Math 111, Introduction to the Calculus, Fall 2011
Midterm II Practice Exam 1 Solutions

1. *Find the derivative of each of the following functions (no explanation is necessary, but it may help you get the correct answer if you show your work):*

   (a) $f(x) = 2x^3 - 3x + 1$

   (b) $g(t) = \dfrac{t^2 - 1}{t^2 + 1}$

   (c) $P(r) = \sqrt{r^2 + 1}$

   (d) $f(\theta) = \theta \sin\left(\dfrac{1}{\theta}\right)$

   (e) $H(t) = \tan^2\left(\frac{t}{2}\right)$

   (a) $f'(x) = 6x^2 - 3$

   (b) $g'(t) = \dfrac{(t^2+1)(2t) - (t^2-1)(2t)}{(t^2+1)^2} = \dfrac{4t}{(t^2+1)^2}$

   (c) $P'(r) = \dfrac{1}{2}(r^1 + 1)^{-\frac{1}{2}}(2r) = \dfrac{r}{\sqrt{r^2+1}}$

   (d) $f'(\theta) = \sin\left(\dfrac{1}{\theta}\right) + \theta(-\theta^{-2})\cos\left(\dfrac{1}{\theta}\right) = \sin\left(\dfrac{1}{\theta}\right) - \dfrac{1}{\theta}\cos\left(\dfrac{1}{\theta}\right)$

   (e) Here you can quote the fact that $\frac{d}{dx}(\tan x) = (\cos x)^{-2}$ and so get

   $$H'(t) = 2\tan(t/2)(\cos(t/2))^{-2}(1/2) = \frac{\tan(t/2)}{\cos^2(t/2)}.$$

2. *The variables y and x are related by the equation*

   $$y^2 - y = x.$$

   Find the equation of the tangent line to the graph of y against x at the point $(2, 2)$.

   Differentiating this equation we get

   $$2yy' - y' = 1$$

   and so

   $$y' = \frac{1}{2y - 1}.$$

   Therefore, at the point $(2, 2)$, the slope of the tangent line is given by $1/(4-1) = 1/3$. The equation of the tangent line is therefore

   $$(y - 2) = \frac{x - 2}{3}$$

   or

   $$y = \frac{1}{3}x + \frac{4}{3}.$$

3. *The functions $f(t)$ and $g(t)$ are related by the equation*

$$fg = \frac{1}{f+g}.$$

Suppose that $f(0) = 2$ and $f'(0) = -1$. Find $g(0)$ and $g'(0)$.

It's easier here to multiply both sides by $f + g$ to get

$$fg(f+g) = 1$$

or

$$f^2g + fg^2 = 1.$$

Differentiating this, we get

$$2ff'g + f^2g' + 2fgg' + f'g^2 = 0.$$

Rather than solving for g' now I recommend waiting until we can substitute in values we know.

The question asks us to find $g(0)$. From the original equation, we have

$$f(0)g(0) = \frac{1}{f(0)+g(0)}$$

and hence

$$2g(0) = \frac{1}{2+g(0)}$$

so

$$2g(0)^2 + 4g(0) - 1 = 0.$$

Therefore

$$g(0) = \frac{-4 \pm \sqrt{16+8}}{4} = -1 \pm \sqrt{24}/4 = -1 \pm \sqrt{6}/2.$$

For each of these values of $g(0)$ we can now substitute what we know into the equation the tells us the value of g'.

If $g(0) = -1 + \sqrt{6}/2$:

$$2(2)(-1)(-1+\sqrt{6}/2) + 4g'(0) + 2(2)(-1+\sqrt{6}/2)g'(0) + (-1)(-1+\sqrt{6}/2)^2 = 0$$

and so

$$g'(0) = \frac{-4+2\sqrt{6}+1-\sqrt{6}+6/4}{4-4+2\sqrt{6}} = \frac{-3/2+\sqrt{6}}{2\sqrt{6}} = \frac{-3+2\sqrt{6}}{4\sqrt{6}}.$$

If $g(0) = -1 - \sqrt{6}/2$:

$$2(2)(-1)(-1-\sqrt{6}/2) + 4g'(0) + 2(2)(-1-\sqrt{6}/2)g'(0) + (-1)(-1-\sqrt{6}/2)^2 = 0$$

and so

$$g'(0) = \frac{-4-2\sqrt{6}+1+\sqrt{6}+6/4}{4-4-2\sqrt{6}} = \frac{-3/2-\sqrt{6}}{-2\sqrt{6}} = \frac{3+2\sqrt{6}}{4\sqrt{6}}.$$

4. *The populations of wolves (W) and bears (B) in a forest are related by the equation*

$$W + B^3 = \textit{constant}.$$

At some point there are 10 *bears and* 1000 *wolves. If the number of bears is increasing at a rate of 1 bear per year, how fast the number of wolves increasing?*

Differentiating the equation we get

$$\frac{dW}{dt} + 3B^2\frac{dB}{dt} = 0.$$

Therefore, at the point when $B = 10$ and $\frac{dB}{dt} = 1$, we have

$$\frac{dW}{dt} = -3B^2\frac{dB}{dt} = -300.$$

So the number of wolves is *decreasing* at a rate of 300 wolves per year.

5. *A child is flying a kite at an altitude of 200 feet with the kite string at an angle of 30 degrees to horizontal. If the child wants to pull the kite in so that it moves horizontally at a speed of 2 feet per second. How fast should the child pull the string in? (Assume that the kite naturally remains at an altitude of 200 feet.)*

I highly recommend drawing a diagram to see what is happening here. Let $x(t)$ be the horizontal position of the kite at time t and $l(t)$ the length of the string from where the child is holding the string to the kite. We then want to find $\frac{dl}{dt}$ when $\frac{dx}{dt} = -2$. To answer this, we need to know the relationship between l and x. Since the altitude of the kite is 200 feet, we have, by the Pythagorean Theorem

$$200^2 + x^2 = l^2.$$

Differentiating this we get

$$2x\frac{dx}{dt} = 2l\frac{dl}{dt}.$$

To solve for $\frac{dx}{dt}$ we therefore also need to know the values of x and l at the point where the child starts pulling the kite in. since the angle is initially 30 degrees to the horizontal, we have

$$\frac{200}{x} = \tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$$

and so $x = 200\sqrt{3}$. Similarly, we have

$$\frac{200}{l} = \sin(\frac{\pi}{6}) = \frac{1}{2}$$

and so $l = 400$. Therefore we get

$$2(200\sqrt{3})(-2) = 2(400)\frac{dl}{dt}$$

and so

$$\frac{dl}{dt} = -\sqrt{3}.$$

So the child should pull the string in at a rate of $\sqrt{3}$ feet per second.