Recollection and Reflection: *From Euler to Lagrange in 1Q84*

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I still enjoy taking leisurely walks among book shelves in libraries, scanning titles one by one, mostly in the math sections, occasionally in the humanity ones. On a Tuesday morning in October, 2022, while sauntering on the upper floor in the Science & Engineering Library at UMass Amherst, I came across David Rothery's *Moons*, in the beloved Very Short Introductions series. While flipping through the book, I was immediately struck by the question "Can moons have moons?" from Chapter 1 and its negative answer for our solar system, as if I almost knocked down the shelf I leaned against before starting an impressive domino effect. I was reminded by two theories, which are parallel to each other: (1) Given a manifold, be it topological or differentiable, its **`boundary's boundary = empty`**. (2) Given a chain complex, ∂ and 0 being the boundary and zero homomorphisms, $\boldsymbol{\partial \circ \partial = 0}$. Now, (3) **`moon's moon = empty`** was yet another example of this format. While (1) and (2) are explicitly interwoven, I wondered if (3) could hide any currently inexplicable feature shared with (1) and (2) on top of the common formality. In fact, there are complexes with two indices whose boundary homomorphisms don't satisfy (2), but after a certain summation process, a version of (2) still holds. On the other hand, on average, moons' moons, even if they exist, would eventually fall to the moons or be pulled away to the planets. So (3), (2), and thus (2)'s "geometric realization" (1), three similar statements, could share a common "statistical" stability, among other to-be-found properties.

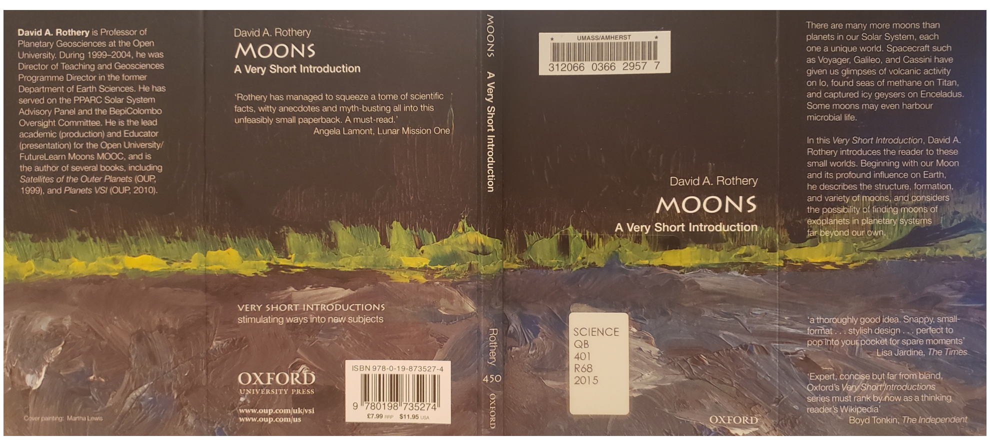

Figure 1: *The cover painting of* `MOONS: A Very Short Introduction` *by Martha Lewis.*

I was tempted to write about these three parallel stories as a short article, and even planned a multi-column format. Page 1: two columns for (1) and (2). Page 2: two columns for (1) and (2). Page 3: three columns for (1), (3) and (2), with the newly found (3) occupying the middle column, like an abrutly introduced character inserting a third narrative to the plot. This must have been inspired by the structure of *1Q84*. Book 1: two story lines (Aomame and Tengo). Book 2: two story lines (Aomame and Tengo). Book 3: three story lines (Ushikawa, Aomame, and Tengo). After all, *1Q84* features moons, which could have provided an unexpected link between formats of mathematical exposition and literary discourse. But it's more than that. The two subjects could be interwoven beyond the level of formats. Their contents could be driven by each other as (1) and (2) do. The short story *From Euler to Lagrange in 1Q84* was such an example: the novel generated a mathematical problem, and an exploration of its solution had furthered the novel's plot from one perspective.

There are two moons in *1Q84*, the normal yellow Moon, and the smaller green moon. Since I read *1Q84* in graduate school[1], I had been remembering the smaller green moon, though the memory of it had been fading. A decade later, having learned more math, while thinking about transferring the format of *1Q84* to mathematical exposition, I wondered about *1Q84* itself: Why couldn't I give a mathematical explanation of the smaller green moon? Now, holding *Moons* in hands, I also wonder if the dreamy aurora green separating the dark night brush strokes from the oceanic rocky terrain on the cover painting (Figure 1) had led me to subconsciously steer away from the `moon's moon = empty` story and be pulled toward the *1Q84* smaller green moon.

So I read *1Q84* again, this time ordered from Amazon.com as three paperbacks wrapped in a transparent plastic box designed by Chap Kidd and Maggie Hinders. I was struck by how little details I remembered from the book as if it was the first time I read it, and the journal I kept with the single-volume copy ten years ago was as if written by anyone else than me, but the feeling about the ending was familiar: "*Until the newly risen sun shone upon it, robbing it of its nighttime brilliance. Until it was nothing more than a gray paper moon, hanging in the sky.*" From pinball machine to mechanical wind-up bird, from artificial satellites continuing on their individual trajectories to the automatic ringing bell at the bottom of a mysterious well, lifeless machines have been used as eternal objects to witness existence, preserve meaning, confront loneliness and present alternatives. This I get. But the inhuman, metallic and cold touch which the gray paper moon shares with them is just different from the overall cathartic warmth Murakami's novels convey. I wondered about a slightly warmer ending. A chirping bluebird, unzipping the dark cover of the bright sky by its warble, as recorded in Henry David Thoreau's journal, rather than a thieving magpie robbing the moonshine, came to mind.

