Introductory Complex Analysis in 30 Hours
———186 problems in 36 lessons

Side A: homework problems
Side B: lesson summary

MATH 345 Functions of a Complex Variable
Fall 2023

MATH 345 Calendar, Fall 2023

	Monday 9:00 SMUD 206	Tuesday	Wednesday 9:00 SMUD 206	Thursday	Friday 9:00 SMUD 206
Week 1	Sept 4	Sept 5 L0 Hello Math 345! College classes begin.	Sept 6 L1 I.1, I.2, I.3 Five views of complex numbers	Sept 7	Sept 8 L2 I.4 The n-th power and n-th root functions
Week 2	Sept 11 L3 I.5, I.6 The exponential and logarithmic functions	Sept 12 **Hw#1 (L0–L2) due**	Sept 13 L4 I.7 Power functions in general	Sept 14	Sept 15 L5 I.8 Trigonometric functions and their inverses
Week 3	Sept 18 L6 II.2 Definition of analytic functions	Sept 19 **Hw#2 (L3–L5) due**	Sept 20 L7 II.3 Equivalence to the Cauchy-Riemann equations	Sept 21	Sept 22 L8 II.4 Jacobian, IFTs, and inverse analytic functions
Week 4	Sept 25 L9 1.II.5 Introduction to harmonic functions	Sept 26 **Hw#3 (L6–L8) due**	Sept 27 L10 II.6 Conformal mappings and analytic functions	Sept 28	Sept 29 L11 II.7 Fractional Linear Transformations (FLT)
Week 5	Oct 2 L12 III.1, III.2, III.3 Real line integrals and harmonic conjugates	Oct 3 **Hw#4 (L9–L11) due**	Oct 4 L13 III.4 The Mean Value Property	Oct 5	Oct 6 L14 III.5 The Maximum Principle
Week 6	Oct 9 <———Mid-Semester	Oct 10 Break———————>	Oct 11 L15 IV.1 Complex line integrals	Oct 12 **Hw#5 (L12–L14) due**	Oct 13 **Exam 1 (L1 — L14)**
Week 7	Oct 16 L16 IV.2 Fundamental Theorem of Calculus for f(z)	Oct 17 **Your project starts by now, at least**	Oct 18 L17 IV.3, IV.4 Cauchy's Theorem & Cauchy's Integral Formula	Oct 19	Oct 20 L18 IV.4 Cauchy's Integral Formula cont'd
Week 8	Oct 23 L19 IV.5 Cauchy Estimates and Liouville's Theorem	Oct 24 **Hw#6 (L15–L18) due**	Oct 25 L20 IV.6, IV.7 Morera's Theorem & Goursat's Theorem	Oct 26	Oct 27 L21 IV.8 An elegant notation & Pompeiu's Formula
Week 9	Oct 30 L22 V.1, V.2 Series of functions in general	Oct 31 **Hw#7 (L19–L21) due**	Nov 1 L23 V.3 Power series in particular	Nov 2	Nov 3 L24 V.4 Expansion of analytic functions as power series
Week 10	Nov 6 L25 V.5 Power series at infinity	Nov 7 **Hw#8 (L22–L24) due**	Nov 8 L26 V.6 Manipulating power series	Nov 9	Nov 10 L27 V.7 Zeros of analytic functions and their magic
Week 11	Nov 13 L28 VI.1 Laurent decomposition and Laurent series	Nov 14 **Hw#9 (L25–L27) due**	Nov 15 L29 VI.2 Classification of isolated singularities	Nov 16	Nov 17 **Exam 2 (L15 — L27)**
Week 12	Nov 20 <—————————	Nov 21 ——————————	Nov 22 — Thanksgiving Break —	Nov 23 —————————	Nov 24 ——————————>
Week 13	Nov 27 L30 VI.3, VI.4 Meromorphic functions and PFD on C*	Nov 28 **Hw#10 (L28-L29) due**	Nov 29 L31 VII.1 The residue theorem	Nov 30	Dec 1 L32 VII.2—7 The residue calculus
Week 14	Dec 4 L33 VII.2—7 The residue calculus cont'd	Dec 5 **Hw#11 (L30-L32) due**	Dec 6 L34 VIII.1 The argument principle	Dec 7	Dec 8 **Paper+video due** L35 VIII.2 Rouche's Theorem
Week 15	Dec 11 L36 IX.1, IX.2 Automorphisms of the unit disk	Dec 12	Dec 13 L37 Award Ceremony **Hw#12 (L33-L36) due**	Dec 14 <——Reading/Study	Dec 15 Period———————>
Week 16	Dec 18 **<———————————**	Dec 19 **—————Final Exam**	Dec 20 **Period—————————**	Dec 21 **—————————>**	Dec 22

MATH 345-01, Fall 2023: Functions of a Complex Variable

Class meetings: **MWF** 9:00 – 9:50 AM @ SMUD 206
Instructor: Yongheng Zhang
Office: SMUD 510
Office Hours: **MW** 11:00 – 11:50 AM, 1:00 – 2:50 PM, **Th** 9:00 – 10:50 AM
Email: yzhang@amherst.edu
Text: Theodore W. Gamelin, *Complex Analysis*, Undergraduate Texts in Mathematics, Springer, New York, 2001 (corrected printing 2003).
Four copies of the textbook are on reserve in the Science Library.
Also available as an e-book: type "Gamelin complex analysis" in Amherst's Library Search.

Calendar: See the top of our Moodle page. It lists the exam, project, and homework due dates. It also lists the daily topic and the corresponding textbook sections.

Topics: "Complex analysis is a splendid realm within the world of mathematics, unmatched for its beauty and power." This course will testify this claim of Gamelin. We will study all core topics in classical complex analysis, including but not limited to holomorphic functions, harmonic functions, conformal mappings, meromorphic functions, power series and Laurent series, and the theorems of Cauchy, Riemann, Liouville, Morera, Goursat, Weierstrass, Casorati, Rouché, Schwarz and many more. We will follow the textbook fairly closely, covering most sections from Chapter I to Chapter IX, so that the rest can be left to your self-study and enjoyment in the next many years.

Attendance: You are expected to attend **every** class and **take notes**, as Lesson N depends on Lessons i for all $i < N$. If you have to miss Lesson N, please make sure you spend double time making up for it before Lesson $N + 1$, where $0 \le N \le 36$. We will take a group photo in Lesson 37.

Project: In addition to attending classes, taking notes, doing homework (more on this soon), you will also write a paper (no page requirement) and make a video presentation (10 to 12 minutes) in this class. You can either work independently or form teams with others in class. It's not supposed to be a burden but rather to be fun. I will suggest a list of topics but you can also choose your own. Just anything related to complex analysis you feel an urge to write and speak about, as long as it seriously explores your chosen topic. Your papers will be made into an anthology and distributed to the class. Your video presentation will be made available to everyone in our class to watch. Novel and experimental formats are encouraged. For example, students wrote poems, programmed online games, made softwares, taught us cool applications in the past. Some further explored class topics and went quite far. Some presented complete proofs of difficult theorems such as the prime number theorem. Paper and video are graded by completion. I will say more about this after the first Midterm, but you should start to think about which part of mathematics really means something to you and your project now.

Grading: Your grade will be determined by the weighted scores as follows:
Better Midterm **15%**
The other Midterm **10%**
Final exam **30%** (Yes! We do have a final exam!)
Homework **30%**
A project paper on a topic you feel passionate about **10%**
Project video presentation **5%**

Exams: Midterm 1: **Friday, October 13, in class.**
Midterm 2: **Friday, November 17, in class.**
Final exam: Time and location to be announced.
Only pencils/pens, eraser, and notes on a double-sided paper are allowed in exams.
Abide by the **Statement of Intellectual Responsibility**.

Homework: Doing homework is the **most important** part of this class. One can only learn mathematics by getting hands dirty. There is one assignment sheet for each lesson, which contains five problems, except Lesson 1, which contains six. Homework are assigned for L0 to L36, so you will work on 186 homework problems in Math 345 this semester. Each problem may have several parts. They fill missing steps from class, verify formulas, extend results and explore new situations. Homework problems are generally hard. However, ample hints, partial solutions, and sometimes full solutions are provided. Exam problems are much easier, with the final slightly harder than the midterms.

Problems assigned during a whole week are due in Moodle at 23:59 PM on Tuesday night the following week. See the calendar for details and a few exceptions. Start working on the problems **early**. After attending lecture and taking your notes, reorganize your notes, annotate them, and see if you can teach the core concepts to yourself and others. Then first work through the homework problems on your own, and then talking to your friends and working in groups are **highly recommended**: we can seek help from each other and we also understand ourselves better by explaining it to others. Sometimes, we also make collective progress which is so much fun but not possible if one works alone. Our Math Fellow and I will also hold office hours to answer your questions. Coming to office hours is an important part of learning. I encourage you to go to as many as you can, as turning abstract math into precise verbal communication is one of the secrets of learning math.

