

(a) Compute $\nabla f(a,b)$

(b) Show that $(0,0)$ is a saddle point of $f(x,y)$

8. (a) State the Mean Value Theorem.

(b) State the Intermediate Value Theorem.

(c) Use parts (a) and (b) to prove that the equation $x \ln x = 2$ has exactly one solution in the interval $[1, e]$

9. Let $T : V \to V$ be a linear transformation such that $T \circ T$ is the zero linear transformation. Also let $v \in V$ satisfy $T(v) \neq 0$. Prove that the set $\{v, T(v)\}$ is linearly independent.

10. Compute the inverse of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 4 & 1 & 2 \end{pmatrix}.$$

Check your answer by matrix multiplication.

11. Let $M_n(\mathbf{R})$ be the vector space of all $n \times n$ matrices with real entries. We say that $A, B \in M_n(\mathbf{R})$ commute if $AB = BA$.

(a) Fix $A \in M_n(\mathbf{R})$. Prove that the set of all matrices in $M_n(\mathbf{R})$ that commute with A is a subspace of $M_n(\mathbf{R})$.

(b) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbf{R})$ and let $W \subseteq M_2(\mathbf{R})$ be the subspace of all matrices of $M_2(\mathbf{R})$ that commute with A. Find a basis of W.

12. Let $T : V \to V$ and $U : V \to V$ be linear transformations that commute, i.e. $T \circ U = U \circ T$. Let $v \in V$ be an eigenvector of T such that $U(v) \neq 0$. Prove that $U(v)$ is also an eigenvector of T.

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 26
February 3, 2006

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. Let H and K be subgroups of a group G. The set HK is defined to be $\{hk : h \in H, k \in K\}$.

   (a) Show that if H is normal in G, then HK is a subgroup of G.

   (b) Show that if both H and K are normal in G, then HK is a normal subgroup of G.

2. Let φ be a homomorphism from the group G to the group G'.

   (a) Prove that $\varphi(e) = e'$, where e and e' are the identities of G and G' respectively.

   (b) Prove that $\varphi(x^{-1}) = (\varphi(x))^{-1}$ for each $x \in G$.

   (c) Suppose that G' is abelian and that H is a subgroup of G containing $\ker\varphi$. Prove that H must be normal in G.

3. (a) Define what it means for a subset I of a ring R to be an **ideal**.

   (b) Define what it means for an ideal of a ring R to be **maximal**.

   (c) Let R be the ring of all real-valued functions defined on the real numbers, under the usual pointwise operations. Let $I = \{f \in R : f(0) = 0\}$. Show that I is an ideal of R and that it is maximal.

4. Let R be a commutative ring with $1 \neq 0$.

   (a) Define what it means for R to be an **integral domain**.

   (b) Show that if R is an integral domain containing an ideal I such that $\{0\} \neq I \neq R$, then R must be infinite.

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
February 3, 2006

Work the following four problems. Record your answers in the blue book provided.
PLEASE SHOW ALL YOUR WORK.

1. State a theorem from the course, having the hypothesis: "If f is a continuous, real-valued function on a compact set, then ..."

2. Prove that for every positive integer n, the following inequality is true:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

3. Let $\{f_n\}_{n\geq 1}$ be a sequence of real-valued functions of a real variable, continuous on the interval $[a, b]$.

   i) Complete the following definition: $\{f_n\}_{n\geq 1}$ *converges uniformly to* g *on* $[a, b]$ if ...

   ii) If each f_n is continuous on $[a, b]$ and $\{f_n\}_{n\geq 1}$ converges uniformly to g on $[a, b]$, then prove that $\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b g(x)\, dx$

4. Recall that a compact set C has the property that if it is "covered" by a family $\{O_\alpha\}_{\alpha\in A}$ of open sets: $C \subset \bigcup_{\alpha\in A} O_\alpha$, then some finite subfamily, $\{O_{\alpha_1}, O_{\alpha_2}, O_{\alpha_3}, \ldots, O_{\alpha_n}\}$, suffices to cover C : $C \subset \bigcup_{k=1}^{n} O_{\alpha_k}$. Using this characterization of compactness, show that the interval $[0, \infty)$ of real numbers is not compact.

AMHERST COLLEGE

Department of Mathematics and Computer Science

COMPREHENSIVE EXAMINATION

Mathematics 11, 12, 13, 25

2:00 p.m. Friday, January 28, 2005

Seeley Mudd 205

1. Evaluate the following limits.

   (a) $\lim_{x \to 0} \dfrac{x \tan(x)}{\sin(x^2)}$

   (b) $\lim_{x \to \infty} \left(1 - \dfrac{\pi}{x}\right)^x$

   (c) $\lim_{n \to \infty} \sum_{i=0}^{n} (-1)^i \dfrac{\pi^{2i}}{(2i)!}$

2. Determine whether each series converges absolutely, converges conditionally, or diverges. Justify your answers.

   (a) $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^2 - 1}{n^2 + 1}$

   (b) $\sum_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{n}$

   (c) $\sum_{n=1}^{\infty} (-1)^n \dfrac{n^n}{4^n n!}$

3. Evaluate the following integrals.

   (a) $\int \sin^2 \theta \cos^3 \theta \, d\theta$

   (b) $\int_0^{\infty} \dfrac{e^x}{(1 + e^x)^2} \, dx$

   (c) $\int_0^2 \int_0^{\sqrt{4-x^2}} \dfrac{1}{1 + x^2 + y^2} \, dy \, dx$

   (d) $\int_C -y \, dx + x \, dy$, where C is a simple closed curve, oriented counterclockwise, bounding a closed region A in the plane

4. Determine all real numbers x for which the series $\sum_{n=0}^{\infty} \dfrac{(x-3)^n}{3^n(n+1)}$ converges.

5. In hydrogeology, one studies water flow in an underground aquifer. At a point (x, y) on the surface, dig down until you hit water, and let $h(x, y)$ be the height of the water above sea level. In what direction does the vector $-\nabla h$ point? How should this relate to how water flows in the aquifer?