As I read through the book in a nonlinear order, I had also been reading papers, books, dissertations, and technical reports about the math, engineering and technology of celestial mechanics and interplanetary transfer. I came across "The smaller moon was like a child hiding in its mother's skirts." in Book 3, which reminded me of the periodic orbits of an artificial satellite with negligible mass around the Lagrange point behind the Moon in the Earth-Moon restricted three-body problem. I imagined writing about the smaller green moon traversing small ear-shaped loops behind the Moon. Back to Book 2, when asked if the two moons ever overlap, Tengo said, "I don't know why, but the distance between the two moons always stays the same." This is a feature of the two triangular Lagrange points[2]: objects there are stable and so tend to stay at a relatively fixed distance to the Moon under small perturbation. This could be the trajectory of the smaller green moon: it started from the equilibrium point behind the Moon. As this point is unstable, a calculated perturbation would send it off track, and if we are lucky, it would be captured by a triangular point.

Indeed, luck was with us, and I leave it to the readers to find out what happened. What surprised me the most, however, was the shape of the smaller green moon's trajectory around the triangular Lagrange point: it looked like a cocoon!! (It was called *chrysalis* in *1Q84*.)

From Euler to Lagrange in 1Q84 was formulated as an eight-page paper, with a two-page front-matter and a six-page novel. The front-matter contains a one-and-half page derivation of the differential equations of motion of the smaller green moon in a rotating coordinate system from scratch. Seasoned mathematicians may find it unnecessary. Readers with little knowledge of college mathematics may find it difficult to follow. I included it for the need of the plot: the symbolic derivation of these equations was part of the novel. Aomame

[1]I read *1Q84* twice in graduate school, once in English, once in Chinese. Trying as hard as I could, I could not remember which version I read first, though I recall I bought the English single-volume hardcover with the semi-transparent jacket design by Chap Kidd at the university bookstore at Vanderbilt while attending a summer program there in May, 2012. At checkout, the cashier was somewhat puzzled by the 30% discount red sticker above the letter *Q* over Aomame's eye.

[2]Any triangular point, the Earth, and the Moon form an equilateral triangle. The mass ratio of the Earth-Moon system is small enough which also avoids three special values so that the triangular Lagrange points are stable. See §35 of Siegel and Moser, *Lectures on Celestial Mechanics* and Chapter 12 of Meyer and Offin, *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, 3e.

and Tengo went through it before meeting at the beginning of the story. Multivariable calculus is sufficient background for following things through, and Stewart's *Calculus* is a more accessible and modern textbook for learning calculus than those of Spivak, though the latter was an important element in the story.

As I struggled to come up with a concise but self-contained exposition of the equations, what would seem a banal mathematical monologue naturally split into two narratives[3], those of Aomame and Tengo, like a beam of colorless light dispersed into its color components through Newton's prism, or the zero function decomposed into two sinusoidal functions with half-period phase shift.

Among the two characters, Aomame is more powerful. She works unusually hard, has strong will and is exceptionally intelligent. In contrast, Tengo is more of an auxiliary role, weak in character, receptive in demeanor, even though serious about his work, and can be quite tough to himself. Indeed, the plot was mostly driven by Aomame. She was the one who proposed to investigate the motion of the smaller green moon mathematically, knowing Tengo took a good number of classes as a math major while she herself took Calculus I and learned multivariable calculus on her own. With unequal backgrounds in mathematics, they played equal roles in their mathematical exploration. After all, it's quite unusual for someone to work through all problems in both Spivak's *Calculus* and *Calculus on Manifolds*. Aomame could prove the existence of five and only five Lagrange points and then find them with high precision using a calculator. Her math and tech savvy were even more obvious when she could independently write a numerical program with graphical-user interface and take charge of the simulation after reading Tengo's program only once. Aomame's prowess is further demonstrated by her solution of the conundrum that the smaller green moon could not settle to a triangular Lagrange point even though it was close to it, among other things which the reader would find out.

Nonetheless, the reader would have some impression that gender stereotypes were present in Aomame as a hardworking female teacher and Tengo as a male mathematical *whiz* who can see the answer simply by closing his eyes, "like Euler". Euler, who became blind during his last years, also became more prolific in his mathematical output. This contradicts an assumption that blindness inhibits mathematical reasoning. However, the opposite may be true. It was argued that blind people's brain may recycle the powerful visual cortex for internal mathematical thinking, which is independent of senses[4]. Tengo closed his eyes twice in the story, nearly so did Aomame once. Indeed, they imitated Euler, but they were doing it scientifically, not by magic. Closing one's eyes in silence helps one to focus most intensely. During this process, laborious work goes on. Carefully deforming a 3D fiber space in a 4D space, which could not be directly visualized, but be imagined in total darkness with minimal distraction, is such an example. So Tengo is not a whiz. Instead, he also worked hard. As I hoped to convey in the story, dedication to hardworking is of the utmost value, and dedication to teaching is among the most respected. I'm lucky and honored to be part of this profession.

I use *From Euler to Lagrange in 1Q84* to tell a single mathematical story about two beloved characters, but I also try to put different pieces of mathematics together, in historical contexts, which I wish my teachers could tell me when I first learned them. But more importantly, I hope I had continued the theme exploration of open and closed systems and finding connections among seemingly unrelated worlds. The Japanese last name Kodaira on the first page of the story is spelled Xiaoping in Chinese. Indeed, both Kodaira Kunihiko and Deng Xiaoping's open thoughts solved many impossible problems, even though in different areas.

"The owl of Minerva spreads its wings only with the coming of the dusk." I hope you would find the same joy as I do, at night, by reading through a few pages of a book in math, no matter what we went through during the day. The story, finished on 12.31.2022, will start soon on the next page, and I dedicate it to you.

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[3]I recall two other times this happened, once when a friend and I wrote a derivative work of *Dragon Ball* together back in elementary school when Hong Kong was still a British colony, and once in 2018, when I gave a job talk for a teaching position at Amherst College. About the former, I only remember the illustration I made of an imagined red monster. About the latter, the talk was titled *The strange Library*. Instead of being threatened by a brain-eating librarian, the "me" met with a 4D *MATHEMATICIAN*, with whose help he learned some math and was also finally able to enter the mysterious study room in the library on the third floor of the Mathematical Sciences Building at Purdue University, but ended up appearing in the basement of the Frost Library.