However, your homework solution must be totally **your own work**. That means you must write down your solution in your own words, without looking at your group members' work. Mathematics is ultimately learned by individuals. Copying others' work spoils all the fun of learning math. It also violates the **Statement of Intellectual Responsibility**. Furthermore, it is a requirement that you must **show complete solution** to problems. Writing down answers with insufficient justification or writing down proofs with big gaps will receive little credit.

As a courtesy to the graders and for your own benefit of developing neat writing styles, please (1) do the problems in order and label their lesson number and problem number; (2) write in logical and complete English sentences; (3) write legibly (it will be particularly pleasing to everyone if you strive for the standard of calligraphy). Once you are done, scan your solution to the problems as **a single pdf file** before uploading it to our Moodle site.

Homework sets are due at 11:59 PM ET on Tuesdays. Your graders will then immediately start to grade it. If you expect illness or emergency will prevent you from submitting your homework on time, let me know in advance so that I will ask them to extend your homework due date.

In Hindsight: Your notes will actually be your most precious work if you look back after a few years. Treat it seriously. It's your journal on which you record your mathematical journey, and document your thoughts, feelings, and personal growth. You want to admire yourself when you recall this four months' learning experience in the future. I'm very fond of my notes when I took Functions of Several Complex Variables in graduate school on which I found many interesting thoughts.

Prerequisite: Even though MATH 211 is the main prerequisite for this course, MATH 345 is **significantly harder** in that this is a proof-based course. If you have insufficient formal proof experience before (e.g., $\epsilon - N$, $\epsilon - \delta$ arguments), you will find this course too challenging. Talk to me if you feel unprepared. Usually, if you want to see how proof is done, your textbook is a good resource to turn to. However, Gamelin's proof style is not what you should mimic at this stage of education. It is too terse, resulting in outlines of proofs rather than actual proofs. In your homework and exams, you should supply enough details and make every sentence logical.

Suggestion: How hard should we work? **The Four-Hour Rule:** (less time $\rightsquigarrow$ lower quality of learning)

One hour for attentively reviewing your notes.
One hour for doing the problems on your own.
One hour for talking (or typing in emails) to your professor, your TA, and your friends.
One hour for thinking about your notes and your solution on your own again when you walk to Val, watch movies, take showers, and possibly during sleep.

That's enough. Keep a balance while pouring energy into learning math!

Contents

Hello, Math 345 students!

Welcome to Complex Analysis! If you are eager to be at the next level of mathematics and has been working unusually hard to achieve this goal, then Math 345 is for you!

This course starts from complex numbers. You would likely have heard of i, the mysterious "number" whose square is -1, so that i is a solution of $x^2+1=0$. Indeed, some people said complex numbers originated from our endeavor to create new numbers in order to solve this quadratic equation, but others argued that complex numbers really grew only from solving third degree polynomial equations. Whatever the historical fact is, the invention of complex numbers is a revolution, not just in math, but in science, society, and culture at large. Not too long after the focus on solving algebraic equations, mathematicians realized that complex numbers are indispensable when describing the motion of astronomical bodies. Some questions, such as the famous three-body problem formulated by Weierstrass, first falsely solved by Poincaré, later nailed by Sundman, used complex numbers in essential ways. Some advances in complex analysis itself, such as the theorem of Rouché, which we will learn toward the end of the semester, was inspired by staring into the sky. You may find complex numbers used in many aspects of everyday life. It is used in safely sending electricity to your wall socket. It appears in the Schrödinger equation supporting the digital age. Imagine what the world would be like without the second and third industrial revolutions.

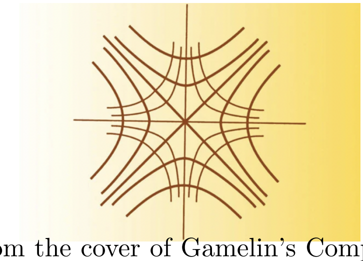
Applications of complex analysis to science, engineering and technology are far more versatile than previously scratched. In this course, instead, we will focus on the mathematical aspect. One underlying motif out of many is the surprising solution of difficult questions about functions of a real variable. Consider this classical one. The function $f(x)=\dfrac{1}{x^2+1}$ can be expanded as a Taylor series at any $x_0\in\mathbb{R}$, and it can be shown that the radius of convergence of this series is $\sqrt{x_0^2+1}$, though this is quite difficult if you do this by computing the coefficients $\frac{f^{(n)}(x_0)}{n!}$. Later, we will see that if we extend the real line to the entire complex plane, this is so merely because $\pm i$ are the zeros of the denominator! Take another example. Many real integrals are inconvenient to solve, like $\displaystyle\int_{-\infty}^{\infty}\frac{\cos x}{x^2+1}dx$, $\displaystyle\int_{0}^{\infty}\frac{x^{\frac{1}{\pi}}}{(1+x)^2}dx$, $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x}{x}dx$ and $\displaystyle\int_{-\infty}^{\infty}\sin(x^2)dx$, if it's even possible. Using complex numbers, they can be done with ease. Another motif is the close connection of complex analysis to other fields of mathematics. Proof of the prime number theorem uses the Riemann zeta function $\zeta(s)$ of the complex variable s. The Riemann hypothesis is about $\zeta(s)$ itself. Many kinds of transforms in analysis use complex numbers. Topological surfaces grew from trying to understand complex functions, which we will discuss right during the first week. Its higher dimensional versions properly fall in geometry and algebra.

This course will cover the core of classical complex analysis, by going through the multitude of topics meticulously. Please read the syllabus and calendar carefully, and do homework problems in Lesson 0. Their solutions have also been posted so that after comparing yours with them, you know the level of mathematical writings I expect of you, which you will further develop as we compute, prove, understand, and discover.

I'm really really excited to see you in class on Wednesday, when we will learn five ways of viewing complex numbers!

August 28, 2023
Yongheng Zhang (Math 345 instructor)

Lesson 0 Welcome to MATH 345!



1. The above image is taken from the cover of Gamelin's Complex Analysis, our textbook to be used in MATH 345. It consists of two classes of curves, the thinner ones and the thicker ones. What may be surprising is that, except at the origin, any two curves, one from each class, intersect at a right angle. We'll prove this fact in this problem as a way to review multivariable calculus.
   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function. What does it mean to say $f(x, y)$ is differentiable at (a, b)?
   (b) Prove that at the point (a, b) on the level curve $f(x, y) = k$, where $f(x, y)$ is differentiable (so the chain rule holds) and k is a constant, if $\nabla f(a, b)$ is not the zero vector, then it is a normal vector to the curve at (a, b), i.e., it is perpendicular to a tangent vector of the curve here. You used this fact in the Lagrange multiplier method.
   (c) In the image, the thicker curves are $C_1 : x^2 - y^2 = k$ and the thinner ones are $C_2 : 2xy = h$. Show that for any $h, k \neq 0$, if C_1 and C_2 intersect at (x, y), then they are perpendicular at this point, i.e., their tangent vectors are perpendicular. [1]

2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be C^n if all of its derivatives up to order n exist and are continuous. Show that $f(x) = x|x|$ is C^1 but not C^2. [2]

3. Use Internet to find a function $f : \mathbb{R} \to \mathbb{R}$ which is C^∞ but not analytic, i.e., its Taylor series expansion at a point exists but it's not the same as the function itself. No justification is needed. [3]

4. Find two C^1 functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f and g are the same over a small interval in $\mathbb{R}$ but f and g are different over $\mathbb{R}$. [4]

5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x^2+1}$.
   (a) Show that f is C^1.
   (b) Show that f is bounded on $\mathbb{R}$. [5]
   (c) Show that f is not an open mapping. [6]

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[1] Use dot product to show the two normal vectors are perpendicular. In Lesson 2, we will learn to generate infinitely many such pictures with ease. In Lesson 10, we will understand why the angles are 90^o.

[2] In contrast, if a function of a complex variable is C^1 (actually, existence of derivative is enough), then it's C^∞, i.e., C^n for all $n \in \mathbb{N}$. We will learn this in Lesson 18. C^∞ functions are called smooth functions.

