Footnote to *Geometry and Algebra Merry-go-round*

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Abstract

This footnote unrolls the circular mathematics from the vinyl records of the album *Geometry and Algebra Merry-go-round* to traditional rectangular pages with more words.

Contents

1 Side A: The dot product

Geometric Definition. Given vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^n$, let $|\mathbf{a}|, |\mathbf{b}|$ be their lengths and θ their angle[a]. Then

$$\mathbf{a} \cdot \mathbf{b} := |\mathbf{a}||\mathbf{b}| \cos \theta.$$

[a] Strictly speaking, θ is only defined when both $\mathbf{a}$ and $\mathbf{b}$ are nonzero. In the case that one of them is $\mathbf{0}$, we let $\mathbf{a} \cdot \mathbf{b} = 0$, which is consistent with the above definition in the sense that $\mathbf{a} \cdot \mathbf{b}$ is continuous in both $\mathbf{a}$ and $\mathbf{b}$. In our later discussion of the dot product, cross product, and scalar triple product, the special case when any vector is $\mathbf{0}$ will be omitted. It can be checked that all statements hold for this special case.

Algebraic Definition. Given vectors $\mathbf{a} = \langle a_1, \cdots, a_n \rangle, \mathbf{b} = \langle b_1, \cdots, b_n \rangle$ in $\mathbb{R}^n$,

$$\mathbf{a} \cdot \mathbf{b} := a_1 b_1 + \cdots + a_n b_n.$$

1.1 *From geometry to algebra*

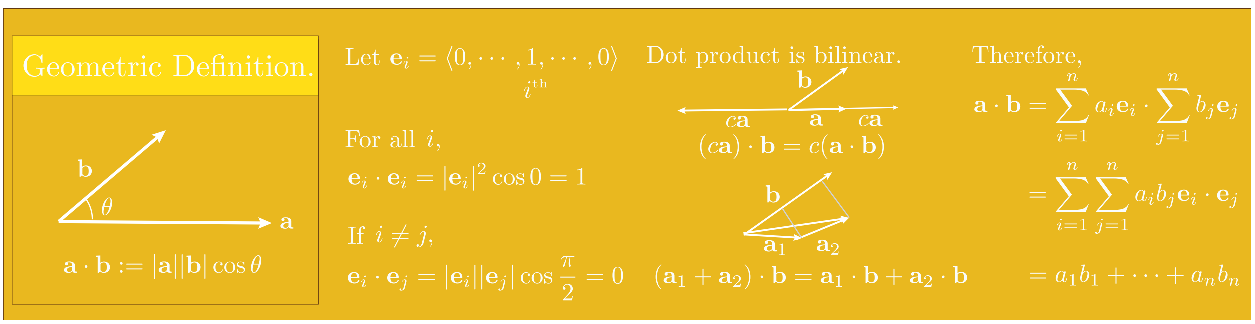


Figure 1: *Dot product: from geometry to algebra*

Assume the geometric definition of $\mathbf{a} \cdot \mathbf{b}$. Define the unit vectors $\mathbf{e}_i = \langle 0, \cdots, 1, \cdots, 0 \rangle$, where the only 1 is the ith entry. Then the angle θ between $\mathbf{e}_i$ and $\mathbf{e}_j$, where $i \neq j$, is $\pi/2$, as a natural generalization from dimensions 2 and 3 to general n. Thus, $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$ and $\mathbf{e}_i \cdot \mathbf{e}_j = 1$ if $i = j$. These conditions can be summarized by saying that the vectors $\mathbf{e}_i$, $1 \leq i \leq n$, form an orthonormal basis of $\mathbb{R}^n$.

Proposition. Dot product is bilinear.

Proof. We will show that dot product is linear in the first entry. As dot product is commutative from its geometric definition, it will also follow that dot product is linear in the second entry.

Consider $c\mathbf{a}$ and $\mathbf{b}$. The angle between them is either θ or $\pi - \theta$, depending on whether $c > 0$ or $c < 0$. Using the geometric definition of scalar product, if $c > 0$, then we have $(c\mathbf{a}) \cdot \mathbf{b} = |c\mathbf{a}||\mathbf{b}| \cos \theta = c|\mathbf{a}||\mathbf{b}| \cos \theta$, and if $c < 0$, then we have $(c\mathbf{a}) \cdot \mathbf{b} = |c\mathbf{a}||\mathbf{b}| \cos(\pi - \theta) = -c|\mathbf{a}||\mathbf{b}|(-\cos \theta) = c|\mathbf{a}||\mathbf{b}| \cos \theta$. Both are $c(\mathbf{a} \cdot \mathbf{b})$.

Consider $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$. Let the angle formed between $\mathbf{b}$ and each of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_1 + \mathbf{a}_2$ be θ_1, θ_2 and θ, respectively. Then by the geometric definition of vector addition, $|\mathbf{a}_1 + \mathbf{a}_2| \cos \theta = |\mathbf{a}_1| \cos \theta_1 + |\mathbf{a}_2| \cos \theta_2$. Therefore, $(\mathbf{a}_1 + \mathbf{a}_2) \cdot \mathbf{b} = |\mathbf{a}_1 + \mathbf{a}_2||\mathbf{b}| \cos \theta = |\mathbf{a}_1||\mathbf{b}| \cos \theta_1 + |\mathbf{a}_2||\mathbf{b}| \cos \theta_2 = \mathbf{a}_1 \cdot \mathbf{b} + \mathbf{a}_2 \cdot \mathbf{b}$. ◻

Using the bilinearity above, we have

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i \mathbf{e}_i \sum_{j=1}^{n} b_j \mathbf{e}_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = \sum_{i=1}^{n} a_i b_i = a_1 b_1 \cdots + a_n b_n.$$

1.2 From algebra to geometry

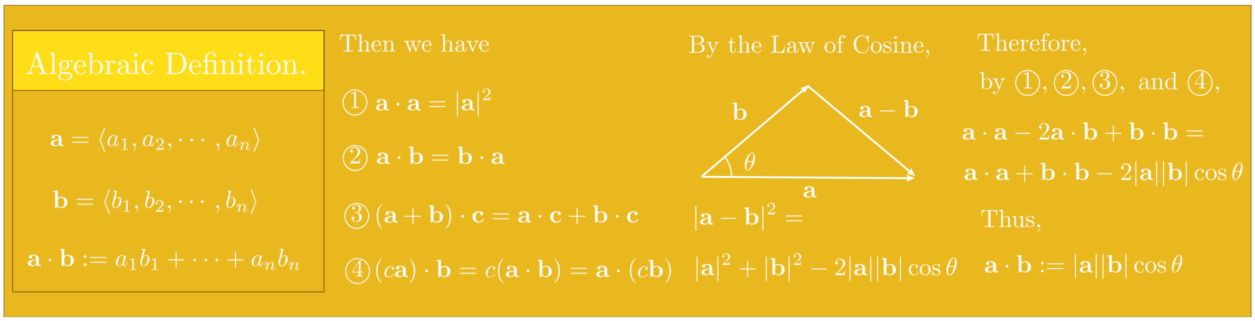


Figure 2: *Dot product: from algebra to geometry*

Starting from the algebraic definition of $\mathbf{a} \cdot \mathbf{b}$, we can show that