[4]Chapter 6 of Stanislas Dehaene, *How we learn*, presented neuroscientific discoveries about the brains of blind mathematicians.

From Euler to Lagrange in *1Q84*

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Abstract

This paper discusses a possible motion of the smaller moon in Murakami's *1Q84*, using the model of the restricted three body problem, in the form of an imagined chapter to the third volume of this novel.

Introduction

1Q84 is a three-volume novel by writer Haruki Murakami, first published in Japan in 2009 and 2010, followed by translated versions in many other languages. In this novel, parallel to 1984, the fictional year 1Q84 features the appearance of two moons. I wonder if the presence of the smaller green moon could be explained by mathematics, as the protagonist Tengo Kawana is himself a mathematician. In this paper, a mathematical model is built to explain possible motions of the smaller moon under the gravitational pull of the Earth and the normal Moon. Aomame and Tengo will discuss the consequences in an imagined chapter of Book 3. Though writing one's own follow-up seems encouraged [2], I express my apologies to Murakami readers and celestial mechanics experts for any shoddy writing or shallow mathematics; any objections only encourage the current author to work harder.

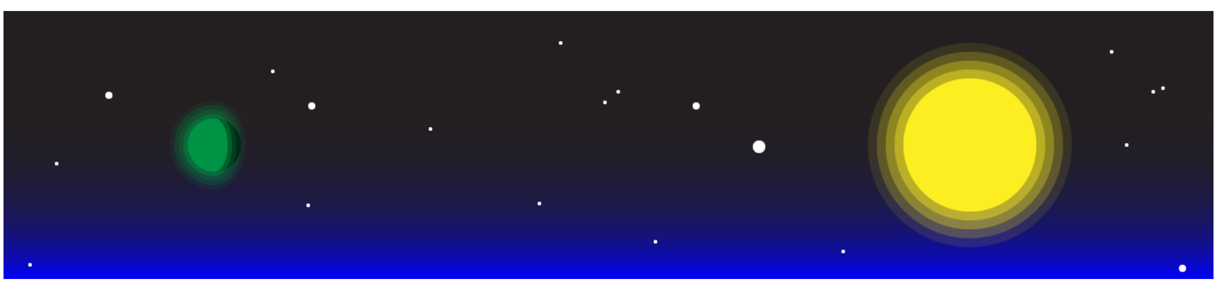

Figure 1: *To our eyes, the angle between the normal Moon and the* 60% *smaller green moon is* $60^\circ \pm 2^\circ$.

The story starts from equation (U) on the next page, so I will first explain where it came from. Assume that the Earth (with mass m_1), the Moon (with mass m_2), and the smaller moon (with mass m) move on a common plane with origin at 0. As no one else noticed the smaller moon except the main characters, we assume that the presence of the smaller moon does not influence the motion of the Earth and the Moon, as if "*It's (really) Only a Paper Moon*," as alluded to several times in this novel, and thus we model it as a geometric object with $m \ll m_1$ and $m \ll m_2$. Following [3], we use complex numbers to model vectors on this plane, for simpler notation, but more for the ease of describing planar rotation by complex number multiplication dictated by $i^2 = -1$ ($90^\circ + 90^\circ = 180^\circ$), as we will do momentarily. Thus, let $z_1(t)$ and $z_2(t)$ be the position vectors of the Earth and the Moon as time t elapses, then *Newton's Second Law of Motion* and the *Universal Law of Gravitation*, as expounded in the three-volume *Principia*, applied to m_1 and m_2, give the following equations, where the gravitational pull from the smaller moon has been neglected, for the reason just mentioned.

$$m_1 \frac{d^2 z_1}{dt^2} = \frac{-Gm_1 m_2 (z_1 - z_2)}{|z_1 - z_2|^3} \qquad \text{(N1)}$$

$$m_2 \frac{d^2 z_2}{dt^2} = \frac{-Gm_2 m_1 (z_2 - z_1)}{|z_2 - z_1|^3} \qquad \text{(N2)}$$

The inverse square law, which was separately explored by Halley, Hooke, and others besides Newton, is more explicit once we separate $\frac{z_i - z_j}{|z_i - z_j|^3}$ into $\frac{1}{|z_i - z_j|^2}$ and the unit direction vector $\frac{z_i - z_j}{|z_i - z_j|}$. The negative sign means gravitational pull points away from the mass m_i. G is the gravitational constant.

In general, solutions to (N1) and (N2) are any conic sections if we don't specify initial conditions. In particular, if we let $\mu = \frac{m_2}{m_1 + m_2}$, the relative mass of the Moon in the Earth-Moon system, R a constant distance between the Earth and the Moon, and Ω a constant angular speed of the Moon around the Earth, then one can readily check that (z_1), (z_2) and Kepler's Third Law (K) solve (N1) and (N2), where $e^{i\theta} = \cos\theta + i \sin\theta$.

$$z_1 = -\mu R e^{i\Omega t} \quad (z_1) \qquad z_2 = (1-\mu) R e^{i\Omega t} \quad (z_2) \qquad G(m_1 + m_2) = \Omega^2 R^3 \quad \text{(K)}$$

Then it's immediately clear that $(1-\mu)z_1 + \mu z_2 = 0$. So the center of mass is the origin 0 through which a line rotates with constant angular speed Ω. On opposite sides of this turning line are located the Earth and the Moon, whose distances to 0 are the constants μR and $(1-\mu)R$, respectively. This simple circular motion of the Earth and the Moon, rather than following general conic sections, is our starting point, which holds throughout our story.