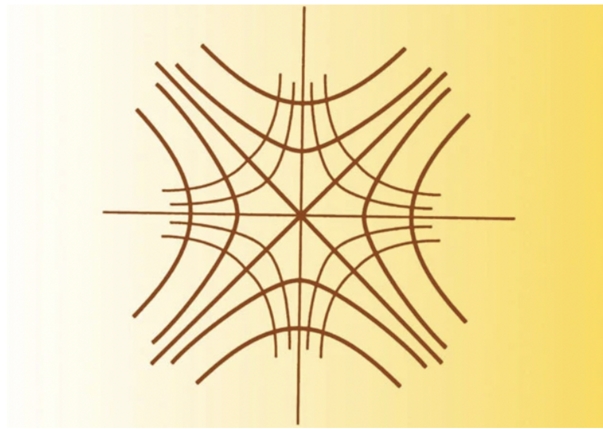
[3] In contrast, if a function of a complex variable is C^1, then it's analytic. We will learn this in Lesson 24.

[4] In contrast, if two functions of a complex variable are C^1 , and they are equal over a sub-region, then no matter how small this sub-region is, the two functions are the same function. This means if we know such a function locally, then we know it globally. It's as if if you know Amherst, you know the whole world. We will learn this in Lesson 27.

[5] A function $f : \mathbb{R} \to \mathbb{R}$ is bounded on $\mathbb{R}$ if there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. In contrast, if a function of a complex variable is C^1 and bounded over the entire complex domain, then f must be a constant. This is called Liouville's Theorem 0. We will learn this in Lesson 19. There are three other Liouville's Theorems which we will also learn.

[6] A subset E of $\mathbb{R}$ is open if for all $x \in E$, there is an open interval $(x - \epsilon, x + \epsilon)$ contained in E. A function $f : \mathbb{R} \to \mathbb{R}$ is called an open mapping if its image $f(\mathbb{R}) = \{f(x) \in \mathbb{R} : x \in \mathbb{R}\}$ is open. In contrast, if a non-constant function of a complex variable is C^1, then it's an open mapping. We will learn this in Lesson 27.

Solution to Lesson 0 homework



1. The above image is taken from the cover of Gamelin's Complex Analysis, our textbook to be used in MATH 345. It consists of two classes of curves, the thinner ones and the thicker ones. What may be surprising is that, except at the origin, any two curves, one from each class, intersect at a right angle. We'll prove this fact in this problem as a way to review multivariable calculus.

   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function. What does it mean to say $f(x, y)$ is differentiable at (a, b)?

   Solution. According to Stewart, Multivariable Calculus, 8e, $f(x, y)$ is differentiable at (a, b) means both partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist, and there are functions $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ satisfying $\lim\limits_{(x,y)\to(a,b)} \epsilon_i(x, y) = 0$ for $i = 1, 2$ such that $f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \epsilon_1(x, y)(x - a) + \epsilon_2(x, y)(y - b)$ for (x, y) close to (a, b).

   (b) Prove that at the point (a, b) on the level curve $f(x, y) = k$, where $f(x, y)$ is differentiable (so the chain rule holds) and k is a constant, if $\nabla f(a, b)$ is not the zero vector, then it is a normal vector to the curve at (a, b), i.e., it is perpendicular to a tangent vector of the curve here. You used this fact in the Lagrange multiplier method.

   Proof. Let $\vec{r}(t) = \langle x(t), y(t)\rangle$ be a smooth parametrization [1] of the curve $f(x, y) = k$ such that $\vec{r}(0) = \langle a, b\rangle$. So $f(x(t), y(t)) = k$. As f is differentiable, the chain rule holds. Thus, if we take the derivative with respect to t on both sides of $f(x, y) = k$, then we have $f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0$. So,

   $$\langle f_x(x(t), y(t)), f_y(x(t), y(t))\rangle \cdot \langle x'(t), y'(t)\rangle = 0,$$

   i.e., $\nabla f(x(t), y(t)) \cdot \vec{r}\,'(t) = 0$. Plugging in $t = 0$, we then have $\nabla f(a, b) \cdot \vec{r}\,'(0) = 0$, showing $\nabla f(a, b)$ is perpendicular to $\vec{r}\,'(0)$, the tangent vector to the curve at (a, b). $\square$

[1] The condition $\nabla f(a, b) \neq \langle 0, 0\rangle$ guarantees that such a smooth parametrization exist and the tangent vector $\vec{r}\,'(0) = \langle x'(0), y'(0)\rangle \neq \langle 0, 0\rangle$. Older textbooks on advanced calculus proves this. If you look into what Amherst College math majors learned in the 1970s, you will find such a book.

(c) In the image, the thicker curves are $C_1 : x^2 - y^2 = k$ and the thinner ones are $C_2 : 2xy = h$. Show that for any $h, k \neq 0$, if C_1 and C_2 intersect at (x, y), then they are perpendicular at this point, i.e., their tangent vectors are perpendicular.

Proof. Let $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$. Suppose C_1 and C_2 intersect at (x, y), and we consider $\nabla f(x, y) = \langle 2x, -2y \rangle$ and $\nabla g(x, y) = \langle 2y, 2x \rangle$. Neither is the zero vector. (If they were, then $x = y = 0$, which means $k = h = 0$, contradicting to the assumption that $h, k \neq 0$.) So by (b), $\nabla f(x, y)$ is the normal vector to C_1 at (x, y) and $\nabla g(x, y)$ is the normal vector to C_2 at (x, y). Then, as $\nabla f(x, y) \cdot \nabla g(x, y) = \langle 2x, -2y \rangle \cdot \langle 2y, 2x \rangle = 4xy - 4xy = 0$, these two vectors are perpendicular to each other. But by (b), for each curve, its tangent vector is perpendicular to its normal vector. Therefore, the two tangent vectors are perpendicular. (A picture would be helpful for us to see this more clearly.) $\square$

2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be C^n if all of its derivatives up to order n exist and are continuous. Show that $f(x) = x|x|$ is C^1 but not C^2.

Proof. The function $f(x)$ is a piecewise function which is x^2 if $x \geq 0$ and $-x^2$ if $x < 0$. Let's calculate $f'(x)$ first. When $x > 0$, $f'(x) = 2x$. When $x < 0$, $f'(x) = -2x$. We simply used the power rule for each branch of this piecewise function away from the origin. At the origin, we have to use the definition of derivative. $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h|h| - 0}{h} = \lim_{h \to 0} |h| = 0$. Summarizing, $f'(x) = 2|x|$, whose graph is a V-shaped "continuous" curve. Thus, f' is continuous, which means f is C^1. (We can also use the $\epsilon - \delta$ definition of continuity.[2] We will do it in the future.) However, f is not C^2 because $f'(x) = 2|x|$ has a sharp corner at the origin, but if $(f')'$ exists, then the graph should be smooth everywhere, and particularly at the origin. Alternatively, $(f')'(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0} \frac{2|h|}{h}$, which is 2 if $h \to 0$ from the right and -2 if $h \to 0$ from the left. This shows $(f')'(0)$ does not exist. $\square$

3. Use Internet to find a function $f : \mathbb{R} \to \mathbb{R}$ which is C^∞ but not analytic, i.e., its Taylor series expansion at a point exists but it's not the same as the function itself. No justification is needed.

Solution. After typing "smooth but non-analytic function" into Google (Or use Chat-GPT?), the wikipedia article suggested the function $f(x)$ defined by $e^{-\frac{1}{x}}$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$. Its derivatives not at 0 exist and are continuous up to all orders as they are derivatives of the 0 function and the composite function $e^{-1/x}$ over $(-\infty, 0)$ and $(0, \infty)$ respectively. Its derivatives at 0 are 0 for all orders. This can be proved by induction as shown in the wikipedia article. Thus, its Taylor series at the origin $f(0) + f'(0)x + f''(0)/2!x^2 + f'''(0)/3!x^3 + \cdots$ is the 0 function, which is different from $f(x)$ to the right of 0. This function is very useful in many areas of mathematics. It's worth remembering it.

[2] It goes like this. For any $a \in \mathbb{R}$ and any $\epsilon > 0$, let $\delta = \epsilon/2$, which is > 0. Then if $|x - a| < \delta$, then $|f'(x) - f'(a)| = |2|x| - 2|a|| = 2||x| - |a|| \leq 2|x - a| < 2\epsilon/2 = \epsilon$. Thus, the function $f'(x)$ is continuous at any $a \in \mathbb{R}$.

4. Find two C^1 functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f and g are the same over a small interval in $\mathbb{R}$ but f and g are different over $\mathbb{R}$.

   Solution. There are so many such pairs. For example, let f be the function in Problem 3. Let g be the translation of f by 1 unit to the right defined by $g(x) = f(x-1)$. Then $f(x) = g(x) = 0$ over $(-\infty, 0)$, but f and g are different over $\mathbb{R}$.

   For the same f above, we can also let $g(x) = 0$ for all $x \in \mathbb{R}$.

   Another choice is to let $g(x) = 0$ if $x < 0$ and $g(x) = x^2$ if $x \geq 0$.

5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x^2+1}$.

   (a) Show that f is C^1.

