AMHERST COLLEGE

Department of Mathematics and Computer Science

# COMPREHENSIVE EXAMINATION

## Calculus and Linear Algebra

2:00 pm Friday, January 28, 2011

Seeley Mudd 206

There are 12 problems on this examination. Record your answers in the blue book provided. SHOW ALL WORK.

1. [10 points] Compute the following limits.

   (a) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\ln(1+x)}\right)$

   (b) $\lim_{x\to 0}\left(e^x - xe^x\right)^{1/x}$

2. [15 points] Limits are used in many definitions in calculus, including (A) the definition of derivative, (B) the definition of Riemann integral, and (C) the definition of the sum of an infinite series.

   For each of the problems below, label the problem with the appropriate letter (A), (B), or (C) and use tools related to the corresponding concept (derivative, integral, infinite sum) to compute the limit:

   (a) $\lim_{n\to\infty}\sum_{i=1}^{n} 3\cdot 2^{1-i}$

   (b) $\lim_{h\to 0}\frac{\int_2^{2+h} e^{t^2}\,dt}{h}$ Hint: Let $f(x)=\int_0^x e^{t^2}\,dt$

   (c) $\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2$ Hint: $\frac{i^2}{n^3}=\left(\frac{i}{n}\right)^2\cdot\frac{1}{n}$

3. [15 points] Compute the following integrals:

   (a) $\int \tan^2 x\,dx$

   (b) $\int \sin^{-1} x\,dx$

   (c) $\int \frac{x^2+1}{x^2-1}\,dx$

4. [18 points] For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges. Justify your reasoning.

   (a) $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$

   (b) $\sum_{n=1}^{\infty}\frac{1}{e^{1/n}}$

   (c) $\sum_{n=1}^{\infty}\frac{(-2)^n(n!)^2}{(2n)!}$

5. [10 points] Find all values of $x$ for which the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n n^2}$ converges.

6. [8 points] Let $C$ be the boundary (oriented counterclockwise) of the region under $y = e^x$ for $0 \leq x \leq 2$. Compute $\displaystyle\int_C 2xy^3 dx + (xy + 3x^2y^2)\ dy$.

7. [15 points] Let $V$ be the region in $\mathbf{R}^3$ inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.

   (a) Express the volume of $V$ in cartesian, cylindrical and spherical coordinates.

   (b) Evaluate one of the integrals found in part (a).

8. [9 points] Let $f(x, y)$ be differentiable on $\mathbf{R}^2$.

   (a) State the definition of the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$.

   (b) Suppose that $f_x(0, 0) = 2$ and that the directional derivative of $f$ at $(0, 0)$ in the direction $\vec{u} = \frac{1}{\sqrt{2}}(1, 1)$ is $5/\sqrt{2}$. Determine the value of $f_y(0, 0)$.

9. [10 points] Let $f(x, y) = 4xy - x^4 - y^4$.

   (a) Find the critical points of $f(x, y)$.

   (b) Use the second derivative test to classify the critical points as local maxima, local minima or saddle points.

10. [10 points] Let $T : \mathbf{R^2} \to \mathbf{R^2}$ be a linear transformation such that $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

    (a) Find the matrix of $T$ with respect to the standard basis of $\mathbf{R}^2$.

    (b) What is the rank of the matrix of part (a)? Is $T$ one-to-one? Onto? Justify your answers.

11. [10 points] Let $T : V \to W$ be linear. Also assume that $v_1, \ldots, v_k$ form a basis of the nullspace of $T$ and that these vectors can be extended to a basis $v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$ of $V$.

    (a) Express the dimension of the range of $T$ in terms of $n$ and $k$.

    (b) Prove that $T(v_{k+1}), \ldots, T(v_n)$ are linearly independent in $W$.

12. [10 points] Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Find an invertible matrix $Q$ such that $Q^{-1}AQ$ is diagonal.

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*Instructions:* Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. **(30 points).** Let $G$ be a group, and let $H \subseteq G$ be a subset of $G$.
   (a) (15 points). Suppose that $H \subseteq G$ is a subgroup with the property that for every $x, y \in G$, we have $xyx^{-1}y^{-1} \in H$. Prove that $H$ is a **normal** subgroup of $G$.
   (b) (15 points). Given $G$ and $H$ as in part (c), prove that the quotient group $G/H$ is abelian.

2. **(20 points).** Let $G$ be a finite group, and suppose that there is an element $a \in G$ with the property that $a^{-1} = a$ but $a$ is not the identity. Prove that $G$ has an even number of elements.

3. **(25 points).** Consider the group $S_6$ of permutations of the set $\{1, 2, 3, 4, 5, 6\}$. Let $\sigma \in S_6$ be the permutation

$$\sigma = (3\ 6\ 5)(1\ 6\ 3)(1\ 4\ 5\ 2)(4\ 6).$$

   (a) (8 points). Write $\sigma$ as a product of **disjoint** cycles.
   (b) (8 points). Compute the **order** of $\sigma$.
   (c) (9 points). Is $\sigma$ **even** or **odd**? Why?

4. **(25 points).** Let $R$ be a ring.
   (a) (10 points). Define what it means for a subset $I \subseteq R$ to be an **ideal** of $R$.
   If you use other terms like "closed" or "coset" or "subgroup" or "subring" or "maximal" in your definition, you must define those terms as well.
   (b) (15 points). Let $R$ be the ring of **differentiable** functions $f : \mathbb{R} \to \mathbb{R}$ from the real line to itself, under the usual operations of multiplication and addition of functions. Let

$$I = \{f \in R : f(3) = f'(3) = 0\}.$$

   Prove that $I$ is an ideal of $R$.

AMHERST COLLEGE
Department of Mathematics and Computer Science
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Work the following four problems, each worth 10 points.
Record your answers in the blue book provided.
PLEASE SHOW ALL OF YOUR WORK.

1. (a) State the Axiom of Completeness (also known as the Axiom of Continuity for the Real Numbers or Axiom C).
   (b) Use the Axiom of Completeness to prove that an increasing bounded above sequence of real numbers converges.

2. Consider the sequence defined by $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3 + a_n}$ for $n \geq 1$.
   (a) Prove that $a_n < a_{n+1}$ for all $n \geq 1$.
   (b) Prove that $a_n < 1 + \sqrt{3}$ for all $n \geq 1$.
   (c) Explain why $\lim_{n\to\infty} a_n$ exists and find the limit.

3. Consider the sequence of functions on $[0, \pi]$ defined by $f_n(x) = (\sin x)^n$ for $x \in [0, \pi]$ and $n \geq 1$.
   (a) Compute $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [0, \pi]$.
   (b) Does $\{f_n(x)\}_{n=1}^{\infty}$ converge uniformly to $f(x)$? Explain your reasoning.

4. (a) State the definition of $\lim_{x\to 0} f(x) = L$.
   (b) Assume that $\lim_{x\to 0} f(x) = L$ for some real number $L > 0$. Prove that there is $\delta > 0$ with the property that $f(x) > \frac{1}{2}L$ for all $x \in (-\delta, \delta)$, $x \neq 0$.