Let $z(t)$ be the position vector of the smaller green moon. Recall its mass is m. As both the Earth and the Moon exert gravitational pull to m, its equation of motion is

$$m \frac{d^2 z}{dt^2} = \frac{-Gmm_1(z - z_1)}{|z - z_1|^3} + \frac{-Gmm_2(z - z_2)}{|z - z_2|^3} \tag{N3}$$

Then m can be cancelled right away. Once we substitute (z_1) and (z_2), the right-hand side of (N3) explicitly depends on t. To eliminate t, and also to reduce the number of parameters, we introduce the rotating w complex plane such that $z(t) = Rw(t)e^{i\Omega t}$, and the derivative with respect to angle $\dot{(\,)} = d(\,)/d(\Omega t)$ so that $d(\,)/dt = \Omega\dot{(\,)}$. With product rule and chain rule used, and Kepler's Law (K) also substituted, (N3) then simplifies to the following, with only μ as the parameter, while Ω, R, G, m_1, m_2 and $e^{i\Omega t}$ had been cancelled.

$$\ddot{w} + 2i\dot{w} - w = \frac{-(1-\mu)(w+\mu)}{|w+\mu|^3} + \frac{-\mu(w-(1-\mu))}{|w-(1-\mu)|^3} \tag{w}$$

So on the w-plane, the Earth is fixed at $-\mu$ with mass $1-\mu$, the Moon at $1-\mu$ with mass μ, and the distance between them is 1. The two terms on the right-hand side are still in the form of the gravitational pull from the Earth and the Moon. Now we move the the last two terms on the left-hand side to the right-hand side, so that the equation is still in the form of *Newton's Second Law*, then $-2i\dot{w}$ and $+w$ are two fictitious forces. The Coriolis force $-2i\dot{w}$ is obtained by turning the doubled velocity $2\dot{w}$ clockwise through angle $\pi/2$. The centrifugal force $+w$ points away from the origin along the position vector of the smaller moon. The Euler force is absent as the w-plane turns at constant speed Ω.

Let $w(t) = x(t) + iy(t) \equiv \langle x(t), y(t)\rangle$, and $U(w) = \frac{|w|^2}{2} + \frac{1-\mu}{|w+\mu|} + \frac{\mu}{|w-(1-\mu)|}$, the sum of the "centrifugal potential" and the gravitational potentials, then (w) can be written in component form, where U_x and U_y denote partial derivatives of U with respect to x and y, respectively.

$$\begin{aligned} \ddot{x} &= 2\dot{y} + U_x \\ \ddot{y} &= -2\dot{x} + U_y \end{aligned} \tag{U}$$

Indeed, one can check that (U) expands to (w) in vector form. There is no need to combine the Coriolis terms $2\dot{y}$ and $-2\dot{x}$ into the potential, as we shall see in the story that they disappear once we start to integrate.

Seeking solutions to (U) is called the circular planar restricted three body problem (CPR3BP) for the Earth-Moon system. Though lifting to the full 3D space opens up other opportunities, e.g., the existence of stable Halo orbits [1], which the Queqiao relay satellite followed in the Earth-Moon system between 2018 and 2023, and the James Webb Space Telescope has been traversing in the Sun-Earth system since 2022, we shall restrict ourselves on a plane to keep our story simple, but not simpler, due to the unruly nature of CPR3BP.

Chapter 32 (Aomame and Tengo cont'd): From Air Chrysalis to Pea Pod

"How'd your study go? Questions about equation (U)?" Tengo asked Aomame with a feigned erudite tone. Urged by Aomame, they had set out to find a mathematical interpretation of the appearance and disappearance of the smaller green moon. Tengo had browsed through a dozen books on celestial mechanics in the university library not far from his apartment. Then he read Victor Szebehely's *Orbital Mechanics* in detail, but had been spending more time and making slow progress on Moser and Siegel's *Lectures on Celestial Mechanics*. The latter book's cyber yellow hard cover reminded him of the Moon. Now they met in a small study room next to the dome theater on the 4th floor at the Tokyo Science Museum. Aomame suggested this place. Once in a while, she came here to read the books she liked but hadn't finished while taking evening classes at her college for the degree in physical education. Among them was Michael Spivak's *Calculus*, which their Calculus I instructor mentioned when they used a set of mimeographed notes by Kunihiko Kodaira instead. She also read Spivak's *Calculus on Manifolds*, in order to learn multivariable calculus on her own as only Calculus I was required for her degree. As near as she could tell, she worked through all problems in both books. It took quite some time. It also took quite some winding around to get to this room. No one else would be patient enough, so each time Aomame was here, the room had always been empty. It was no different now.

"Reading through your notes, I imagined being your student at the cram school where you teach math. I'm so much excited to learn anything from you. I even brought a notebook to take notes. Let me show you :)" Aomame then reached out her hand to the leather shoulder bag, but a faint ray of sarcasm was caught by Tengo, who shared Aomame's instinct gained in the professions of teaching. To Tengo, math and sports are not that different. Both train the mind and the body, and both Thomas Jefferson and Mao Zedong seem to have said something similar along these lines. It's even more so when it comes to teaching math and sports, even though Kodaira could disagree with them all. "Equation (U) is fine, but it took me more time to arrive at (w). I missed the factor of 2 on the left-hand side the first time I tried it myself," Aomame answered.

"The factor 2 is a binomial coefficient. Even though taking the second order derivative of $f \cdot g$ is different from expanding $(a+b)^2$, the coefficients in front of the terms follow the same combinatorial rule. Same thing holds for $(f \cdot g)^{(n)}$ and $(a+b)^n$." Tengo patiently explained. Of course, Aomame had understood it. While failing to see it the first time, she went back to Spivak's *Calculus* and found an identical rule named "Leibniz Formula" as a problem in Chapter 10. She could not understand how she neglected it as she worked through the problems. If she was this careless in her other job as a killer, it could be fatal. Last night, she proved it by two methods, but now she pretended it's the first time she heard of this kawaii formula and patiently waited for Tengo to finish his mini-lesson while sparkling her eyes to signal encouragement, before she continued.