   Proof. $f'(x) = \frac{-2x}{(x^2+1)^2}$, which as a rational function with non-vanishing denominator, is continuous on $\mathbb{R}$. Thus, f is C^1. $\square$

   (b) Show that f is bounded on $\mathbb{R}$.

   Proof. Let $M = 1$. Then for all $x \in \mathbb{R}$, $|f(x)| = |\frac{1}{x^2+1}| \leq \frac{1}{1} = M$. This shows f is bounded on $\mathbb{R}$. $\square$

   (c) Show that f is not an open mapping.

   Proof. By inspecting the bell-shaped graph of f, we see that $f(\mathbb{R}) = (0, 1]$. The point 1 is in this set. However, no interval of the form $(1-\epsilon, 1+\epsilon)$ can be contained in $(0, 1]$. The part $(1, 1+\epsilon)$ is always outside. Therefore, f is not an open mapping. $\square$

1. For any three complex numbers $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$, prove that
$$(z_1 z_2) z_3 = z_1 (z_2 z_3),$$
by brute force. [7]

2. Define $\phi : \mathbb{C} \to M(2 \times 2, \mathbb{R})$ by $\phi(x+iy) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$. Prove that $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$ and $\phi(z_1 z_2) = \phi(z_1)\phi(z_2)$. [8]

3. (a) Prove that $|\mathrm{Re} z| \le |z|$ for all $z \in \mathbb{C}$.

   (b) Show that for any $z, w \in \mathbb{C}$, $|z+w|^2 = |z|^2 + |w|^2 + 2\mathrm{Re}(z\overline{w})$. [9]

   (c) Use (a) and (b) to show the triangle inequality $|z+w| \le |z| + |w|$. [10]

4. Given a fixed element $a \in \mathbb{C}$ with $|a| < 1$, show that if $|z| = 1$, then
$$\frac{|z-a|}{|1-\overline{a}z|} = 1.$$ [11]
Hint: Write $|z-a|$ as $|\overline{z} - \overline{a}|$ and multiply the numerator by $|z|$.

5. Prove that if $|z| < 1$, then $1 + \dfrac{z}{1-z} + \dfrac{\overline{z}}{1-\overline{z}} = \dfrac{1-|z|^2}{|1-z|^2} = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$. [12]

6. In this problem, we find the formula for stereographic projection and its inverse between $S^2 \backslash \{N\}$ and $\mathbb{C}$. Let $N = (0,0,1)$ be the north pole on the unit sphere S^2 centered at the origin. Let (X,Y,Z) be any point on S^2 but N. Let $z = x + iy$ be a point on the xy-plane, so this point is $(x,y,0)$. By writing down a parametric equation for the line from N to (X,Y,Z) which intersects the xy-plane at $(x,y,0)$, show that [13]

   (a) $z = \dfrac{X}{1-Z} + i\dfrac{Y}{1-Z}$.

   (b) $(X,Y,Z) = (\dfrac{2\mathrm{Re}(z)}{|z|^2+1}, \dfrac{2\mathrm{Im}(z)}{|z|^2+1}, \dfrac{|z|^2-1}{|z|^2+1})$.

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[7] You have seen that in polar form, or interpreted as a 2×2 matrix or a polynomial class, associativity is immediate. You always get something if you view an object from different angles.

[8] From this, we learn that, interpreted as matrices, $i^2 = -1$ corresponds to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

[9] $|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} = z\overline{z} + z\overline{w} + \overline{z\overline{w}} + w\overline{w} = \cdots$

[10] This course has another name: Complex Analysis. According to a mathematician, analysis is all about $\le$ while algebra is all about $=$. This problem helps us to practice analysis.

[11] The expression $(z-a)/(1-\overline{a}z)$ plays an important role. We will of course come back to it.

[12] This expression is called the Poisson kernel, the soul of harmonic functions. We will of course come back to it.

[13] The solution is in I.3 of the textbook.

Lesson 1 Summary

In this lesson, we were introduced to the notion of complex numbers, from multiple perspectives.

(1) The first is to view $\mathbb{C}$ as $\mathbb{R}^2$: each complex number $z = x + iy$ is a point (x, y) on the xy-plane. As each point (x, y) can further be identified as a vector $\langle x, y\rangle$ as was done in multivariable calculus, each complex number can be viewed as a vector on the plane. What distinguishes $\mathbb{C}$ from $\mathbb{R}^2$ is that $\mathbb{C}$ is not merely a vector space of dimension 2 over $\mathbb{R}$: $\mathbb{C}$ has the additional structure of complex number multiplication defined by $(x_1 + iy_1)(x_2 + iy_2) := x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$, which also follows from distributivity and the simpler $i^2 = -1$. With this richer structure, $\mathbb{C}$ is a field, containing the smaller field $\mathbb{R}$ as the x-axis on the xy-plane. The constructions of length $|z|$ and conjugate $\overline{z}$ interact with complex number multiplication very well, which are important tools we will use quite often in the next four months.

(2) The second is to consider $z = re^{i\theta}$ where r and θ are the same r and θ in polar coordinates we learned in calculus. The peculiar notation $e^{i\theta}$ for now is defined as $\cos\theta + i\sin\theta$ and thus $z = r\cos\theta + ir\sin\theta$, consistent with the polar to rectangular transformation $x = r\cos\theta$, $y = r\sin\theta$. Later, it will be seen that using $e^{i\theta}$ to mean $\cos\theta + i\sin\theta$ is most natural. This polar view of complex number makes it clear that complex number multiplication multiplies lengths and adds angles, the latter consistent with sum angle formula for cosine and sine. Later we will see that the pleasant inconvenience of nonuniqueness of θ opens up opportunities for us to broaden our conception of shapes.

(3) The third is to view $\mathbb{C}$ as the surface S^2 of the earth with the north pole N removed. Conversely, we can view the north pole N as *the* ∞ of $\mathbb{C}$. Here "*the*" signifies that there is only one infinity, in contrast to the $\pm\infty$ for $\mathbb{R}$. Going from S^2 to $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$ is called the stereographic projection, which translates things we do on the sphere to the plane and vice versa. This dual view is quite rich.

(4) The fourth is to view $z = x + iy \in \mathbb{C}$ as a 2×2 matrix $\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \in \mathcal{C}$, which defines a field isomorphism $\phi : \mathbb{C} \to \mathcal{C}$. In particular, $\phi(z_1z_2) = \phi(z_1)\phi(z_2)$. As matrix multiplication is associative, so is complex number multiplication via ϕ and ϕ^{-1}.

(5) The last is to view $z = a_0 + ia_1$ as an element of the quotient ring $\mathbb{R}[X]/(X^2 + 1)$ given by the isomorphism $\varphi(a_0 + ia_1) = [a_0 + a_1X]$, the class represented by $a_0 + a_1X$. Essentially, we are sending i to X and complex multiplication dictated by $i^2 = -1$ is translated to the law enforcement $X^2 + 1 = 0$, i.e., $X^2 = -1$. As polynomial multiplication is associative, so is complex number multiplication via φ and φ^{-1}.

This last viewpoint generalizes to define Hamilton's quaternions and Caley's octonions, which give $\mathbb{R}^4$ and $\mathbb{R}^8$ richer structures, respectively. Curiously, no other $\mathbb{R}^n$ except $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$ possess similar structures. This is beyond undergraduate mathematics, though ambitious students would want to start to explore it. It has application as near as calculus.

Lesson 2 The n^{th} power and n^{th} root functions and the emergence of Riemann surfaces

1. Hand sketch the images of each of the following set under the function $f(z) = z^2$.

    (a) the circle $|z| = 2$
    (b) the half line $\theta = \frac{\pi}{3}$
    (c) the line $\operatorname{Re} z = 1$
    (d) the annular sector $\frac{1}{2} < r < 2$, $-\pi/2 < \theta < \pi/2$
    (e) the circle $|z - 1| = 1$

2. (a) With branch cut taken as $(-\infty, 0]$, define the principal branch $f_0(z)$ and the other branch $f_1(z)$ of $f(z) = z^{\frac{1}{2}}$.
    (b) Hand sketch the image of the line $\operatorname{Re} z = 1$ under each of the branches you defined above.
    (c) Describe in pictures and words how you would construct the Riemann surface S of $z^{\frac{1}{2}}$ such that $f : S \to \mathbb{C}$ is a continuous function whose restriction to each of the two constituent $\mathbb{C}\backslash(-\infty, 0]$ on S are f_0 and f_1, respectively.

3. (a) With branch cut taken as $(0, \infty]$, define the principal branch $f_0(z)$ and the other branch $f_1(z)$ of $f(z) = z^{\frac{1}{2}}$.
    (b) Hand sketch the image of the line $\operatorname{Im} z = 1$ under each of the branches you defined above.

