1. Calculate the following integrals

(a) \[ \int_1^4 \frac{1}{\sqrt{x}} \, dx \]

(b) \[ \int_{-\pi/2}^0 \sin(2t) \, dt \]

(a) Since \( \frac{1}{\sqrt{x}} = x^{-1/2} \) we have

\[
\int_1^4 x^{-1/2} \, dx = \left[ \frac{x^{1/2}}{1/2} \right]_{x=1}^{x=4} = 2(4^{1/2}) - 2(1^{1/2}) = 4 - 2 = 2
\]

(b) We can guess that the antiderivative of \( \sin(2t) \) involves \( \cos(2t) \). In fact, the derivative of \( \cos(2t) \) is

\[-2 \sin(2t)\]

so the antiderivative of \( \sin(2t) \) is \(-\frac{1}{2} \cos(2t)\). Therefore

\[
\int_{-\pi/2}^0 \sin(2t) \, dt = \left[ -\frac{1}{2} \cos(2t) \right]_{t=-\pi/2}^{t=0} = (-\frac{1}{2} \cos(0)) - (-\frac{1}{2} \cos(-\pi)) = -\frac{1}{2} - \frac{1}{2} = -1
\]

2. Find the critical points of the function

\[ f(x) = x \sin(x) + \cos(x) \]

that are in the range \( -\pi \leq x \leq \pi \). Classify each critical point as either a local maximum, a local minimum or neither.

To find the critical points we find \( f'(x) \):

\[
f'(x) = \sin(x) + x \cos(x) - \sin(x) = x \cos(x)
\]
The critical points therefore satisfy $x \cos(x) = 0$. This means that either $x = 0$ or $\cos(x) = 0$. In the range $-\pi \leq x \leq \pi$ the solutions to $\cos(x) = 0$ are $x = -\pi/2$ and $x = \pi/2$. So the critical points are:

$$x = 0, \quad x = -\pi/2, \quad x = \pi/2.$$ 

To classify these, we use the second derivative test. We have

$$f''(x) = \cos(x) - x \sin(x).$$

Then:

$$f''(-\pi/2) = \cos(-\pi/2) + \pi/2 \sin(-\pi/2) = -\pi/2 < 0,$$

so $x = -\pi/2$ is a local maximum;

$$f''(0) = \cos(0) = 1 > 0,$$

so $x = 0$ is a local minimum;

$$f''(\pi/2) = \cos(\pi/2) - \pi/2 \sin(\pi/2) = -\pi/2 < 0,$$

so $x = \pi/2$ is a local maximum.

3. For the function

$$f(x) = \frac{x^2}{x + 1},$$

we have

$$f'(x) = \frac{x(x + 2)}{(x + 1)^2}$$

and

$$f''(x) = \frac{2}{(x + 1)^3}.$$ 

(a) Where does the graph of $f(x)$ have a vertical asymptote?

(b) On which intervals is $f(x)$ increasing or decreasing?

(c) On which intervals is $f(x)$ concave up or concave down?

Sketch a graph of the function $f$ for the range $-4 \leq x \leq 4$ that displays your answers to the above questions. (Your graph should also show that $f(0) = 0$ and $f(-2) = -4$. There is extra space on the next page for this question so you can make your graph nice and big.)

(a) $x = -1$

(b) Since $(x + 1)^2$ is always positive, it is sufficient to look at the sign of $x(x + 2)$.
   For $x < -2$, $x < 0$ and $x + 2 < 0$, so $f'(x) > 0$ and $f$ is increasing.
   For $-2 < x < -1$, $x < 0$ and $x + 2 > 0$ so $f'(x) < 0$ and $f$ is decreasing.
   For $-1 < x < 0$, $x < 0$ and $x + 2 > 0$ so $f'(x) < 0$ and $f$ is decreasing.
   For $x > 0$, $x > 0$ and $x + 2 > 0$ so $f'(x) > 0$ and $f$ is increasing.
(c) For $x < -1$, we have $(x + 1)^3 < 0$ so $f$ is concave down. For $x > -1$, $(x + 1)^3 > 0$ so $f$ is concave up.

The graph looks as follows:

![Graph](image)

4. (a) Show how to calculate
\[ \int_{-1}^{2} |2x| \, dx \]
by finding the relevant area (not by finding antiderivatives). You should include a diagram and explain your answer fully.

(b) Suppose you know that
\[ \int_{0}^{2} g(x) \, dx = -4, \quad \int_{0}^{4} g(x) \, dx = 5. \]

Find:

i. \[ \int_{0}^{2} 2g(x) - 1 \, dx \]

ii. \[ \int_{2}^{4} g(x) \, dx \]

(a) The graph of the function is

![Graph](image)
The integral is given by adding the two shaded areas. The one on the left is a triangle with base 1 unit and height 2 units, so its area is 1. The one of the right is a triangle with base 2 units and height 4 units, so its area is 4. Therefore the integral is equal to $1 + 4 = 5$.

(b) i. We have

$$
\int_0^2 2g(x) - 1 \, dx = \int_0^2 2g(x) \, dx - \int_0^2 1 \, dx
$$

$$
= 2 \int_0^2 g(x) \, dx - [x]_{x=0}^{x=2}
$$

$$
= 2(-4) - 2
$$

$$
= -10
$$

ii. We have

$$
\int_2^4 g(x) \, dx = \int_0^4 g(x) \, dx - \int_0^2 g(x) \, dx = 5 - (-4) = 9.
$$

5. If the sum of two positive numbers $x$ and $y$ is 10, find the maximum possible value of $2y + x^2$.

(As part of your answer, explain how you know there is a maximum value.)

[This question should have said that $x, y \geq 0$ so that they are allowed to be 0.]

We want to find the maximum value of $2y + x^2$ where $x + y = 10$. Then $y = 10 - x$ we are trying to maximize

$$
f(x) = 2(10 - x) + x^2 = x^2 - 2x + 20.
$$

The range of possible $x$-values is $0 \leq x \leq 10$. We have to test the critical points:

$$
0 = f'(x) = 2x - 2
$$

so that $x = 1$, where $f(1) = 1 - 2 + 20 = 19$. There are no points where $f$ is not differentiable. We also have to test the endpoints $x = 0$ where $f(0) = 20$ and $x = 10$ where $f(10) = 100 - 20 + 20 = 100$.

Therefore the maximum value is 100 when $x = 10$ and $y = 0$. 