"I would rather write out the right-hand side of (U) as detailed as these so that I could think more concretely," Aomame pointed Tengo to the formulas in her notebook:

$$\ddot{x} = 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{((x+\mu)^2+y^2)^{3/2}} - \frac{\mu(x-(1-\mu))}{((x-(1-\mu))^2+y^2)^{3/2}} \tag{U1}$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{((x+\mu)^2+y^2)^{3/2}} - \frac{\mu y}{((x-(1-\mu))^2+y^2)^{3/2}} \tag{U2}$$

"Brilliant format." Tengo nodded. A reciprocated encouragement, Tengo's reassurance was still pleasant to Aomame's unbalanced ears, the left much bigger than the right, like the two moons used to be in the same sky. "Let's start from the equations you have now," Tengo said, "but later, (U) will gain its advantage once we dig deeper into the world created by these equations." Aomame did not know what it would be. Through the window, she looked *to the Moon and back*. Aomame decided to focus her attention as sharp as laser beams to the reflectors left by the Apollo mission on the Moon. "Even though we have two equations, this is a fourth order system. If we consider x, y, $\dot{x}$ and $\dot{y}$ as four variables, then the derivative of each of them can be written as an algebraic expression of these four variables only, without their derivatives. Two of these four equations are just your (U1) and (U2). The other two are the definitions of $\dot{x}$ and $\dot{y}$." Tengo looked at Aomame. She had already written down the four differential equations in her notebook and was waiting for Tengo to go on. "One of the first things we could do is to look for equilibrium points," Tengo continued. "They are where the four variables don't change. In other words, they are where the derivatives of the four variables all vanish."

"In other words, they are where $\dot{x}, \dot{y}, \ddot{x}$ and $\ddot{y}$ are 0," Aomame imitated Tengo's tone, "so now (U1) and (U2) become simpler while the other two equations contain no new information."

"There are five points (x, y) as solutions to the simplified algebraic equations you obtained. Joseph-Louis Lagrange proved this result for a more general case in 1771, so can you for these two equations." These are the type of manipulations Tengo taught at the cram school for students to prepare for the college entrance examinations. Aomame noticed that y could be factored out of (U2), then there were two cases to consider: (1) $y \neq 0$ and (2) $y = 0$. "In the former case, the other factor of (U2) says

$$1 = \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}, \tag{U21}$$

where r_1 and r_2 are the distances from (x, y) to $(-\mu, 0)$ and $(1-\mu, 0)$, i.e., from the smaller moon to the Earth and the Moon, respectively. Substituting (U21) into (U1) to cancel the x terms immediately gives $r_1 = r_2$. Then we have $r_1 = r_2 = 1$ using (U21) again. As the distance between the Earth and the Moon is also 1, they and the smaller moon form an equilateral triangle. So we get two solutions $(1/2 - \mu, \pm\sqrt{3}/2)$. Indeed $y \neq 0$. Is this the same Lagrange as in the Lagrange Multiplier Method?" Aomame remembered this method. It's Problem 16 in Chapter 5 of *Calculus on Manifolds*, right before the problem on its nice application to self-adjoint operators in linear algebra.

"The same guy," Tengo answered. "These two solutions are called Lagrange triangular points L_4 and L_5." Aomame was excited. She repeated what a mathematician who lived two centuries ago did. "What about L_1, L_2, and L_3?" Aomame felt she had flew up by the infinite thrust one typically gains after any initial success in mathematics, no matter how small it is. "What about your Case (2)?" Tengo hinted to Aomame.

"In the latter case, (U1) can be further simplified by substituting $y = 0$. Then each of the two numerators can be cancelled with a factor in the denominators in each of the three cases (i) $-\mu < x < 1 - \mu$ (ii) $x > 1 - \mu$ and (iii) $x < -\mu$. In each case, the solution occurs at the unique intersection of the graphs of an increasing function and a decreasing function," Aomame said. "Good job! Indeed, they are L_1, L_2, and L_3." Tengo was impressed by Aomame's algebra skill. Aomame went on to solve for the exact locations of these points. After clearing the denominators, she had three fifth degree polynomials with coefficients containing μ. She could not remember any formula like the quadratic formula but for the roots of polynomials of this high degree.

"Leonard Euler ran into the same problem. In fact, Euler also knew these three points, called collinear solutions, as they lie on the line through the masses at $-\mu$ and $1 - \mu$, further back in 1750," Tengo said.

"Then we should call these points E_1, E_2, and E_3!" Aomame complained for Euler. "OK. We do what you suggest, but Euler and Lagrange were not in competition. Lagrange was about thirty years younger than Euler, who mentored the former mathematically, as Euler himself was also taught and befriended by members of the Bernoulli family. In 1772, they shared the prize of the French Academy for their work on lunar theory and the three body problem, advancing the work started by Newton in Book 3 of *Principia*." Tengo continued, "Even though both Euler and Lagrange made contributions, it was only proved and further developed by future generations, notably Ruffini, Abel and Galois, that general fifth and higher degree polynomial equations have no formulas like the quadratic formula. Ours is a concrete real life example where finding solutions to polynomial equations is of interest, which motivated advances in abstract mathematics."

"Then how do we find the solutions?" Aomame resisted the temptation of dashing to a library to learn more about the theories of Ruffini, Abel and Galois, but she wondered what to do with the equations now. "We could solve them numerically," Tengo said. "Both Euler and Lagrange developed various numerical methods for solving equations by iterating a formula hundreds of times. This is especially favorable in modern days as we can use calculator or even a computer to help us with the actual calculation. However, Euler was able to do all these in his head. This is particularly impressive as he became totally blind in the later part of his life."

"I suspect blindness actually boosted his mental power, " Aomame realized. "I would also say so," Tengo said. "Indeed, Euler's mathematical productivity increased after he became blind, though I also wonder if other factors contributed to it. For example, his son and students helped elaborating his chalk work on slate. He was also saved by a neighbor in a devastating fire the year Lagrange found all five equilibrium points."