4. Consider the function $f(z) = z^{-1} = \frac{1}{z}$, which is defined on $\mathbb{C}\backslash\{0\}$. Note that (1) if $|z| \leq 1$, then $|f(z)| \geq 1$ and if $|z| \geq 1$, the $|f(z)| \leq 1$, so f maps anything inside the circle $|z| = 1$ to the outside of it and vice versa[14], and (2) $f(z) = \frac{\bar{z}}{z\bar{z}} = \frac{1}{|z|^2}\bar{z}$, i.e., $f(re^{i\theta}) = \frac{1}{r}e^{i(-\theta)}$, so $f(z)$ reflects the angle about the x-axis (and change the radius r to $\frac{1}{r}$).

    (a) Hand sketch the image of the circle $|z| = 2$ under f.
    (b) Hand sketch the image of another circle $|z - 3| = 1$.
    (c) Hand sketch the image of the line $\operatorname{Re} z = 1$. [15]
    (d) Hand sketch the image of the region $0 < |z| < \frac{1}{2}$.

5. Finally, we can consider other negative powers by combining z^n where $n \in \mathbb{N}$ and z^{-1}.

    (a) Define $f(z) = z^{-2} := (z^{-1})^2 = (z^2)^{-1}$. Sketch the image of the half circle $|z| = 2$, where $\operatorname{Re} z > 0$, under f.

    (b) Define $f(z) = z^{-\frac{1}{2}} := (z^{-1})^{\frac{1}{2}} = (z^{\frac{1}{2}})^{-1}$. So we need to do branch cut in order to appreciate this function. Using $\mathbb{C}\backslash(-\infty, 0]$ as the domain, define the two branches f_0 and f_1 of f. Sketch the images of the half circle $|z| = 2$, where $\operatorname{Re} z > 0$, under f_0 and f_1.

[14] This property of $f(z) = \frac{1}{z}$ is very useful when we will discover a striking fact of complex analytic functions later.
[15] Probably, you may suspect that images of these lines and circles are circles. This is indeed the case in general.

Lesson 2 Summary

We begin our study of complex-valued *functions of a complex variable* of the form $f : A \subseteq \mathbb{C} \to \mathbb{C}$ in this lesson, consistent with the title of MATH 345. What such functions do can be appreciated by drawing where $f(z)$ is on the codomain complex plane for each z in the domain complex plane. If you put a collection of such points together, you typically get a sophisticated picture of points, curves and regions, even for functions as simple as $f(z) = z^n$ and $f(z) = z^{\frac{1}{n}}$.

Even though homework problems were mostly about $f(z) = z^2$ and $f(z) = z^{\frac{1}{2}}$, in class, we scrutinized $f(z) = z^3$ and $f(z) = z^{\frac{1}{3}}$. For the former function, as $f(re^{i\theta}) = r^3e^{i3\theta}$, what f does is to cube its length and triple its angle. This alone can be used to hand sketch the images of many sets under f. Note that f is three-to-one away from 0: for any $w_0 \neq 0$, if $f(z_0) = w_0$, then $e^{\frac{i2\pi}{3}}z_0$ and $e^{\frac{i4\pi}{3}}z_0$ are also mapped to w_0. This causes trouble when we consider the latter function: the inverse $f(z) = z^{\frac{1}{3}}$ is multi-valued, in our case, triple valued. This can be solved by simply defining three inverses f_0, f_1 and f_2. However, another more serious problem is that any such inverse would be discontinuous on the entire complex plane $\mathbb{C}$. Luckily, this can also be solved by restricting the domain, i.e., throwing away the trouble-making points. In our case, throwing away any half-curve starting from the origin, e.g., $(-\infty, 0]$, would do. So we have three branches f_0, f_1, f_2, each defined on the slit plane, as the inverses of the cubic power function.

Here is a motivation of Riemann surface: even though we solved the above inverse problem, there were two regrets: (1) we had three, instead of one inverse. (2) we lost some points in the domain. To avoid such deficiencies, we can start from three copies of the slit domain $\mathbb{C}\backslash(-\infty, 0]$, with two intervals $(-\infty, 0)$ thrown back to the upper edge and lower edge respectively, and then glue the top edge of copy 0 to the lower edge of copy 1, glue the top edge of copy 1 to the lower edge of copy 2, and finally glue the top edge of copy 2 to the lower edge of copy 0. The thing we get is called *the*[16] Riemann surface of $z^{\frac{1}{3}}$. This is a topological space which locally looks like $\mathbb{R}^2$ everywhere[17]. For each point on $\mathbb{C}\backslash\{0\}$, including the interval $(-\infty, 0)$, we have three points on S. So we regained points on the deleted half line $(-\infty, 0)$. Furthermore, we can define a continuous $f : S \to \mathbb{C}$ whose restriction to each of the three copies of the slit plane is f_0, f_1 and f_2.

Riemann Surface is a subject in its own, bridging algebra, analysis, number theory, geometry, and topology. The very last chapter of our textbook is devoted entirely to it. Though a profound chapter and a decent beginning, it only scratched the surface.

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[16] if you do other branch cuts, the Riemann surface you get is of course different from this set-wise, but it looks exactly the same as this, which are dubbed isomorphic.

[17] The origin 0 is different, but we have excluded it.

Lesson 3 The exponential and logarithmic functions

1. Hand sketch the image of each of the following sets under the mapping $f(z) = e^z$.

   (a) the half line $\operatorname{Re} z > 1$, $\operatorname{Im} z = \frac{\pi}{4}$
   (b) the line segment $\operatorname{Re} z = 0$, $-\pi < \operatorname{Im} z \leq 0$
   (c) the rectangular region $0 < \operatorname{Re} z < \ln 2$, $-\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}$

2. Prove the usual identity, but for complex numbers:
$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

3. Hand sketch the image of each of the following sets under the mapping $f(z) = \operatorname{Log} z$, where $\operatorname{Log} z$ is the principal branch of the multi-valued logarithmic function $\log z$, i.e.,
$$\operatorname{Log} z = \ln |z| + i\operatorname{Arg} z,$$
where $-\pi < \operatorname{Arg} z < \pi$.

   (a) the slit annulus $1 < |z| < e^2$, and $z \notin (-e^2, -1)$
   (b) the upper half plane $\operatorname{Im} z > 0$
   (c) the vertical line $\operatorname{Re} z = e$

4. Prove the usual identity, but for complex numbers:
$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2,$$
where z_1 and z_2 are on the right-half plane ($\operatorname{Re} z_1, \operatorname{Re} z_2 > 0$) so that $z_1 z_2$ is still in the slit plane $\mathbb{C}\backslash(-\infty, 0]$ for $\operatorname{Log}(z_1 z_2)$ to make sense.

5. Define a continuous branch $f(z)$ of $\log z$ over $\mathbb{C}\backslash[0, \infty)$ such that $f(1-i) = \ln\sqrt{2} + i\frac{7\pi}{4}$. What is $f(-i)$?

Lesson 3 Summary

Continuing our study of complex-valued functions of a complex variable, now we turn to the complex analogues of the real exponential and logarithmic functions, which we learned in single-variable calculus.

As James Clerk Maxwell unified the study of electricity and magnetism, it is said that Leonard Euler unified exponential and trigonometric functions via the complex exponential $f(z) = e^{\operatorname{Re} z}e^{i\operatorname{Im} z}$, where the first part is the real exponential function and the second part contains sine and cosine. What e^z does is to wrap the entire complex plane around the origin infinitely many times such that horizontal lines go to rays converging to the origin (but never touching it) and vertical lines go to circles centered at the origin each of which is wrapped infinitely often. Note that these image rays and circles are perpendicular. In fact, if you draw any perpendicular curves on the domain, their image will be perpendicular. It's a little hard to see for now, but we will do soon after learning the miraculous property of complex differentiability in the next chapter.

The complex logarithmic function $\log z$ is expected to be inverse of e^z, but we run into the same problem as last time when we studied $z^{\frac{1}{n}}$: $\log z$ is multi-valued, and it's not continuous over its domain $\mathbb{C}\backslash\{0\}$. To solve these two problems, we resort to the same technique: branch cut and gluing: we remove the points in $(-\infty, 0)$ from $\mathbb{C}\backslash\{0\}$ to get the slit plane $\mathbb{C}\backslash(-\infty, 0]$ and then define the continuous principal branch $\operatorname{Log} z$ of $\log z$ by the "obvious" inverse formula $\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z$, then we generate infinitely many branches $f_k(z) = \operatorname{Log} z + i2\pi k$, $k \in \mathbb{Z}$ over infinitely many copies of the slit plane $\mathbb{C}\backslash(-\infty, 0]$. To get a continuous single-valued function $f(z)$, we first construct the domain Riemann surface S by gluing the top edge of the k^{th} copy of $\mathbb{C}\backslash(-\infty, 0]$ to the bottom edge of the $k+1^{\text{th}}$ copy of $\mathbb{C}\backslash(-\infty, 0]$ and then define $f(z)$ to be $f_k(z)$ over each copy of $\mathbb{C}\backslash(-\infty, 0]$.