Algebraic Properties of Dot Product

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

The law of cosine applied to the triangle with the edge $\mathbf{a} - \mathbf{b}$ opposite to the angle θ is

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta.$$

Using the above algebraic properties, we have

$$\mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2|\mathbf{a}||\mathbf{b}| \cos\theta.$$

Thus,

$$\mathbf{a} \cdot \mathbf{b} := |\mathbf{a}||\mathbf{b}| \cos\theta.$$

2 Side B: The cross product

As opposed to dot product, cross product is typically only defined when $n = 3$. In the last section, we will see that it can also be defined for $n = 1, 7$, but those will be it: dot product does not exist in dimensions other than 1, 3, and 7.

Geometric Definition. Given vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$, let $|\mathbf{a}|, |\mathbf{b}|$ be their lengths and θ their angle. Then $\mathbf{a} \times \mathbf{b}$ is defined to be the vector whose direction and length are given as follows:

- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. More specifically, $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ in this order satisfy the right-hand-rule.
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$, the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Algebraic Definition. Given vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle, \mathbf{b} = \langle b_1, b_2, b_3 \rangle$ in $\mathbb{R}^3$,

$$\mathbf{a} \times \mathbf{b} := \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

2.1 *From geometry to algebra*

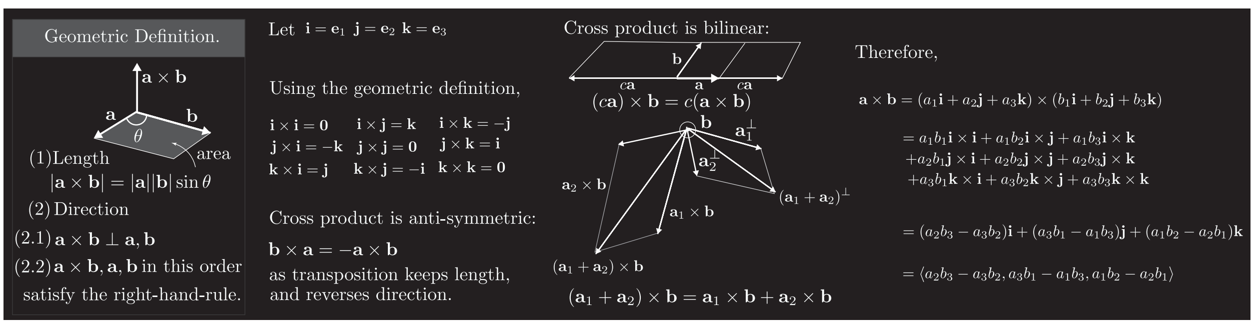


Figure 3: *Cross product: from geometry to algebra*

Assume the geometric definition of $\mathbf{a} \times \mathbf{b}$. Let $\mathbf{i} = \mathbf{e}_1$, $\mathbf{j} = \mathbf{e}_2$, and $\mathbf{k} = \mathbf{e}_3$. Then

$$\begin{array}{ccc} \mathbf{i} \times \mathbf{i} = \mathbf{0} & \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{j} \times \mathbf{j} = \mathbf{0} & \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{k} \times \mathbf{k} = \mathbf{0} \end{array}$$

Same as dot product, cross product is linear in both entires.

Proposition. Cross product is bilinear.

Proof. Using its geometric definition, cross product is antisymmetric, i.e., $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$. Thus, if we can show $\times$ is linear in the first entry, then it is also linear in the second.

Let c be real. If $c > 0$, then $(c\mathbf{a}) \times \mathbf{b}$ has the same direction as $\mathbf{a} \times \mathbf{b}$, which in turn has the same direction as $c(\mathbf{a} \times \mathbf{b})$. Furthermore, $|(c\mathbf{a}) \times \mathbf{b}| = |c\mathbf{a}||\mathbf{b}| \sin\theta = c|\mathbf{a}||\mathbf{b}| \sin\theta = c|\mathbf{a} \times \mathbf{b}| = |c(\mathbf{a} \times \mathbf{b})|$. Therefore, $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b})$. If $c < 0$, then $(c\mathbf{a}) \times \mathbf{b}$ has the opposite direction as $\mathbf{a} \times \mathbf{b}$, which in turn has the opposite direction as $c(\mathbf{a} \times \mathbf{b})$. Thus, $(c\mathbf{a}) \times \mathbf{b}$ has the same direction as $c(\mathbf{a} \times \mathbf{b})$. Furthermore,

$|(c\mathbf{a}) \times \mathbf{b}| = |c\mathbf{a}||\mathbf{b}| \sin(\pi - \theta) = -c|\mathbf{a}||\mathbf{b}| \sin\theta = -c|\mathbf{a} \times \mathbf{b}| = |(-c)(\mathbf{a} \times \mathbf{b})| = |c(\mathbf{a} \times \mathbf{b})|$. Therefore, $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b})$.

Now consider $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{b}$. Let the three vectors start from the same point, and orient them in such a way that $\mathbf{b}$ is perpendicular to the page and points out of it. Then $\mathbf{a}_1$ and $\mathbf{a}_2$ appear as the their projected images $\mathbf{a}_1^\perp$ and $\mathbf{a}_2^\perp$ on the page, and $\mathbf{b}$ appears as a single point 0. Consider the parallelogram spanned by $\mathbf{a}_1^\perp$ and $\mathbf{a}_2^\perp$, then its diagonal starting at 0 is the projection of $\mathbf{a}_1 + \mathbf{a}_2$ on the page. Now turn this parallelogram clockwise by $\pi/2$ and stretch it in all directions by a factor of $|\mathbf{b}|$ at 0. Then in this new parallelogram, the two edge vectors starting from 0 are $\mathbf{a}_1 \times \mathbf{b}$ and $\mathbf{a}_2 \times \mathbf{b}$, and the diagonal starting at 0 is $(\mathbf{a}_1 + \mathbf{a}_2) \times \mathbf{b}$, which is also $\mathbf{a}_1 \times \mathbf{b} + \mathbf{a}_2 \times \mathbf{b}$ as the sum of the two edge vectors. $\square$