Instead, Aomame solved the equations using the simple but fast Newton's Method on her handheld CASIO calculator which she used to compute clients' muscle indices. Aomame learned this method in Calculus I. The μ value she used were based on the most recent publications on the masses of the Earth and the Moon Tengo found in the library: $\mu = 0.012150584460351$. Then $E_1 = (0.836915131427382, 0)$, $E_2 = (1.1556821610245677, 0)$ and $E_3 = (-1.005062645331442, 0)$, taking 8, 17 and 3 iterations from $(1,0)$, $(1,0)$ and $(-1,0)$, respectively. Imagining blind Euler with chalk in hand, Aomame drew these points $(E_1, E_2, E_3, L_4, L_5$, the Moon and the Earth) on the blackboard next to their table.

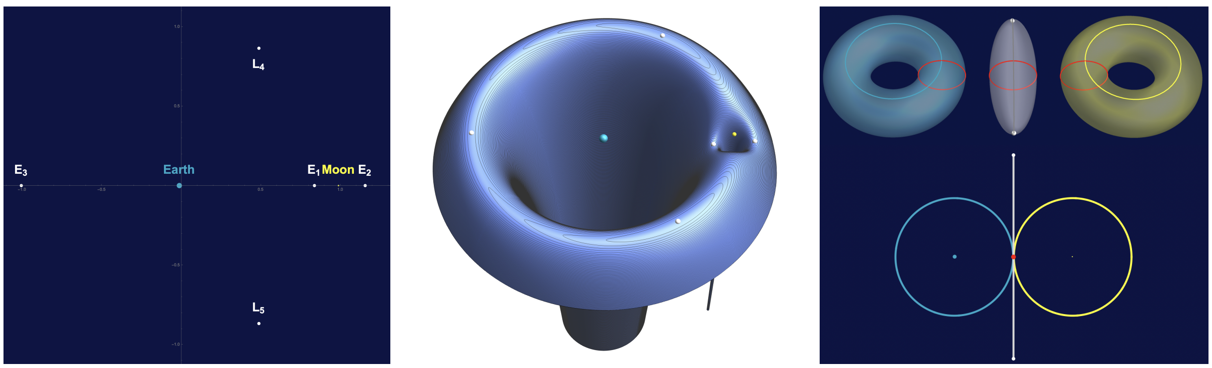

Figure 2: *The Euler and Lagrange points; Hill's hill; A deformation retract of Jacobi's fiber space (top).*

"You said equilibrium points are where the variables $x, y, \dot{x}$ and $\dot{y}$ don't change, so once we put an object at (x, y), it stays there forever?" Aomame said with assurance. Her question mark meant a period, while a period could be a question mark for Fuka-Eri, who brought this world with two moons through *Air Chrysalis*. "At any of the five points, an object stays there forever," Tengo still reassured Aomame using her words. "Then I get it!" exclaimed Aomame. "The smaller green moon could have been existing in this world for a long time, hiding at E_2 behind the Moon." This did not surprise Tengo, as he immediately realized it when he prepared the notes. On the blackboard, the Moon stood between the Earth and E_2. If the smaller green moon was at E_2, then it would be completely blocked by the Moon to us living on the Earth.

"The smaller green moon was a timid child hiding behind its parent, the Moon. Perhaps they are *dohta* and *maza*," Tengo speculated. "Then how did it escape *maza*'s shadow to arrive at another equilibrium point to the side of the Moon?" Aomame pondered, "E_1 is right in front of the Moon. E_3 is on the other side of the Earth. The only possibilities are L_4 and L_5. How did the the smaller moon travel from E_2 to $L_{4/5}$?"

"Now is the time to go back to equation (U). After multiplying the first equation by $\dot{x}$ and the second by $\dot{y}$, we add them up to get $\ddot{x}\dot{x} + \ddot{y}\dot{y} = U_x\dot{x} + U_y\dot{y}$, where the Coriolis terms had been cancelled. Note that both sides are 'time' derivative of a single entity, where we have to assume sufficient differentiability so that multivariable chain rules can be used. The left-hand side is the derivative of $\frac{\dot{x}^2+\dot{y}^2}{2}$ and the right-hand side is $\dot{U}$," Tengo explained, "then moving both terms to the left, and then integrating, $\frac{\dot{x}^2+\dot{y}^2}{2} - U(x, y)$ becomes a constant, which is called the Jacobi integral J. So once the smaller moon starts to move, no matter where it is and how fast it travels, $J(x, y, \dot{x}, \dot{y}) := \frac{\dot{x}^2+\dot{y}^2}{2} - U(x, y)$ stays as the same constant, where $U(x, y) = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$."

"Therefore, if traveling to $L_{4/5}$ and then staying there were possible, then the following must have a solution, where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the initial speed the smaller moon has to acquire before leaving E_2 for $L_{4/5}$."

Aomame scribbled down this expression on the blackboard:

$$\frac{0^2+0^2}{2} - U(L_{4/5}) = \frac{v^2}{2} - U(E_2)$$

Using her CASIO, she found that this equation indeed has a positive real solution: $v = 0.4291426332296729$. After multiplied by $R = 384400$ kilometers and $\Omega = 2\pi/(27.321661 \times 86400)$ radians per second, it is 0.4390804171840862 kilometers per second. "So $-U(L_{4/5})$ is larger than $-U(E_2)$, and to keep J constant, the initial velocity at E_2 must have this magnitude," Aomame concluded.

"$-U(L_{4/5})$ is actually larger than $-U(x, y)$ for all (x, y) not at the Earth or the Moon, where $-U(x, y)$ is not defined. It's the global maximum." Tengo unfolded an oversized poster from his bag. It covered the entire lab table. He sometimes pulled out similar posters and then stuck them to the blackboard using magnets in his classes. On this poster printed the graph of the function $-U(x, y)$, attributed to William George Hill, with cross sections perpendicular to the z-axis so fine that you had to zoom in to see them. If you did, you could see that $-U(x, y)$ indeed has global maximum at $L_{4/5}$. Furthermore, $-U(E_1) < -U(E_2) < -U(E_3) < U(L_{4/5})$. Now it made sense the Euler points E_1, E_2, and E_3 and the Lagrange points $L_{4/5}$ were arranged in this order.