Same as for $z^{\frac{1}{n}}$, we can cut off other half curves, as long as it starts from the origin, to construct the slit plane.

Lesson 4 What i^i is and complex power functions in general

1. Compute the following complex powers.

   (a) i^0

   (b) $i^i i^{-i}$

   (c) $(1+i\sqrt{3})^{1-i}$

   (d) $(i^i)^i$

   (e) $i^{i\cdot i}$

2. Prove that $z_1^\alpha z_2^\alpha = (z_1 z_2)^\alpha$, where both sides are considered sets.

3. The following statements are FALSE! Give a counterexample to each of them[18].

   (a) $z^\alpha z^\beta = z^{\alpha+\beta}$

   (b) $(z^\alpha)^\beta = z^{\alpha\beta}$

4. Let $f_0(z)$ be the principal branch of z^i where the cut is along $(-\infty, 0]$. Sketch the image of the annular sector $e^{-\frac{\pi}{2}} < |z| < e^{\frac{\pi}{2}}$, $-\frac{\pi}{2} \le \operatorname{Arg} z \le \frac{\pi}{2}$ of f_0. Label which edge in the domain is mapped to which edge in the codomain.

5. In Lesson 5, we will study the complex versions of the trigonometric functions like the cosine function, which is actually a composition of an exponential function with the following function
$$J(z) = z + \frac{1}{z},$$
which is called the Jourkowsky (Zhukovsky) map, in honor of the mathematician, scientist and engineer who is a founding father of aerodynamics. Indeed, this map is useful when studying air flows around airplane wings. In this problem, we explore a basic property of this map.

   (a) Show that $J(re^{i\theta}) = (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta$. [19]

   (b) Draw the images of the following counterclockwise oriented circles: $|z| = 5$, $|z| = 3$, $|z| = 1$, $|z| = \frac{1}{2}$, and $|z| = \frac{1}{4}$ under J. Label the direction of your images.

   (c) Following similar process, draw the images of these circles under this function $K(z) = z - \frac{1}{z}$, which is slightly different from $J(z)$, but useful for the sine function next time.

[18] You have found two already in Problem 1.

[19] Thus, J maps a circle $|z| = r$ to the ellipse $\frac{x^2}{(r+\frac{1}{r})^2} + \frac{y^2}{(r-\frac{1}{r})^2} = 1$ if $r \ne 1$ and the circle $|z| = 1$ to the line segment $[-2, 2]$ traversed twice.

Lesson 4 Summary

i^i doesn't mean multiplying i copies of the base i together, even though 5^3 means multiplying three copies of 5 together. In order to generalize 5^3 to i^i, we rewrite 5^3 as $e^{\ln 5^3}$, which is the same as $e^{3\ln 5}$. Having learned the complex versions of the exponential and logarithmic functions in Lesson 3, we can now define $i^i = e^{i \log i}$. In general,

$$z^\alpha = e^{\alpha \log z},$$

where $z \neq 0$, as we don't want $\log z$ to be not defined.

The function $\log z$ is multivalued, so do we expect z^α to be. However, that's not always the case, as the base e may absorb the multivaluedness, depending on what α is. To see this, continuing the above definition, we have

$$z^\alpha = e^{\alpha(\ln|z| + i(\operatorname{Arg} z + 2\pi k))} = e^{\alpha(\ln|z| + i\operatorname{Arg} z)}(e^{i2\pi\alpha})^k.$$

So z^α is multivalued if and only if $e^{i2\pi\alpha} \neq 1$. Thus, z^α is single-valued precisely when α is an integer, which is not usually the case. Indeed, this definition overlaps with those of z^n, $n \in \mathbb{Z}$.

Because of the multivalued nature of z^α in general, familiar identities like $z^\alpha z^\beta = z^{\alpha+\beta}$ and $(z^\alpha)^\beta = z^{\alpha\beta}$ do not hold. This is mainly because considered as sets, one side contains more complex numbers than the other.

Complex power functions are useful when use residue calculus to solve some difficult single variable integrals problems later in the course.

Lesson 5 Trigonometric functions and their inverses

1. In class, we proved that $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$. Now it's your turn: show that
$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

2. Prove that the zeros of $\sin z$ are $k\pi$, $k \in \mathbb{Z}$.

3. Prove that $\sin(z + T) = \sin z$ for all $z \in \mathbb{C}$ if and only if $T = 2k\pi$, where $k \in \mathbb{Z}$. [20]

4. Let $w = \sin^{-1} z$.

   (a) From $\sin w = z$ and the definition of $\sin w = \frac{e^{iw} - e^{-iw}}{2i}$, solve for w to show that
$$w = \frac{1}{i} \log(iz \pm \sqrt{1 - z^2}).$$

   (b) Let $z = x$ be real and $-1 \leq x \leq 1$. Show that by letting $\sqrt{\ }$ be the usual positive square root, log be the principal branch Log, and choosing $+$ instead of $-$, w is the usual inverse trigonometric function $\arcsin(x)$ where $-\frac{\pi}{2} \leq w \leq \frac{\pi}{2}$.

5. Define $\tan w = \frac{\sin w}{\cos w}$, and let $w = \tan^{-1} z$.

   (a) From $\tan w = z$ and the definitions of $\sin w = \frac{e^{iw} - e^{-iw}}{2i}$ and $\cos w = \frac{e^{iw} + e^{-iw}}{2}$, solve for w to show that
$$w = \frac{1}{2i} \log(\frac{1 + iz}{1 - iz}).$$

   (b) Let $z = x$ be real. Show that w is the usual inverse trigonometric function $\arctan(x)$ where $-\frac{\pi}{2} < w < \frac{\pi}{2}$, by letting log be the principal branch Log.

Remark. When doing parts (b) for both Problems 4 and 5, draw the right triangles relating θ and x. We can certainly do calculations during the last steps, but they can be avoided if we simply stare at the pictures.

[20]For the forward direction, letting z be real, then $T = 2k\pi$ from what we know in calculus. For the backward direction, given $T = 2k\pi$, we verify the equality.

Lesson 5 Summary

Trigonometric and inverse trigonometric functions can also be extended to complex numbers. Their definitions follow from the real formulas, which has to be the case, as we will see later in the course. So do familiar trigonometric identities.

In more details, as

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i},$$

we define

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

Recall from the homework of Lesson 4 that $J(z) = z + \frac{1}{z}$ and $K(z) = z - \frac{1}{z}$. Thus, we can also write

$$\cos z = \frac{1}{2}J(e^{iz}) \text{ and } \sin z = \frac{1}{2i}K(e^{iz}).$$

Since we know what the mappings J, K and $e^{(\ )}$ do, and that multiplication by i is counter-clockwise rotation by $\pi/2$, dividing by 2 is radial shrinking by a factor of 2, we have a pretty good picture of what $\cos z$ and $\sin z$ do to points on the complex plane.

Also recall that $\sin z$ and $\cos z$ are periodic functions, with period 2π along the real direction, and that's it: there are no period along the imaginary direction. This is guaranteed, as we will later learn that if a complex analytic function is defined on the entire $\mathbb{C}$ and it has periods along two directions, then it has to be constant! We know sin and cos are not constants, so we never see things like $\cos(z+iT) = \cos(z)$ and $\sin(z+iT) = \sin(z)$, where iT is pure imaginary.

Lastly, you may wonder why log appeared in inverse trigonometric functions. Here is the deal: similar to that complex exponential functions unify real exponential functions and trigonometric functions, complex logarithmic functions unify real logarithmic functions and inverse trigonometric functions (The $\arg z$ in $\log z = \ln|z| + i\arg z$ is an angle, which is the output of inverse trigonometric functions). Therefore, it's expected that complex inverse trigonometric functions have something to do with complex logarithm.

LESSON 6 DEFINITION OF ANALYTIC FUNCTIONS

1. Prove that if $u(x,y)$ is continuously differentiable on a domain (open and path-connected) D, such that $\nabla u := \langle u_x, u_y\rangle = \langle 0,0\rangle$ on D, then u is a constant function on D. [21]

2. Consider the function $f(z) = \overline{z}$. In this problem, we show that even though f is continuous, it is not differentiable at any point and thus is not analytic.