Using the multiplication table and bilinearity above, $\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$. Thus,

$$\mathbf{a} \times \mathbf{b} := \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

2.2 *From algebra to geometry*

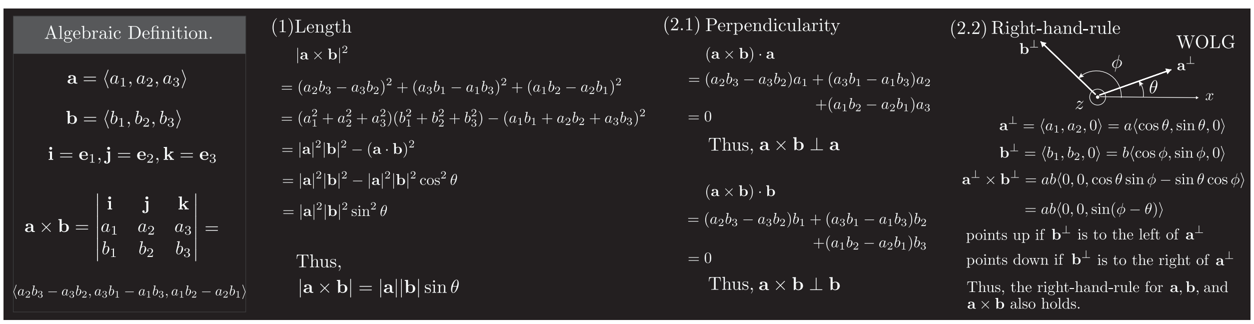


Figure 4: *Cross product: from algebra to geometry*

Starting from the algebraic definition of $\mathbf{a}\times\mathbf{b}$, we have $|\mathbf{a}\times\mathbf{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta = (|\mathbf{a}||\mathbf{b}|\sin\theta)^2$. Thus, $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$, which is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

Then, from $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (a_2b_3 - a_3b_2)a_1 + (a_3b_1 - a_1b_3)a_2 + (a_1b_2 - a_2b_1)a_3 = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = (a_2b_3 - a_3b_2)b_1 + (a_3b_1 - a_1b_3)b_2 + (a_1b_2 - a_2b_1)b_3 = 0$, we see that $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Finally, we show $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$, in this order, when $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, satisfy the right-hand-rule. First, there must be one of the coordinate planes $x = 0$, $y = 0$ or $z = 0$, which is not perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. Without loss of generality, due to the cyclic symmetry, say it's the $z = 0$ plane. Consider the projections $\mathbf{a}^\perp = \langle a_1, a_2, 0 \rangle$ and $\mathbf{b}^\perp = \langle b_1, b_2, 0 \rangle$ of $\mathbf{a}$ and $\mathbf{b}$ onto $z = 0$. Then the handedness of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}\times\mathbf{b}$ is the same as the handedness of $\mathbf{a}^\perp$, $\mathbf{b}^\perp$, and $\mathbf{a}^\perp\times\mathbf{b}^\perp = \langle 0, 0, a_1b_2 - a_2b_1 \rangle$. Writing $\mathbf{a}^\perp = a\langle \cos\theta, \sin\theta, 0 \rangle$ and $\mathbf{b}^\perp = b\langle \cos\phi, \sin\phi, 0 \rangle$, where θ and ϕ are the angles from $\mathbf{i}$ to $\mathbf{a}^\perp$ and $\mathbf{b}^\perp$, respectively, we have $\mathbf{a}^\perp \times \mathbf{b}^\perp = \langle 0, 0, ab\sin\gamma \rangle$, where $\gamma = \phi - \theta$ is the angle from $\mathbf{a}$ to $\mathbf{b}$. Thus, viewed from above, if $\mathbf{a}$ is to the right of $\mathbf{b}$, then $\sin\gamma > 0$ and thus $\mathbf{a}^\perp \times \mathbf{b}^\perp$ points up, and if $\mathbf{a}$ is to the left of $\mathbf{b}$, then $\sin\gamma < 0$, and thus $\mathbf{a}^\perp\times\mathbf{b}^\perp$ points down. In both cases, $\mathbf{a}^\perp$, $\mathbf{b}^\perp$, and $\mathbf{a}^\perp\times\mathbf{b}^\perp$ satisfy the right-hand-rule. Therefore, so do $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}\times\mathbf{b}$. This proof was learned from Eric Thurschwell, *An Even Simpler Proof of the Right-Hand Rule*, College Mathematics Journal, Vol. 46, No. 3 (2015), pp. 215-217.

3 Side C: The scalar triple product

Geometric Definition. Given vectors **a**, **b**, and **c** in $\mathbb{R}^3$, the scalar triple product $\mathrm{T}(\mathbf{a},\mathbf{b},\mathbf{c})$ is defined to be the signed volume of the parallelepiped spanned by **a**, **b**, and **c**, where the sign $\pm$ is determined by whether $\mathbf{a}\times\mathbf{b}$ forms an acute or obtuse angle with **c**.

Algebraic Definition. Given vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ in $\mathbb{R}^3$, the scalar triple product $\mathrm{T}(\mathbf{a},\mathbf{b},\mathbf{c})$ is defined to be the following determinant.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3.1 *From geometry to algebra*

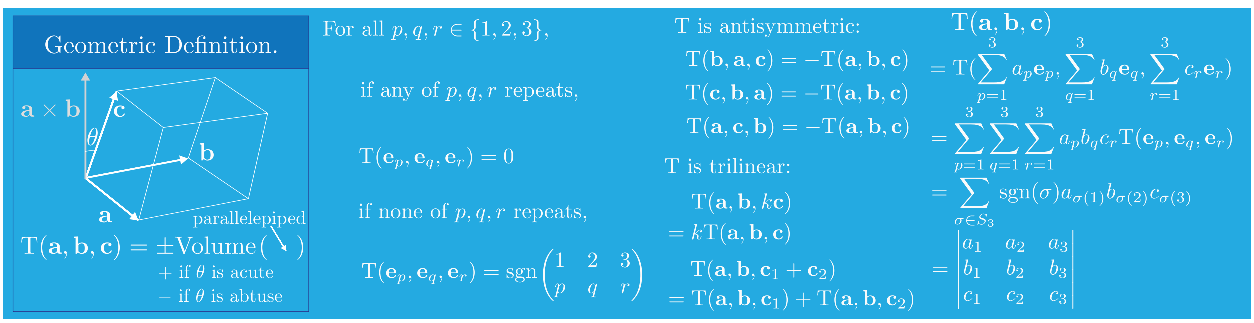


Figure 5: *Scalar triple product: from geometry to algebra*

Assume the geometric definition. First of all, for $p, q, r \in \{1, 2, 3\}$, $\mathrm{T}(\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r) = 0$ if any two of p, q, r are equal. Otherwise, it is the sign of the permutation $1 \mapsto p$, $2 \mapsto q$, $3 \mapsto r$. Furthermore, we also have the following.

