Math 111, Introduction to the Calculus, Fall 2011
Midterm II Practice Exam 2 Solutions

1. *(a) State the product rule for finding the derivative of a product $f(x)g(x)$ of two functions f and g.*

   (b) Use the product rule to show that the derivative of

$$f(x) = x^{\frac{3}{2}}$$

   is given by

$$f'(x) = \frac{3\sqrt{x}}{2}.$$

   (In your answer, you may only use the power rule to differentiate x^n where n is a positive integer.)

(a) $\dfrac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

(b) We have

$$f(x)f(x) = x^3.$$

Differentiating both sides we get

$$f'(x)f(x) + f(x)f'(x) = 3x^2$$

and so

$$2f'(x)f(x) = 3x^2.$$

Therefore,

$$f'(x) = \frac{3x^2}{2f(x)} = \frac{3x^2}{2x^{\frac{3}{2}}} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}.$$

2. *Find the derivative of the following function*

$$P(t) = t^2 \cos\left(\frac{2t+1}{2t-1}\right).$$

By the product rule, we have

$$P'(t) = 2t\cos\left(\frac{2t+1}{2t-1}\right) + t^2\frac{d}{dt}\left(\cos\left(\frac{2t+1}{2t-1}\right)\right).$$

By the chain rule

$$\frac{d}{dt}\left(\cos\left(\frac{2t+1}{2t-1}\right)\right) = -\sin\left(\frac{2t+1}{2t-1}\right)\frac{d}{dt}\left(\frac{2t+1}{2t-1}\right)$$

and by the quotient rule

$$\frac{d}{dt}\left(\frac{2t+1}{2t-1}\right) = \frac{(2t-1)(2)-(2t+1)(2)}{(2t-1)^2} = \frac{-4}{(2t-1)^2}.$$

Putting all of this together we have

$$\begin{aligned} P'(t) &= 2t\cos\left(\frac{2t+1}{2t-1}\right) - t^2\sin\left(\frac{2t+1}{2t-1}\right)\frac{-4}{(2t-1)^2} \\ &= 2t\cos\left(\frac{2t+1}{2t-1}\right) + \frac{4t^2}{(2t-1)^2}\sin\left(\frac{2t+1}{2t-1}\right) \end{aligned}$$

3. *The height h (measured in meters) of a toy rocket at a time t seconds after taking off satisfies the equation*

$$\frac{h^2}{t^2} + 5h = 100 - 50t.$$

How fast is the rocket moving 1 second after take off?

Differentiating the given equation with respect to t, we get

$$\frac{t^2(2hh') - h^2(2t)}{t^4} + 5h' = -50.$$

We want to find the value of $h'(1)$ so we evaluate this at $t = 1$ to get

$$\frac{2h(1)h'(1) - 2h(1)^2}{1} + 5h'(1) = -50.$$

To figure this out we need to find $h(1)$. From the original equation we have

$$\frac{h(1)^2}{1} + 5h(1) = 100 - 50$$

and so

$$h(1)^2 + 5h(1) - 50 = 0$$

and so

$$(h(1) + 10)(h(1) - 5) = 0$$

so either $h(1) = -10$ or $h(1) = 5$. Since the height cannot be negative we must have $h(1) = 5$. Therefore the previous equation becomes

$$10h'(1) - 50 + 5h'(1) = -50$$

and so

$$h'(1) = 0.$$

Therefore, the speed of the rocket 1 second after take off is zero.

4. *Find the equation of the tangent line to the function*

$$H(r) = \sin(2r + \pi)$$

at the point $r = 0$.

The slope of the tangent line is given by $H'(0)$. We have

$$H'(r) = 2\cos(2r + \pi)$$

so

$$H'(0) = 2\cos(\pi) = -2.$$

Since $H(0) = \sin(\pi) = 0$, the line passes through the point $(0, 0)$. Therefore, the equation of the line is

$$y = -2r.$$

(Since r was the input variable of the original function, it makes sense to give the equation of the line in terms of r instead of x.)

5. *A television camera watching the launch of the Space Shuttle is 500 meters from the launch site. If the take-off speed of the shuttle is* 100 *meters per second, how fast does the camera initially need to turn (in radians per second) to keep the Shuttle in the center of the picture?*

It's a very good idea to draw a diagram here to show what is going on. Let $h(t)$ be the height of the Shuttle (in meters) above the launch pad after t seconds and let $\theta(t)$ be the angle between the camera direction and the horizontal after t seconds. Then we have

$$\tan(\theta) = \frac{h}{500}.$$

We are interested in the value of $\theta'(0)$.

Differentiating the equation we get

$$\frac{1}{\cos^2(\theta)}\theta' = \frac{h'}{500}.$$

At the point of launch, $t = 0$, we have $\theta = 0$ and $h' = 100$. Therefore, this equation gives

$$\frac{1}{\cos^2(0)}\theta'(0) = \frac{100}{500}.$$

Therefore

$$\theta'(0) = \frac{1}{5}.$$

So the initial rate at which the camera needs to turn is 0.2 radians per second.