   (a) Prove that f is continuous on $\mathbb{C}$ using the definition of continuity.
   (b) Prove that f' does not exist at any $z \in \mathbb{C}$ by choosing two directions of Δz along which $\displaystyle\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$ are not equal. [22]

The function
$$F(z) = \int_0^1 \frac{f(t)}{t-z}dt,$$
where $f : [0,1] \to \mathbb{C}$ is continuous, is the main player in the next three problems. Note that F is a function from $D = \mathbb{C}\backslash[0,1]$ to $\mathbb{C}$. F itself is not very useful in this course, but it is of good educational value as some techniques we will use later are contained in the following exercises.

3. As a warm-up, use definition to show that F is continuous on D. [23]

4. Use definition to prove that $F'(z_0) = \displaystyle\int_0^1 \frac{f(t)}{(t-z)^2}dt$ for all $z_0 \in D$. [24]

5. Use definition to prove that $F'(z)$ is continuous at all $z_0 \in D$. Problems 4 and 5 show that F is analytic on D. [25]

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[21] The proof is on page 38 of the textbook. To fill in more detail, feel free to use the 2D Mean-Value Theorem: if $u(x,y)$ is differentiable on a domain containing the line from (x_0,y_0) to (x,y), then there is a point (x_1,y_1) on this line such that $u(x,y) - u(x_0,y_0) = \nabla u(x_1,y_1)\cdot\langle x-x_0, y-y_0\rangle$

[22] One choice is on page 43 of the textbook.

[23] That is, for any $z_0 \in D$, and any $\epsilon > 0$, we need to find $\delta > 0$ such that for any $z \notin [0,1]$, if $|z-z_0| < \delta$, then $|F(z) - F(z_0)| < \epsilon$. For Problems 3, 4 and 5, it is useful to use the estimate $|\int_0^1 g(t) - h(t)dt| \le \int_0^1 |g(t)-h(t)|dt$.

[24] That is, for any $z_0 \in D$, and any $\epsilon > 0$, we need to find $\delta > 0$ such that for any $z \notin [0,1]$, if $0 < |z-z_0| < \delta$, then $|\frac{F(z)-F(z_0)}{z-z_0} - \int_0^1 \frac{f(t)}{(t-z)^2}dt| < \epsilon$.

[25] You may find it useful that the function $|t-z_0|$ is continuous where t is on the compact $[0,1]$ and thus $|t-z_0| < N$ for some $N > 0$.

Lesson 6 Summary

When people say that $D \subseteq \mathbb{C}$ is a *domain*, they don't just mean that D can be the domain of a function $f : D \to \mathbb{C}$. Here, more specifically, D is open and path-connected. Open means at any point in D, there is an open disk centered at this point which is completely contained in D. So an open set is a union of open disks. Path-connected means any two points in D can be joined by a polygonal line segment which is completely contained in D. In this course, most of our functions' domains are *domain* in this sense.

Given a function $f : D \to \mathbb{C}$,

(1) $\lim_{z \to z_0} f(z) = L$ for some $z_0 \in D$ if for any $\epsilon > 0$, there is $\delta > 0$ such that if $z \in D$ and $0 < |z - z_0| < \delta$, then $|f(z) - L| < \epsilon$.

(2) We say f is continuous at $z_0 \in D$, if $\lim_{z \to z_0} f(z) = f(z_0)$, where the limit L is the function's value $f(z_0)$ at z_0.

(3) We say f is continuous (on D), if f is continuous at each $z_0 \in D$.

(4) We say f is differentiable at $z_0 \in D$, if $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists. This limit is denoted by $f'(z_0)$. Equivalently, by letting $\epsilon(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$, we have
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)(z - z_0),$$
where $\lim_{z \to z_0} \epsilon(z) = 0$.

(5) We say f is differentiable (on D), if f is differentiable at each $z_0 \in D$.

(6) Finally, we say f is analytic on D if f is differentiable on D and f' is continuous on D.

Why is it called an *analytic* function? The delightful property of such functions is that if f is differentiable once and its derivative is continuous (which actually is implied from the differentiability of f, as we will see later, but it will take a big detour), then f is infinitely differentiable, which means f', f'', f''', f'''', f''''', ... exist and are continuous. Such functions are dubbed C^∞. But more is true: we will see that **the Taylor series (power series) of f equals f**. In real analysis/calculus, a function is called analytic if its Taylor series equals the function itself. Same thing here, but for us, the mere existence of f' implies the analyticity of f. To distinguish the consequent analyticity from its less demanding definition, most people call a function which is differentiable on D (and whose derivative is continuous on D) a *holomorphic* function, which means f' exists everywhere on the *whole* D. I like this name better, also because it's paired with *meromorphic* functions, which we will learn later.

Lesson 7 Equivalence to the Cauchy-Riemann equations

1. Use the Cauchy-Riemann equations to show that the function $f(z) = \overline{z}$ we considered last time is not analytic. How about $f(z) = |z|^2$?

2. Use the Cauchy-Riemann analyticity criterion to show that the following are analytic functions. Also find a formula for each function in the form of $f(z)$. [26] For example, $x^2 - y^2 + i(2xy) = z^2$ because it is $(x+iy)^2$.

   (a) $x^3 - 3xy^2 + i(3x^2y - y^3)$

   (b) $\frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$

   (c) $\frac{1}{2}\ln(x^2+y^2) + i\text{Arctan}(\frac{y}{x})$ on $\operatorname{Re} z > 0$.

3. In class, we derived the Cauchy-Riemann equations by evaluating the limit of the difference quotient along the horizontal ($\Delta z = \Delta x$) and vertical ($\Delta z = i\Delta y$) directions. Being the simplest directions, they are not the only directions we can take. In fact, we can choose any direction. Let's try another: Prove that if $f'(z)$ exists, then
$$f'(z) = \frac{u_x + u_y}{1+i} + i\frac{v_x + v_y}{1+i}$$
Hint: Let $\Delta z = \Delta t + i\Delta t$. [27]

4. Show that if both $f = u + iv$ and $\overline{f} = u - iv$ are analytic on D, then f is constant. [28]

5. Show that if $f = u + iv$ is analytic on D and $|f|$ is constant, then f has to be constant. [29]

[26] *Hint*: These are all familiar functions.

[27] Further hints. One way to do it is as follows. $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{u(z+\Delta z) + iv(z+\Delta z) - u(z) - iv(z)}{\Delta z} = \lim_{\Delta t + i\Delta t \to 0} \frac{u(x+\Delta t, y+\Delta t) - u(x,y) + i(v(x+\Delta t, y+\Delta t) - v(x,y))}{\Delta t + i\Delta t} = \lim_{\Delta t \to 0} \frac{1}{1+i} \frac{u(x+\Delta t, y+\Delta t) - u(x,y)}{\Delta t} + \lim_{\Delta t \to 0} \frac{i}{1+i} \frac{v(x+\Delta t, y+\Delta t) - v(x,y)}{\Delta t} = \frac{1}{1+i} \frac{du(x+t,y+t)}{dt}\Big|_{t=0} + \frac{i}{1+i} \frac{v(x+t,y+t)}{dt}\Big|_{t=0}$. It remains to do the last-step calculation by using the chain rule.

[28] Set up four C-R equations. Show $u_x = u_y = 0$, and $v_x = v_y = 0$ to conclude that both u and v are constant using Problem 1 of Lesson 6.

[29] If $|f|$ is the constant 0, then $f = 0$. Otherwise ($|f| \neq 0$ and thus f is never 0), consider the well-defined expression $\overline{f} = \frac{|f|^2}{f}$, which is analytic. The connection with Problem 4 should be clear at this moment.

Lesson 7 Summary

Last time, we defined analytic (holomorphic) functions. Thus, simple power functions z^n, polynomials and rational functions are analytic as we can show explicitly that their derivative exist and are continuous. But what about functions like e^z? It's not immediately clear what $(e^z)'$ is using definition.

The Cauchy-Riemann Analyticity Criterion saves us. It says that $f = u + iv$ is analytic if and only u and v are C^1 (partial derivatives of u and v exist and are continuous) and they satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. I really don't recommend just memorizing these two equations, but to understand where they came from: along the horizontal and vertical directions, we have $f' = u_x + iv_x = \frac{1}{i}(u_y + iv_y)$.

Using this criterion, we see e^z is analytic and using either of the last two formulas above, we have $(e^z)' = e^z$, the same formula we learned in calculus.

Thus, the C-R equations and the C-R criterion provide us with computable tools to deal with analytic functions. It's worth mentioning that this criterion holds, especially in the reverse direction, because of a miracle to go from the real to the complex.

