1. Sketch a graph of the function

\[ f(x) = \frac{x^2 - 1}{x^2 + 1} \]

to show the horizontal asymptote, as well as the intervals on which \( f \) is increasing or decreasing. You should explain how you worked out each part of your answer.

To find the horizontal asymptote we have to look at the limit of \( f(x) \) as \( x \to \pm \infty \). We have

\[
\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} = \lim_{x \to \infty} \frac{1}{1} = 1.
\]

Similarly

\[
\lim_{x \to -\infty} = 1.
\]

Therefore, there is a horizontal asymptote at \( y = 1 \).

To decide where \( f \) is increasing or decreasing, we have to look at \( f' \). We have

\[
f'(x) = \frac{(x^2 + 1)(2x) - (x^2 - 1)2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}.
\]

Since the denominator is always positive, we see that \( f'(x) \) is negative when \( x < 0 \), zero when \( x = 0 \) and positive when \( x > 0 \). Therefore, \( f(x) \) is decreasing for \( x < 0 \) and increasing for \( x > 0 \) with a local minimum at \( x = 0 \). At \( x = 0 \) the value of \( f \) is \(-1\) and (although the question did not ask for it) the graph crosses the \( x \)-axis at \( x = \pm 1 \). Therefore the graph looks as follows:
2. Find the critical points of the function

\[ f(x) = x^2 \sqrt{x+1} \]

and classify each critical point is a local maximum, local minimum, or neither.

The derivative is

\[ f'(x) = 2x \sqrt{x+1} + \frac{x^2}{2 \sqrt{x+1}} = \frac{4x(x+1) + x^2}{2 \sqrt{x+1}} = \frac{x(5x + 4)}{2 \sqrt{x+1}}. \]

The critical points are when this is zero, that is

\[ x(5x + 4) = 0 \]

so \( x = 0 \) or \( x = -\frac{4}{5} \).

Rather than using the Second Derivative Test, we can just see where \( f'(x) \) is positive or negative. For \( x < -4/5 \), we have \( x < 0 \) and \( 5x + 4 < 0 \), so \( f'(x) > 0 \). Therefore, \( f \) is increasing on this region. For \( -4/5 < x < 0 \), we have \( x < 0 \) but \( 5x + 4 > 0 \), so \( f'(x) < 0 \). Therefore \( f \) is decreasing on this region. For \( x > 0 \), we have \( x > 0 \) and \( 5x + 4 > 0 \), so \( f'(x) > 0 \) and \( f \) is increasing again on this region. This means that \( x = -4/5 \) is a local maximum and \( x = 0 \) is a local minimum.

3. Find the point on the curve \( y = \sqrt{2x+9} \) that is closest to the origin.

We are trying to minimize the quantity

\[ d = \sqrt{x^2 + y^2} \]

where \( x, y \) are related by the equation

\[ y = \sqrt{2x + 9}. \]

Therefore

\[ d(x) = \sqrt{x^2 + 2x + 9}. \]

The range of possible \( x \) values is \( x \geq -9/2 \) (because we cannot take the square-root of a negative number). To find the minimum we therefore have to compare the value of \( d(x) \) at any critical points and at the endpoint \( x = -9/2 \).

We have

\[ d'(x) = \frac{2x + 2}{2 \sqrt{x^2 + 2x + 9}} = \frac{x + 1}{\sqrt{x^2 + 2x + 9}}. \]

The critical points are when \( x + 1 = 0 \), that is \( x = -1 \). Notice that as long as \( 2x + 9 > 0 \) then \( x^2 + 2x + 9 > 0 \) so the function \( d(x) \) is differentiable at all points we are considering.

Our two candidates for the closest point are therefore \( x = -9/2 \) where \( d = 9/2 \) and \( x = -1 \) where \( d = \sqrt{8} < 3 \). Therefore, the closest point is when \( x = -1 \), that is, at the point \((-1, \sqrt{7})\).
4. Calculate the integral \[ \int_{-1}^{2} (1 - x) \, dx \]
in two ways:

(a) by drawing a graph and finding the appropriate area or areas;

(b) using the Fundamental Theorem of Calculus.

(a) The graph looks as follows

Areas of regions below the \( x \)-axis count negative so the integral is equal to

\[ \frac{1}{2} (2)(2) - \frac{1}{2} (1)(1) = \frac{3}{2}. \]

(b) The integral is equal to

\[ \left[ x - \frac{x^2}{2} \right]_{x=-1}^{x=2} = (2 - \frac{2^2}{2}) - ((-1) - \frac{(-1)^2}{2}) = 0 + 1 + 1/2 = \frac{3}{2}. \]

5. (a) Find the derivative of the function

\[ f(t) = \int_{1}^{t^2} \frac{1}{x} \, dx. \]

(b) Find an antiderivative \( G(x) \) for the function

\[ g(x) = 3 \sin(x + \pi) \]

that has the property that \( G(0) = 5 \).
(a) We can write
\[ f(t) = F(t^2) \]
where
\[ F(u) = \int_1^u \frac{1}{x} \, dx. \]
By the Chain Rule we have
\[ f'(t) = F'(t^2)(2t) \]
and by the Fundamental Theorem of Calculus (Part I) \( F'(u) = \frac{1}{u} \), so
\[ f'(t) = \frac{1}{t^2}(2t) = \frac{2}{t}. \]

(b) We might guess that an antiderivative involves \( \cos(x + \pi) \). The derivative of \( \cos(x + \pi) \) is \( -\sin(x + \pi) \), so the derivative of \(-3 \cos(x + \pi) \) is \( 3 \sin(x + \pi) \). This means that our antiderivative of \( g(x) \) must be of the form
\[ G(x) = -3 \cos(x + \pi) + c \]
for some constant \( c \). We want \( G(0) = 5 \) which means that
\[ -3 \cos(0 + \pi) + c = 5 \]
which tells us that
\[ 3 + c = 5 \]
or \( c = 2 \). Therefore, the antiderivative that we require is
\[ G(x) = -3 \cos(x + \pi) + 2. \]