Proposition. Scalar triple product is antisymmetric, i.e., (1) $\mathrm{T}(\mathbf{b},\mathbf{a},\mathbf{c}) = -\mathrm{T}(\mathbf{a},\mathbf{b},\mathbf{c})$, (2) $\mathrm{T}(\mathbf{a},\mathbf{c},\mathbf{b}) = -\mathrm{T}(\mathbf{a},\mathbf{b},\mathbf{c})$, and (3) $\mathrm{T}(\mathbf{c},\mathbf{b},\mathbf{a}) = -\mathrm{T}(\mathbf{a},\mathbf{b},\mathbf{c})$.

Proof. For (1), if we interchange **a** with **b**, then the direction of their cross product is reversed. Thus, the angle between the cross product and **c** changes to its complementary angle, resulting the sign change.

For (2), $\mathbf{a}\times\mathbf{b}$ and $\mathbf{a}\times\mathbf{c}$ are both perpendicular to **a**, each of which is also perpendicular to **b** and **c**, respectively. Viewing all the aforementioned vectors projected onto the plane perpendicular to **a**, we see that if the angle between $\mathbf{a}\times\mathbf{c}$ and **b** is acute, then the angle between $\mathbf{a}\times\mathbf{b}$ and **c** is obtuse, and vice versa.

(3) can be proved similarly to (2). □

Proposition. Scalar triple product is multi-linear.

Proof. As T is antisymmetric, it suffices to show T is linear in the last entry.

First of all, $\mathrm{T}(\mathbf{a}, \mathbf{b}, k\mathbf{c}) = k\mathrm{T}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ as the rescaling of $\mathbf{c}$ by k has the effect of multiplying the volume by $|k|$ and keeping and reversing the sign depending on if $k > 0$ or $k < 0$.

Then, similarly to the proof that dot product is distributive, if we recognize that the signed volume $\mathrm{T}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the area of the base spanned by $\mathbf{a}$ and $\mathbf{b}$ multiplied by the projection of $\mathbf{c}$ onto $\mathbf{a} \times \mathbf{b}$, then we see that $\mathrm{T}(\mathbf{a}, \mathbf{b}, \mathbf{c}_1) + \mathrm{T}(\mathbf{a}, \mathbf{b}, \mathbf{c}_2) = \mathrm{T}(\mathbf{a}, \mathbf{b}, \mathbf{c}_1 + \mathbf{c}_2)$, as the projection of $\mathbf{c}_1 + \mathbf{c}_2$ is the sum of the projections of $\mathbf{c}_1$ and $\mathbf{c}_2$ onto $\mathbf{a} \times \mathbf{b}$. The sign also carries through. $\square$

Therefore, by the above two results,

$$\mathrm{T}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathrm{T}\Big(\sum_{p=1}^{3} a_p \mathbf{e}_p, \sum_{q=1}^{3} b_q \mathbf{e}_q, \sum_{r=1}^{3} c_r \mathbf{e}_r\Big) = \sum_{p=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3} a_p b_q c_r \mathrm{T}(\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r) = \sum_{\sigma \in S_3} \mathrm{sign}(\sigma) a_{\sigma(1)} b_{\sigma(2)} c_{\sigma 3},$$

which is the determinant in the algebraic definition of the scalar triple product.

3.2 From algebra to geometry

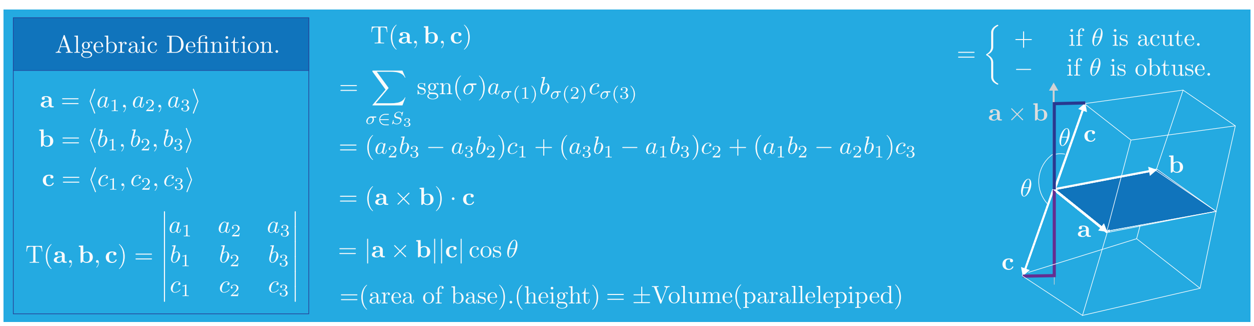


Figure 6: *Scalar triple product: from algebra to geometry*

Assume the algebraic definition of $\mathrm{T}(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Note that it is $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, which is $|\mathbf{a} \times \mathbf{b}||\mathbf{c}| \cos\theta$, where θ is the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$. Thus, $|\mathbf{c}| \cos\theta$ is the signed distance from the opposite face to the parallelogram face spanned by $\mathbf{a}$ and $\mathbf{b}$ in the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Thus, $|\mathbf{a} \times \mathbf{b}||\mathbf{c}| \cos\theta$ is the volume of the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, and it's positive if θ is acute and negative if θ is obtuse.

4 Side D: Cross product in dimensions 1, 3, and 7

We have defined cross product in $\mathbb{R}^3$, but it also exists in dimensions 1 and 7. In order to describe these structures as well as to understand the uniqueness of these dimensions in which cross product exists, we will formulate cross product in $\mathbb{R}^3$ using quaternions, from which we will also see that, same as in scalar triple product, cross product and dot product are intertwined.

We write a vector in $\mathbb{R}^{n+1}$ by $\langle a_0, a_1, \cdots, a_n \rangle = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$, which is also written $a_0 + \mathbf{a}$ where a_0 means $a_0\mathbf{e}_0$ and $\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$.

Consider $\mathbb{R}^4 = \mathbb{R} \oplus \mathbb{R}^3$. Let $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ be $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ so that any element (called a quaternion) of $\mathbb{R}^4$ is of the form $a_0 + \mathbf{a}$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ lives in the three dimensional subspace $\{0\} \times \mathbb{R}^3$. Quaternion multiplication is defined as follows: $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ multiply the same way as their $\mathbb{R}^3$ counterparts: $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$ and $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$, except that now $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, instead of $\mathbf{0}$. General multiplication is obtained by multi-linearity.