"You cared so much to make this fine poster," Aomame said as she was scrutinizing the cross sections. "It must took quite some time." Aomame reminded Tengo of a poster he saw at a children's museum when he was little: *UNLESS someone like you cares a whole awful lot, nothing is going to get better. It's not.* "Unless someone cares a whole awful lot about the smaller moon and boosts its initial speed v to about 439 meters per second, it is never going to reach $L_{4/5}$. It's not."

"It doesn't take much time if you can find a good computer." Tengo vividly recalled how he sneaked into the newly renovated Computer Science and Engineering Department Lab at Waseda University. The oversized printing was taken care of at a local newspaper press. The college classmate who had substituted twice for Tengo's cram school classes helped him to secure both places. Leading a reclusive life himself, this classmate had versatile connections.

"This only shows the shape of $-U(x, y)$ at each (x, y). I wonder what the $4D$ space of $(x, y, \dot{x}, \dot{y})$ with J value $\frac{0^2+0^2}{2} - U(L_{4/5})$ looks like." Aomame asked. "It's a $3D$ subspace of the $4D$ space of $(x, y, \dot{x}, \dot{y})$. Different J values slice the $4D$ space into $3D$ subspaces," Tengo explained. "However, these $3D$ subspaces are not like the Euclidean space we live in. For each (x, y) not at $L_{4/5}$, there is a circle worth of velocities, as $\dot{x}^2 + \dot{y}^2 = 2(-U(L_{4/5}) - (-U(x, y))) > 0$. At $L_{4/5}$, the velocity is $(0, 0)$. "

"So on the plane with Earth and Moon removed, we have a circle at every point except at L_4 and L_5 where we have only a point," Aomame summarized, and then asked, "however obvious it seems, but how do you prove it doesn't have the same shape as the flat $3D$ space?" "We can calculate its homology," Tengo said.

Tengo closed his eyes. A moment later, he drew two donut surfaces and an ellipsoid on the board, each with a red circle distinguished. "The three surfaces are supposed to be glued together along these red circles. So in total there is only one red circle." Under the surfaces, he drew a circle around the Earth, another circle around the Moon, and then a line segment connecting L_4 and L_5 whose midpoint is also the only point shared by the two circles. "To calculate the homology, we just need to look at the union of circles and points over this simpler $1D$ space. This union is the former space of two donuts and one ellipsoid sharing a red circle, which is a deformation retract of the $3D$ space $J^{-1}(-U(L_{4/5}))$. Clearly, $H_0 = \mathbb{Z}^1$, $H_1 = \mathbb{Z}^2$ and $H_2 = \mathbb{Z}^3$."

"I don't know what homology is." Aomame frowned. "You can think of the exponents 1, 2, 3 this way: the space consists of 'one' component; there are 'two' independent two-dimensional holes; and there are 'three' independent three-dimensional cavities. The two-dimensional holes are represented by the blue circle and the yellow circle, each of which doesn't bound any 2-dimensional disk. The three-dimensional cavities are the donut surfaces and the ellipsoid, each of which doesn't bound any 3-dimensional ball," Tengo said.

"Even though I still don't know what homology is, I see why the red circle doesn't add to the exponent 2: even though it doesn't bound any 2-dimensional disk in either of the two donut surfaces, it bounds both half-ellipsoids with $x \geq 0$ and $x \leq 0$, so it doesn't represent another two-dimensional hole," Aomame pointed out. “This is easy to remember, though. *One, two, and three!*” Aomame made silly faces, which intensified in three stages. Now it’s clear to Tengo that Aomame was a talented teacher. He was not at all.

“Spaces that can be deformed to each other have the same homology. For a flat $3D$ space without holes of any kind, $H_0 = \mathbb{Z}^1$, $H_1 = 0$ and $H_2 = 0$. So these two spaces cannot be deformed into each other. Their shapes are fundamentally different.” Tengo felt connections with Henri Poincaré, who invented algebraic topology a hundred years ago partly to aid his study of the three body problem. When Tengo took Topology II, he used to calculate homologies for topology’s sake, but now it was put to use for its original purpose.

“The smaller moon eventually had to leave Moon’s shadow, but I wonder how.” Aomame was still pondering over this question. “Perhaps by an energy seed,” Tengo joked. “A child eventually had to leave a parent’s shadow. It’s also hard to tell where the energy came from.” The names *ilo* and *milo* popped up in Tengo’s mind, like *dohta* and *maza*. Could they be the *Little People*? Tengo imagined that they installed engines on the smaller moon whose propulsion almost instantaneously ejected it off the Euler point with speed v. E_2 is an unstable equilibrium, which can be readily proved using Lyapunov’s stability theory.

“In which direction should we point v so that the smaller moon would eventually reach $L_{4/5}$?” Aomame wondered. “This is what I have been thinking about,” Tengo said. “Perhaps we can try all directions, and then see which will work,” Aomame said, this time with real sparkles in her eyes. Tengo was amazed by this idea. “That’s a circle worth of directions. Once we let time go, this circle will evolve its shape in the $3D$ space $J^{-1}(-U(L_{4/5}))$. Hopefully, in its shadow on the xy-plane, at least one point on the circle will get really close to $L_{4/5}$ and then eventually stay there.”

Aomame and Tengo met at Waseda’s Computer Science and Engineering Department Lab at 2am three days later. Tengo had written the computer programs for solving the deceptively simple differential equations (U1) and (U2) using the Bulirsch-Stoer method to run in FORTRAN 77 on their IBM machines. He chose $10,000$ points uniformly distributed on the circle of initial velocities, all starting at E_2, and then hit “RUN”. After the Sun rose, these points almost covered the entire computer screen. “I thought we would see the circle deform regularly, like a swimming moon jelly, but it behaved as a broken necklace, with its $10,000$ beads scattered everywhere,” Aomame said. Tengo was super excited. He was impressed by the dense picture even though he had expected it. “Some points are really really close to $L_{4/5}$. This is promising,” Tengo said.