6. Find the critical points of $f(x, y) = 3x^2 - 3xy + y^3$ and for each critical point, determine whether it is a local minimum, a local maximum, or a saddle point.

7. Let $f(x,y) = \begin{cases} \frac{x^2y^2+x^2+y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

   (a) Compute $f_x(0,0)$ and $f_y(0,0)$.

   (b) Prove that $f(x,y)$ is differentiable at $(0,0)$.

   (c) Is $f(x,y)$ continuous at $(0,0)$? Explain your reasoning.

8. (a) State the Mean Value Theorem.

   (b) Use the Mean Value Theorem to prove that $|\sin a - \sin b| \leq |a-b|$ for all real numbers a and b.

9. Consider the 3-dimensional region consisting of all points (x,y,z) satisfying $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x^2+y^2+z^2 \leq 1$.

   (a) Express the volume of this region as a triple integral in cartesian, cylindrical and spherical coordinates.

   (b) Evaluate one of the integrals given in (a).

10. (a) Compute all eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$.

    (b) Compute the inverse of $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

11. Suppose that V is a vector space and that $v_1, \ldots, v_n \in V$ are linearly independent. Also assume that $v \in V$ is not contained in the span of $v_1, \ldots, v_n$. Prove carefully that $v, v_1, \ldots, v_n$ are linearly independent.

12. Let P_n be the vector space of polynomials in x of degree at most n with real coefficients. Define

$$T : P_2 \to P_3$$

    by $T(f) = \int_0^x f(t)\,dt$.

    (a) Prove that T is linear.

    (b) Compute $T(a+bx+cx^2)$, where a, b, c are real numbers.

    (c) Compute the matrix of T with respect to the bases $\{1, x, x^2\}$ of P_2 and $\{1, x, x^2, x^3\}$ of P_3.

13. Let A be a 3×5 matrix with real entries, and assume that A has 3 linearly independent columns. The null space of A is defined to be

$$N(A) = \{v \in \mathbf{R}^5 : Av = 0\}$$

    Determine the dimension of $N(A)$. Justify your reasoning.

Amherst College
Department of Mathematics and Computer Science

Comprehensive Examination: Mathematics 26

Friday, January 28, 2005

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. Let G be a group.

   (a) Let $g, h \in G$. Prove that for all integers $k \geq 1$,

   $$(h^{-1}gh)^k = h^{-1}g^k h.$$

   (b) Let N be a normal subgroup of G, and suppose that N is cyclic. Let H be a subgroup of N. Prove that H is a normal subgroup of G.

2. Recall that S_5 denotes the permutation group on 5 letters.

   (a) Write the permutation $\sigma = (1\ \ 3\ \ 5\ \ 2)(1\ \ 3\ \ 4)$ as a product of disjoint cycles in S_5.

   (b) What is the order of σ?

   (c) Is σ an even or an odd permutation?

3. A nonzero element a of a ring is said to be *nilpotent* if there is a positive integer $n \geq 1$ such that $a^n = 0$. (The element 0 itself is *not* said to be nilpotent.)

   (a) Let $\mathbb{Z}/8\mathbb{Z}$ denote the ring of integers modulo 8 (sometimes written $\mathbb{Z}/8$ or $\mathbb{Z}_8$). Find all nilpotent elements of $\mathbb{Z}/8\mathbb{Z}$.

   (b) Let R be a commutative ring, and let $I \subseteq R$ be an ideal. Prove that the following two statements are equivalent:

      i. The quotient ring R/I contains no nilpotents.

      ii. For every element $b \in R$ such that $b^m \in I$ for some positive integer $m \geq 1$, we have $b \in I$.

4. Let $\mathbb{F}_2 = \{0, 1\}$ denote the field of two elements, and let $\mathbb{F}_2[x]$ be the ring of polynomials in one variable with coefficients in $\mathbb{F}_2$. Find all irreducible polynomials of degree 4 in $\mathbb{F}_2[x]$.

AMHERST COLLEGE
Department of Mathematics and Computer Science
COMPREHENSIVE EXAMINATION: MATHEMATICS 28
January 28, 2005

Work the following five problems. Record your answers in the blue book provided.
PLEASE SHOW ALL YOUR WORK.

1. State the Completeness Axiom for the Real Numbers (also known as Axiom C or the Axiom of Continuity).

2. Complete the following definition:
Let the sequence $\{f_n\}_{n=1}^{\infty}$ of real valued functions of a real variable each have domain [a, b]. $\{f_n\}_{n=1}^{\infty}$ converges uniformly on [a, b] if and only if . . . .

3. Given an explicit example of a sequence $\{f_n\}_{n=1}^{\infty}$ of functions having domain [0, 1] and satisfying
   (a) Each f_n is continuous on [0, 1],
   (b) $\lim_{n\to\infty} f_n(x) = g(x)$ for all x in [0, 1], and
   (c) g is not a continuous function on [0, 1].

4. In the plane, let C_r denote the circle of radius r, centered at the origin. Find the cluster points (also called *points of accumulation*) of the set $\bigcup_{n=1}^{\infty} C_{1/n}$.

5. Prove that for every positive integer n, $7^n - 2^n$ is divisible by 5.