Then one can check that $\mathbf{ab} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$. Thus,

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{ab}.$$

In general, we have

$$(a_0 + \mathbf{a})(b_0 + \mathbf{b}) = a_0 b_0 - \mathbf{a} \cdot \mathbf{b} + a_0\mathbf{b} + b_0\mathbf{a} + \mathbf{a} \times \mathbf{b},$$

which is the prototype of the construction used to pass cross product on $\mathbb{R}^n$ to H-space structure on S^n in Section 4.2.1.

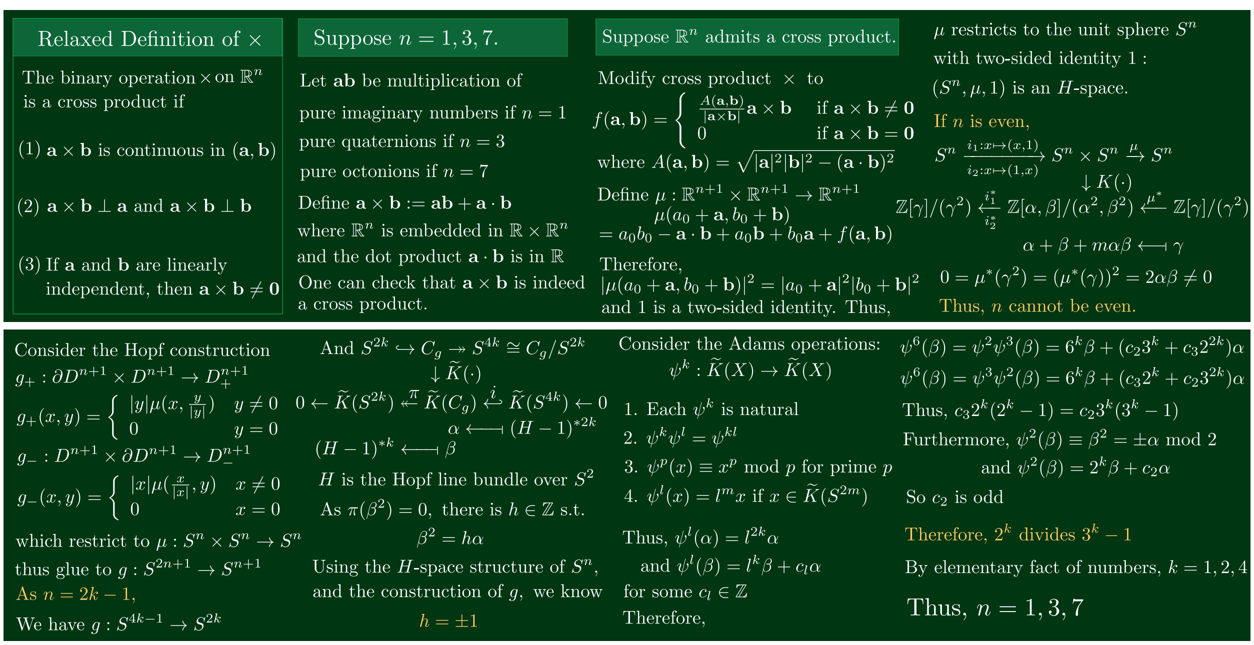


Figure 7: *Cross product exists in $\mathbb{R}^n$ iff $n = 1, 3, 7$.*

From now on, we relax the definition of cross product as follows.

Definition 4.1. The binary operation $\cdot \times \cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a cross product if

- $\mathbf{a} \times \mathbf{b}$ is continuous in $(\mathbf{a}, \mathbf{b})$.
- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, i.e., $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.
- If $\mathbf{a}$ and $\mathbf{b}$ are linearly independent, then $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$.

In addition to $n = 3$, we show cross product also exists in dimensions $n = 1, 7$.

4.1 The existence of cross product also in dimensions 1 *and* 7

As in dimension $n = 3$, when $n = 1$ and $n = 7$, we form $\mathbb{R}^{n+1}$, and define

$$\mathbf{a} \times \mathbf{b} := \mathbf{a} \cdot \mathbf{b} + \mathbf{ab},$$

where $\mathbf{a}$ and $\mathbf{b}$ are identified with pure imaginary complex numbers in $\mathbb{R}^{1+1}$ and pure octonions in $\mathbb{R}^{1+7}$.

Thus, if $n = 1$, then $\mathbf{a} \times \mathbf{b} = 0$, which satisfies the three properties of cross product in Definition 4.1, where the last one is vacuously true as $\mathbf{a}$ and $\mathbf{b}$ are never linearly dependent.

When $n = 7$, the formula for $\mathbf{a} \times \mathbf{b}$ in terms of coordinates is a little tedious to write out, though one can still check that the three conditions hold, where it takes the cancellation of 42 terms when verifying each equality of (2).

4.2 If cross product exists for $\mathbb{R}^n$*, then* $n = 1, 3, 7$

4.2.1. Translation from cross product on $\mathbb{R}^n$ *to* H*-space structure on* S^n

Suppose cross product $\times$ exists for $\mathbb{R}^n$ in the sense of Definition 4.1. First of all, we will modify $\times$ to get a cross product in the usual sense. Following Massey[1], let $A(\mathbf{a}, \mathbf{b}) = \sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2} = |a||b| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. So $A(\mathbf{a}, \mathbf{b})$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Define $f(\mathbf{a}, \mathbf{b}) = \frac{A(\mathbf{a},\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|} \mathbf{a} \times \mathbf{b}$ if $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$ and $f(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ otherwise. Thus defined, $f(\mathbf{a}, \mathbf{b})$ (1) depends on $\mathbf{a}$ and $\mathbf{b}$ continuously, (2) is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ as it points along $\mathbf{a} \times \mathbf{b}$ and (3) has length $A(\mathbf{a}, \mathbf{b})$.

Then include $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as $\{0\} \times \mathbb{R}^n$. Define the continuous product $\mu : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\mu(a_0 + \mathbf{a}, b_0 + \mathbf{b}) = a_0 b_0 - \mathbf{a} \cdot \mathbf{b} + a_0 \mathbf{b} + b_0 \mathbf{a} + f(\mathbf{a}, \mathbf{b}).$$

Then we see that $1 \equiv (1, 0, \cdots, 0)$ is a two-sided identity for μ:

$$\mu(1, b_0 + \mathbf{b}) = b_0 + \mathbf{b} \text{ and } \mu(a_0 + \mathbf{a}, 1) = a_0 + \mathbf{a},$$

and

$$|\mu(a_0 + \mathbf{a}, b_0 + \mathbf{b})|^2 = |a_0 + \mathbf{a}|^2 |b_0 + \mathbf{b}|^2,$$

which holds because of (2), (3), $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ and $\mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|^2$.