“Let’s find which points they are!” Aomame made up her mind to embark on this journey. She divided the $10,000$ initial conditions into 100 adjacent groups. At 2am everyday in the next 20 days, she came back to the lab to run 5 groups, each for slightly more than one Earth year. For each initial value, she used her naked eyes to check if any trajectory eventually settled down to $L_{4/5}$. This was intense work. Without strong body and mind, anyone could give up within a few days. Aomame reminded Tengo of Carl Gustav Jacobi, the mathematician behind the Jacobi integral and other influential ideas. Perhaps he was the most hard-working mathematician of all time, who intimidated his friends by equating a mathematician with good bodily health but without consuming themselves to a cabbage. Tengo stayed at home, could read no more than five pages per day from the last chapter of *Lectures on Celestial Mechanics*, feeling that he was indeed a cabbage.

The Sun and the Moon were both visible in the sky. Aomame had completed the simulations. No trajectories went to $L_{4/5}$, even though some were once close, traversing part of a *counterclockwise* loop around $L_{4/5}$. Tengo sighed with disappointment, feeling sorry for Aomame’s hardwork. “Perhaps we could run the simulations for longer time,” Aomame suggested without the slightest tone for giving up. Tengo stepped back, closing his eyes again. “Compared to E_2, $L_{4/5}$ are stable equilibria for our Earth and Moon. This was a very subtle result, with real progress made only after the work of Kolmogorov, Arnold and Moser,” Tengo continued, “and an object can stay near $L_{4/5}$ by traversing *clockwise* loops around it.”

A moment of silence. “I understand!” Aomame exclaimed after blinking her eyes seven times. “As $L_{4/5}$ is stable, other objects like space debris or asteroids could have been attracted and staying there. If the smaller moon hit one head-on, then it would reverse its direction, like a billiard ball.”

"I'll write the programs right now!" Tengo's mood was also reversed. "Let me do," Aomame said.

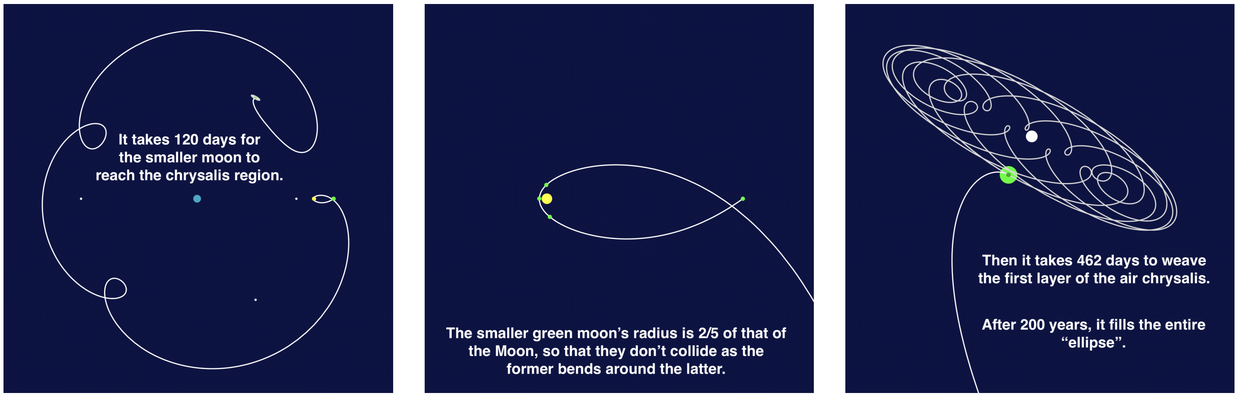

Figure 3: *Initial angle of* ν *is* $\frac{2\pi j}{10000}$, $j = 6000$. *Readers are encouraged to try* $j = 4440$, $j = 5300$ *and* $j = 7730$ *for* ***wilder*** *trajectories from* E_2 *to* L_4, L_4 *and* L_5, *exhibiting sensitive dependence on* j.

An hour later, on the computer screen, once the smaller moon reversed its direction, it started to wind around $L_{4/5}$. To their astonishment, instead of drawing small loops, the smaller moon had been weaving air chrysalises, which eventually enclosed itself. They were speechless. "I would rather call those cocoons," Aomame said. "I agree," Tengo nodded. All of a sudden, the smaller green moon was reemerging in the sky. It indeed was about 60 degrees away from the Moon on the celestial sphere. "Aomame!" Tengo screamed. As he turned to her, Aomame was evaporating. Tengo remembered this phenomenon. If one devoted too much to math, they would slowly dissolve into $\mathbb{R}^3$ and become transparent. And transparency is proportional to assiduity. Jacobi could get 100 percent as often as he wanted. Only once, Tengo achieved 85. Now, Aomame was approaching 98. The green color (`Ao`) was the hard chrysalis shell protecting the tender and precious. Now that the green shell was with the smaller moon again, Aomame became transparent and evanescent.

Time slowed, as if reparametrized. The warmth of Aomame's powerful grip of Tengo's hand when they were ten was still palpable in precise degrees Celsius twenty years later. Now Tengo squeezed Aomame's hand even harder. For five days, he didn't let it go, but it felt like five centuries for light to travel. The smaller moon became invisible again, and Aomame was back to life. Tengo wrapped around Aomame and the little one as if protecting them like green peas in a pod. "If I were transparent for one more year, you would see the green moon weaving air chrysalis to confirm our hypothesis," Aomame said as she opened her eyes. "Enough of it! No more math. Please, don't be another Jacobi." Two tears spilled out of Tengo's eyes and fell audibly onto Aomame's right ear. She giggled. "Live. No matter what," Tengo sobbed. Aomame smiled.

Next to the turntable, *Lectures on Celestial Mechanics* disappeared. The song *ilomilo* started to play on Side B of a cyber yellow record of which Tengo had no impression. It's as if broadcasted from a radio station on the other side of the Moon, spreading wave energy to the smaller moon in its shadow.

It's the last day of the year. Under the serene moonlight, a bluebird suddenly flew by. Its warble was so pure and celestial, like its color, encircling eternal sunshine, melting feathery snow, unveiling a green sprout.

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