AMHERST COLLEGE

Department of Mathematics

COMPREHENSIVE EXAMINATION

Calculus and Linear Algebra

2:00pm Friday, January 27, 2012

Seeley Mudd 206

There are 12 problems (totaling 140 points) on this portion of the examination. Record your answers in the blue book provided. **Show all of your work.**

1. [15 points] Compute the following limits:

   (a) $\lim_{x\to 0} \dfrac{1-\cos(\sin x)}{\sin^2(x)}$

   (b) $\lim_{x\to 0} \Big(\ln(x+1)+1\Big)^{\csc(x)}$

   (c) $\lim_{n\to\infty} \displaystyle\sum_{k=0}^{n} \frac{(-1)^k 2^k}{3^{2k+2}}$

2. [10 points] Compute the following integrals:

   (a) $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1}(x)\,dx$

   (b) $\displaystyle\int \frac{x^2+x+1}{x^3+x}\,dx$

3. [15 points] Determine whether the following series converge or diverge. Justify your answers carefully.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\ln n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{26^n (n!)^3}{(3n)!}$

   (c) $\displaystyle\sum_{n=1}^{\infty} n \ln\left(\frac{n+1}{n}\right)$

4. [10 points] For each real number x, determine whether the series

$$\sum_{n=1}^{\infty} \frac{(1-2x)^n}{n2^n}$$

   converges absolutely, converges conditionally, or diverges.

5. [15 points] Evaluate the following integrals:

   (a) $\displaystyle\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\ dy\ dx.$

   (b) $\int_C (y^2 + 6y)dx + \big(\cos(y^2) + 2x(y+1)\big)dy$, where C is some circle of radius 3 in the xy-plane, oriented counterclockwise.

6. [10 points] Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Note that, while the sphere is not centered at the origin, using spherical coordinates still works nicely for this problem.

7. [12 points] Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Is f continuous at $(0,0)$? Justify your answer.

   (b) Find $f_x(0,0)$ and $f_y(0,0)$.

   (c) Is f differentiable at (0,0)? Justify your answer.

8. [10 points] Assume that the temperature in degrees Celsius at a point (x, y) on the circle $x^2 + y^2 = 4$ is given by $T(x,y) = x^2 + 4x - y^2 + 12$. Find the points on the circle at which the temperature is highest and lowest, and state the temperature at each of these points.

9. [10 points] Let C be a 3×5 real-valued matrix. Answer the following questions about C and briefly justify your answers:

   (a) Can the columns of C be linearly independent?

   (b) Does the equation $C\mathbf{x} = \mathbf{0}$ have a unique solution with $\mathbf{x} \in \mathbb{R}^5$?

   (c) Assume that the span of the columns of C is all of $\mathbb{R}^3$. Can you determine the nullity (= dimension of the null space or kernel) of C?

10. [8 points] Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}.$$

   (a) Determine the eigenvalues and eigenvectors of A.

   (b) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

11. [10 points] Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V. Suppose T is one-to-one (injective). Prove that if $\{v_1, ..., v_n\}$ is a basis for V, then $\{T(v_1), ..., T(v_n)\}$ is also a basis for V.

12. [15 points] Let $T : P_2 \to \mathbb{R}^2$, where $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$, be defined by

$$T(p) = \begin{bmatrix} p(1) \\ p(2) \end{bmatrix}.$$

You may assume that T is linear.

(a) Find bases of the null space (kernel) and range of T.

(b) Find the matrix representation of this transformation with respect to the bases $\{1, t, t^2\}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

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Comprehensive Examination: Algebra

Friday, January 27, 2012

Instructions: Do all four of the following problems. Write your solutions and all scratchwork in your bluebook(s). **Show all your work, and justify your answers.**

1. **(25 points).** Let G be a group, and let $H, K \subseteq G$ be subgroups of G.

   (a) Prove the following standard theorem about subgroups: that $H \cap K$ is a subgroup of G.

   (b) If H and K are both *normal* subgroups of G, prove that $H \cap K$ is also a normal subgroup of G.

2. **(25 points).** Let G and H be groups. Recall that a homomorphism $\phi : G \to H$ is said to be *trivial* if $\phi(g) = e_H$ for all $g \in G$.

   (a) If $|G| = 144$ and $|H| = 25$, prove that any homomorphism $\phi : G \to H$ is trivial.

   (b) Let G be the cyclic group of order 2, and let H be the cyclic group of order 6. Give an example of a nontrivial homomorphism $\phi : G \to H$.

3. **(25 points).**

   (a) List all elements of A_4, the alternating group of degree four, expressing each such element as a product of disjoint cycles.

   (b) For each element you listed, say what its order is.

4. **(25 points).** Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   (b) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$. You may assume that R is a ring under the operations of matrix addition and matrix multiplication.

   Let $I = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Prove that I is an ideal of R.

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COMPREHENSIVE EXAMINATION: ANALYSIS
January 27, 2012

Work the following four problems.
Record your answers in the blue book provided.
PLEASE SHOW ALL OF YOUR WORK.

1. [4 points] State the Axiom of Completeness (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

2. (a) [6 points] The standard triangle inequality states that $|x + y| \leq |x| + |y|$ for $x, y \in \mathbb{R}$. Assuming this result, give a careful proof that if $x_1, \ldots, x_n \in \mathbb{R}$, $n \geq 2$, then $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$.

   (b) [6 points] Recall that a function $f : S \to \mathbb{R}$ is bounded if there is $M \in \mathbb{R}$ with $|f(x)| \leq M$ for all $x \in S$. Now suppose we have bounded functions $f_1, \ldots, f_n : S \to \mathbb{R}$, $n \geq 2$, and define $f_1 + \cdots + f_n : S \to \mathbb{R}$ by $(f_1 + \cdots + f_n)(x) = f_1(x) + \cdots + f_n(x)$ for $x \in S$. Prove that $f_1 + \cdots + f_n$ is bounded.

3. Consider the sequence of functions defined by $f_n(x) = 2 + (1 + \frac{1}{n})x$ for $n \geq 1$. This sequence converges pointwise to $f(x) = 2 + x$.

   (a) [7 points] Prove that the sequence converges uniformly to f on $[0, 10]$.

   (b) [7 points] Prove that the sequence does not converge uniformly on $[0, \infty)$.

4. [10 points] Suppose that we have continuous functions $f, g : \mathbb{R} \to \mathbb{R}$. Prove that the composition $f \circ g : \mathbb{R} \to \mathbb{R}$ is also continuous.