Thus, restricted to the unit sphere S^n in $\mathbb{R}^{n+1}$, μ gives rise to a continuous map $\mu : S^n \times S^n \to S^n$ with two-sided identity 1, which is called an H-space structure on S^n.

[1]W. S. Massey, *Cross Products of Vectors in Higher Dimensional Euclidean Spaces*, Amer. Math. Monthly 90 (1983), no. 10, 697-701

4.2.2. Using K-theories and their Adams operations to show $n = 1, 3, 7$

The implication that $n = 1, 3, 7$ if S^n has an H-space structure was first proved in Adams[2]. The proof was long, using secondary operations of ordinary cohomology. This technique was then also applied in a related problem[3], which was soon completely solved by using primary operations constructed by Adams[4] of topological K-theory, an "extraordinary" cohomology theory introduced by Atiyah and Hirzebruch, adapting a similar construction by Grothendieck, from algebraic geometry to algebraic topology. Less than half a decade later, the two innovators, Adams of K-theory operations, and Atiyah of K-theory itself, produced a short proof of Adams' original Hopf invariant one theorem[5]. This is comparable to the transition from the geocentric model of the solar system for which epicycles are needed to describe trajectories of other planets to the heliocentric models for which first order cycles are sufficient (if we ignore mutual influences of planets).[6]

We will present a brief version of this short proof, in the style of operating a freshly unboxed machine without the courage to look under the hood yet (in case we wouldn't be able to reassemble the pieces). Details can be found in [Aguilar-Gitler-Prieto][7][Atiyah][8] [Hatcher][9][Husemoller][10][Karoubi][11][May][12] [Park][13] and papers by Atiyah, Bott, Hirzebruch et al.

For any compact Hausdorff space X, let $\mathrm{Vect}(X)$ be the isomorphism classes of complex vector bundles over X. Then direct sum $\oplus$ and tensor product $\otimes$ of vector bundles induce a commutative semiring structure on $\mathrm{Vect}(K)$ where sum is denoted by $+$ and product is denoted by juxtaposition. The complex K-theory of X, denoted by $K(X)$, is the best completion of $\mathrm{Vect}(K)$ into a commutative ring, which consists of formal differences $E_1 - E_2$ of complex vector bundles such that $E_1 - E_2$ is equivalent to $E_1' - E_2'$ if $E_1 + E_2'$ is stably isomorphic to $E_1' + E_2$, i.e., $E_1 + E_2' + \epsilon^n$ is isomorphic to $E_1' + E_2 + \epsilon^n$ for some trivial bundle ϵ^n of complex dimension n over X. The zero element of $K(X)$ is $E - E$ for any bundle E over X. Given continuous function $f : X \to Y$, it induces a ring homomorphism $f^* : K(Y) \to K(X)$ obtained by pulling back bundles. Indeed, $K(\cdot)$ is a functor, satisfying all axioms of the ordinary cohomology theory except the dimension axiom. $K(\cdot)$ is dubbed an "extraordinary" cohomology theory.

Pick a point $x_0 \in X$, then we have ring homomorphism $K(X) \to K(x_0) \cong \mathbb{Z}$. Its kernel, denoted by $\widetilde{K}(X)$, consists of all elements $E_1 - E_2$ such that E_1 and E_2 have the same dimension. $\widetilde{K}(\cdot)$ is also a functor, the reduced complex K-theory, an "extraordinary" reduced cohomology theory.

Example 0. Let H be the Hopf line bundle over S^2, i.e., the tautological line bundle of $\mathbb{C}P^1$ and 1 the trivial line bundle over S^2. Then, $K(S^2) \cong \mathbb{Z}[H]/(H-1)^2$ where the relation $(H-1)^2 = 0$, i.e., $H^2 + 1 = H + H$, holds by examining the clutching matrices $\mathrm{diag}(z^2, 1)$ and $\mathrm{diag}(z, z)$ of $H^2 + 1$ and $H + H$ along the equatorial circle: they are homotopic in $GL_2(\mathbb{C})$. The above isomorphism $\cong$, which is part of Bott

[2] J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) 72 (1960), 20-104.

[3] J. F. Adams, *Vector fields on spheres*, Topology 1 (1962) 63-65.

[4] J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) 75 (1962), 603-632.

[5] J. F. Adams, M. F. Atiyah, *K-theory and the Hopf invariant*, Quart. J. Math. Oxford Ser. (2) 17 (1966), 31-38.

[6] https://celebratio.org/Baum_P/article/713/

[7] M. Aguilar, S. Gitler, C. Prieto, *Algebraic topology from a homotopical viewpoint*, translated from Spanish, Universitext, Springer-Verlag, 2002, xxx+478 pp.

[8] M. F. Atiyah, *K-theory*, Notes by D. W. Anderson, 2nd e, Addison-Wesley, 1989, xx+216 pp.

[9] A. Hatcher, *Vector Bundles and K-Theory*, https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html

[10] D. Hosemoller, *Fibre bundles*, Grad. Texts in Math., 20, Springer-Verlag, 1994, xx+353 pp.

[11] M. Karoubi, *K-theory. An introduction*, Classics math. Springer-Verlag, 2008. xviii+308+e-7 pp.

[12] J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Math., U. Chicago Press, 1999, x+243 pp.

[13] E. Park, *Complex Topological K-Theory*, Cambridge Univ. Press, 2008, 218pp.

periodicity, is not easy to show. $\widetilde{K}(S^2)$, is additively infinite cyclic, with generator $H-1$, but with trivial multiplication, as $(H-1)^2 = 0$.

Example 1. $K(S^1) \cong \mathbb{Z}$ and $\widetilde{K}(S^1) \cong 0$ as any complex vector bundle over S^1 is trivial.

Example ∞. By Bott-periodicity, $\widetilde{K}(S^{2k+1}) \cong 0$ while $\widetilde{K}(S^{2k}) \cong \mathbb{Z}$, generated by $(H-1)^{*k} := (H-1) * \cdots * (H-1)$, where $*$ is the reduced external product defined by bundle pull back and tensor product. So $K(S^{2k}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$ where $\alpha = (H-1)^{*k}$.

Proposition. If S^n is an H-space, then n can not be even.

Proof. Suppose n is even, then the H-map $\mu : S^n \times S^n \to S^n$ induces ring homomorphism $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2)$ defined by $\mu^*(\gamma) = 1\alpha + 1\beta + m\alpha\beta + 0$, for some m. This is because the injections $i_1, i_2 : S^2 \to S^n \times S^n$ of S^n into the first and second factors of S^n induce ring homomorphisms i_1^* and i_2^*, sending one of α,β to 0 and the other to γ, which also compose with μ^* to give identity homomorphisms. However, $0 = \mu^*(\gamma^2) = (\mu^*(\gamma))^2 = (\alpha+\beta+m\alpha\beta)^2 = 2\alpha\beta$, which is not 0. Therefore, n has to be odd. □

The Hopf construction $g : S^{2n+1} \to S^{n+1}$ of $\mu : S^n \times S^n \to S^n$ is defined by gluing $g_+ : \partial D^{n+1} \times D^{n+1} \to D_+^{n+1}$ and $g_- : D^{n+1} \times \partial D^{n+1} \to D_-^{n+1}$ along $\mu : \partial D^{n+1} \times \partial D^{n+1} \to \partial D_\pm^{n+1}$, where

$$g_+(x,y) = \begin{cases} |y|\mu(x, \frac{y}{|y|}) & y \neq 0 \\ 0 & y = 0 \end{cases} \quad \text{and } g_-(x,y) = \begin{cases} |x|\mu(\frac{x}{|x|}, y) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Letting $n = 2k-1$, we have $g : S^{4k-1} \to S^{2k}$, which gives a recipe of gluing the boundary of D^{4k} to S^{2k}, obtaining the mapping cone C_g. Collapsing the S^{2k} subspace of C_g to a point, we get S^{4k}. Therefore, we have $S^{2k} \to C_g \to S^{4k}$, which produces the following short exact sequence of reduced K-groups:

$$0 \to \widetilde{K}(S^{4k}) \xrightarrow{i} \widetilde{K}(C_g) \xrightarrow{\pi} \widetilde{K}(S^{2k}) \to 0$$

Let $\alpha = i((H-1)^{*2k})$ and $\pi(\beta) = (H-1)^{*k}$. Then $\pi(\beta^2) = (\pi(\beta))^2 = 0$. Thus, β^2 is in the image of i. So $\beta^2 = h\alpha$ for some $h \in \mathbb{Z}$, which is called the Hopf invariant. This doesn't depend on β.

Proposition. If S^n is an H-space, then the Hopf invariant $h = \pm 1$.

Proof. For any map $\mu : S^n \times S^n \to S^n$, its bidgree is (p,q), where p and q are the degrees of $\mu(\cdot, 1), \mu(1, \cdot) : S^n \to S^n$. It can be shown that $h = pq$. (See [Aguilar-Gitler-Prieto] and [May].) For us, μ is an H-map. Thus, $(p,q) = (\pm 1, \pm 1)$ and so $h = \pm 1$. □

Lastly, consider the Adams operations $\psi^k : \widetilde{K}(X) \to \widetilde{K}(X)$, $k \geq 0$, which are self-ring homomorphisms. We will use the following properties of ψ^k:

1. Each ψ^k is natural.
2. $\psi^k \psi^l = \psi^{kl}$.
3. $\psi^p(x) \equiv x^p \mod p$ for prime p.
4. $\psi^l(x) = l^m x$ if $x \in \widetilde{K}(S^{2m})$.

Using property 4, we have $\psi^l(\alpha) = l^{2k}\alpha$. Using properties 1 and 4, we have $\pi(\psi^l(\beta)) = \psi^l(\pi(\beta)) = l^k\pi(\beta) = \pi(l^k\beta)$. Thus, $\pi(\psi^l(\beta) - l^k\beta) = 0$, and so by exactness at $\widetilde{K}(C_g)$, there is c_l such that $\psi^l(\beta) = l^k\beta + c_l\alpha$.

Now is the magic time. Also using property 2, we have

$$\psi^6(\beta) = \psi^2\psi^3(\beta) = \psi^2(3^k\beta + c_3\alpha) = 3^k(2^k\beta + c_2\alpha) + c_3 2^{2k}\alpha = 6^k\beta + (c_2 3^k + c_3 2^{2k})\alpha.$$

and

$$\psi^6(\beta) = \psi^3\psi^2(\beta) = \psi^3(2^k\beta + c_2\alpha) = 2^k(3^k\beta + c_3\alpha) + c_2 3^{2k}\alpha = 6^k\beta + (c_3 2^k + c_2 3^{2k})\alpha.$$

Thus, $c_3 2^k(2^k - 1) = c_2 3^k(3^k - 1)$, from which we see that $2^k \big| c_2(3^k - 1)$.

Furthermore, by property 3, $\psi^2(\beta) \equiv \beta^2 = \pm\alpha \mod 2$ and we had $\psi^2(\beta) = 2^k\beta + c_2\alpha$. Thus, c_2 is odd. Therefore, $2^k \big| 3^k - 1$.

Proposition. If $2^k \big| 3^k - 1$, then $k = 1, 2, 4$.

Proof. $2^k \big| 3^k - 1$ means the number of 2-factors of $3^k - 1$ is at least k. Let's find all 2-factors of $3^k - 1$, which presumably depends on a more detailed presentation of k. So let $k = 2^l m$, where m is odd. Thus,

$$3^k - 1 = (3^{2^l})^m - 1 = (3^{2^l} - 1)\sum_{i=0}^{m-1}(3^{2^l})^i.$$

As $\sum_{i=0}^{m-1}(3^{2^l})^i$ is the sum of an odd number of odd numbers, all 2-factors of $3^k - 1$ comes from $3^{2^l} - 1$.

If $l = 0$, then $3^{2^l} - 1$ contains one 2. In this case, $k = 2^0 m \leq 1$. Thus, $k = 1$.

If $l \geq 1$, then by using $a^2 - 1 = (a+1)(a-1)$ repeatedly, we have

$$3^{2^l} - 1 = (3^{2^{l-1}} + 1)(3^{2^{l-2}} + 1)(3^{2^{l-3}} + 1)\cdots(3^2 + 1)(3 + 1)(3 - 1).$$

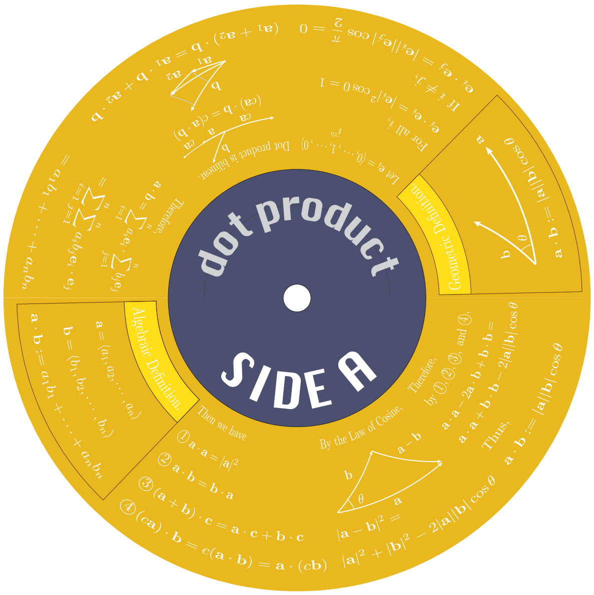
Except $3 + 1$, each of the above factors contains one and only one 2, this is because $3 \equiv -1 \mod 4$ and thus $3^q + 1 \equiv (-1)^q + 1 = 2 \mod 4$ if q is even. Therefore, $3^{2^l} - 1$ contains $l + 2$ factors of 2. In this case, $k = 2^l m \leq l + 2$. By plotting the graphs of $m2^x$ and $x + 2$, we see $m = 1$ and $l = 1, 2$.

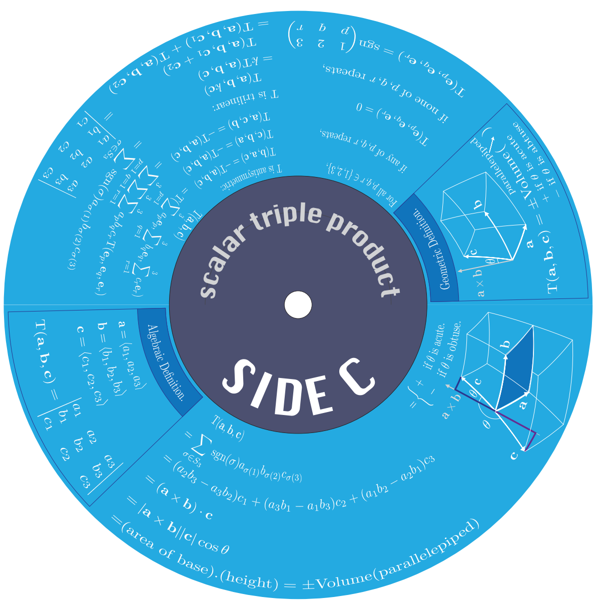
In summary, $k = 2^l$, where $l = 0, 1, 2$, i.e., $k = 1, 2, 4$. □

To conclude, $n = 2k - 1 = 1, 3, 7$.

5 Appendix

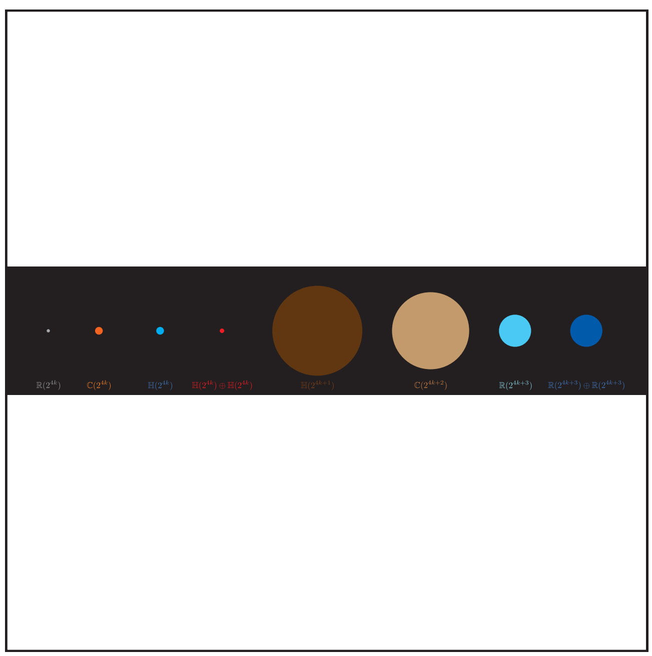
5.1 Sides A, B, C, D



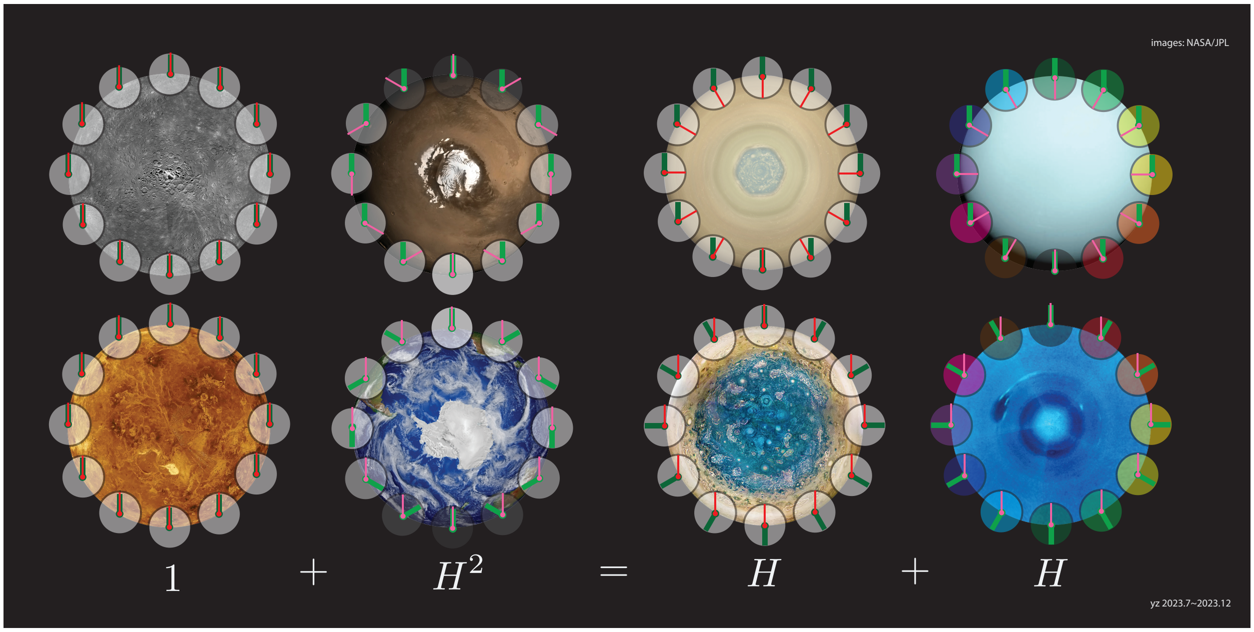


5.2 *Front cover*

5.3 *Back cover*



5.4 Inner jacket



5.5 Final product

5.6 Human-Machine Interaction