Some other applications:

(1) If $f : D \to \mathbb{C}$ is analytic, and $f' = 0$, then f is a constant on D.
(2) If $f : D \to \mathbb{C}$ is analytic, and f is real, then f is a constant on D.
(3) If $f : D \to \mathbb{C}$ is analytic, and f is pure imaginary, then f is a constant on D.
(4) Similarly, if $f : D \to \mathbb{C}$ is analytic, and the image of f is contained on a smooth 1D curve $F(u, v) = k$ whose normal vector ∇F never vanishes, then it's not hard to believe that f is a constant on D.[30]
(5) If both $f, \overline{f} : D \to \mathbb{C}$ are analytic, then f is a constant on D.
(6) If $f : D \to \mathbb{C}$ is analytic and $|f|$ is constant, then f is a constant on D. This is a special case of (4) where the curve is a circle.

All of these are proved using the C-R equations, and Problem 1 of Lesson 6.

Complex analysis is intimately connected with the physical world. This can be seen through the Cauchy-Riemann equations. If $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a 2D vector field, then $\vec{F}(x, y)$ is divergence-free (incompressible) if $P_x + Q_y = 0$ and $\vec{F}(x, y)$ is curl free (irrotational) if $Q_x - P_y = 0$. These two equations are $Q_x = P_y$ and $Q_y = -P_x$, the same as the Cauchy-Riemann equations. If P and Q also have continuous partial derivatives, then we know $Q(x, y) + iP(x, y)$ is analytic. So analytic functions store the components of incompressible and irrotational vector fields. This only reveals the tip of the mysterious iceberg of the connection between complex analysis and real world applications.

[30] Indeed, this can shown by noticing that $0 = F_x = \nabla F \cdot \langle u_x, v_x\rangle = \nabla F \cdot \langle u_x, -u_y\rangle$, $0 = F_y = \nabla F \cdot \langle u_y, v_y\rangle = \nabla F \cdot \langle u_y, u_x\rangle$ and that if two perpendicular vectors are both perpendicular to a third vector in $\mathbb{R}^2$, then these two vectors are $\vec{0}$. On the other hand, this result can also be proved using something we will learn in the end (the open mapping theorem of analytic functions).

1. (a) State the 1D real version of the Inverse Function Theorem (IFT) as we did in class.

   (b) Consider the function $f(x) = x^3$ defined on $\mathbb{R}$. Show that (1) it's C^1 and invertible, but (2) f^{-1} is not differentiable at one point in $\mathbb{R}$. Which condition does not hold in the IFT?

   (c) Consider the function $f(x)$ defined by $f(x) = x + x^2 \sin\frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$. It's a fact that no matter how much we zoom into the origin, f is not invertible. To see what went wrong, (1) Show that $f'(x) = 1 + 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ if $x \neq 0$ (using differentiation rules), and $f'(0) = 1$ (using the limit definition of derivative). (2) Prove that $f'(x)$ is not continuous at 0 by finding a sequence $(x_n) \to 0$ such that $f'(x_n) \to 0$, which is not $f'(0)$. [31]

2. (a) State the 2D real version of the Inverse Function Theorem as we did in class.

   (b) Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (2x - y, x + y)$. Use the above theorem to show that F is locally invertible. Find a formula for F^{-1} and also calculate $J(F^{-1})$. Is it the same as $(JF)^{-1}$?

3. State the complex version of the Inverse Function Theorem as we did in class.

4. Prove that $(\mathrm{Tan}^{-1})'(z) = \frac{1}{1+z^2}$, the familiar formula in calculus, where

$$\mathrm{Tan}^{-1}(z) = \frac{1}{2i}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right).$$

5. This last problem is about another occurrence of Jacobian: it shows up when calculating the area of the transformed image of an analytic function.

   (a) Let $f : D \to \mathbb{C}$ be a 1-1 analytic function. Prove that $\mathrm{Area}(f(D)) = \iint_D |f'(z)|^2 dxdy$. [32]

   (b) Let $f(z) = z^2$ and $D = \{z \in \mathbb{C} \big| |z - 1| \leq 1\}$. Is f 1-1 on D? Sketch the region $f(D)$ and calculate its area. [33]

[31] At what numbers is cosine 1 and sine 0? Answer: $2\pi n$. So we can, for example, let $x_n = \frac{1}{2\pi n}$, which goes to 0.

[32] Recall $\mathrm{Area}(f(D)) = \iint_{f(D)} 1dudv$, which, after a change of variables, is $\iint_D |\frac{\partial(u,v)}{\partial(x,y)}|dxdy$.

[33] As D is a circular disk, it's most convenient to use polar coordinates. Note that D is described by $0 \leq r \leq 2\cos\theta$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Lesson 8 Summary

The 1D real version of the Inverse Function Theorem was used in single variable calculus to show the existence of inverse function and its derivative and how to find this derivative if we know the original function's derivative. The proof of it is not that hard, as $f'(x_0) \neq 0$ and the continuity of $f'(x)$ guarantees that either $f'(x) > 0$ or $f'(x) < 0$ over an interval containing x_0 and thus $f(x)$ is strictly monotone over this interval. Hence, its inverse f^{-1} naturally exists by the intermediate value theorem and is continuous by visual inspection (which immediately generates a rigorous proof). It then follows that f^{-1} is also differentiable with its derivative given by the flip of that of f. The continuity of f^{-1} is used to show this. It's also used to show the continuity of $(f^{-1})'$.

The complex version of the Inverse Function Theorem formally is analogous to the 1D real version of the Inverse Function Theorem. However, the complex version is based on the 2D real version of the Inverse Function Theorem as analytic functions $f : D \to \mathbb{C}$ are mappings from $2D$ plane to itself. The proof of the 2D Theorem is significantly harder, because the monotonicity result doesn't make direct sense in dimensions other than 1. Nonetheless, its proof is a great demonstration of a few techniques in higher dimensional real analysis.

Using the 2D IFT, the complex version of IFT follows naturally, by recognizing the equality $\det(Jf(z)) = |f'(z)|^2$. And thus the non-vanishing of the Jacobian determinant of f is the same as the non-vanishing of f'. The former is a notion in multivariable calculus while the latter is what we just learned in this course.

Using the complex IFT, we quickly recognize the familiar formula $(\operatorname{Log} z)' = \frac{1}{z}$ on $\mathbb{C}\backslash(-\infty, 0]$. Even though we could prove the same using a concrete formula like that in Problem 2(c) of Lesson 7, note that it only holds on the right-half plane. Knowing this formula, many other formulas can also be deduced, for example, $(z^\alpha)' = \alpha z^{\alpha-1}$ and derivatives of inverse trigonometric functions, as they are expressed using the complex logarithmic function.

Inverse Function Theorems is a subject on its own, extending calculus and real and complex analysis to functional analysis, differential topology and differential geometry. The notoriously difficult theorems of John Nash and Jürgen Moser fit into this category. There are much much more to learn.

Lesson 9 Introduction to harmonic functions

1. In this exercise, we check that $\operatorname{Arg} z$, $z \in \mathbb{C}\backslash(-\infty, 0]$ indeed is a harmonic function on its domain by doing concrete calculations on different subregions of its domain, whose union is the domain.

   (a) On the right half plane $\operatorname{Re} z > 0$, use definition to show that $\operatorname{Arg} z$ is harmonic by using $\operatorname{Arg} z = \operatorname{Arctan}(\frac{y}{x})$.

   (b) On the upper half plane $\operatorname{Im} z > 0$, use definition to show that $\operatorname{Arg} z$ is harmonic by using $\operatorname{Arg} z = \operatorname{Arccos}(\frac{x}{\sqrt{x^2+y^2}})$.

   (c) On the lower half plane $\operatorname{Im} z < 0$, use definition to show that $\operatorname{Arg} z$ is harmonic by using $\operatorname{Arg} z = -\operatorname{Arccos}(\frac{x}{\sqrt{x^2+y^2}})$.

2. Use definition to show that the following functions are harmonic on their domains.

   (a) $u(x, y) = e^{2x} \cos 2y$
   (b) $u(x, y) = \dfrac{x}{x^2 + y^2}$

3. Find a harmonic conjugate for each of the functions in Problem 2. What are the associated analytic functions in terms of z?

4. Consider the function $u(x, y) = \begin{cases} \frac{-2xy}{(x^2+y^2)^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0). \end{cases}$ It is a fact that $\Delta u = 0$ on $\mathbb{C}$, which you don't need to verify. However, u is not harmonic on $\mathbb{C}$. To see this, show that $u_x(x, y)$ is not continuous at $(0, 0)$, which is a necessary condition for u to be C^2. [34] [35]

5. Prove that $u(x, y) = \ln \sqrt{x^2 + y^2}$ does not have a harmonic conjugate on its domain $\mathbb{C}\backslash\{0\}$. [36]

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[34] Calculate $u_x(0, 0)$ using the limit definition of derivative. Calculate $u_x(x, y)$, when $(x, y) \neq (0, 0)$, using the quotient rule. Then show the discontinuity by finding a path for (x, y) going to $(0, 0)$ along which the limit of $u_x(x, y)$ is not $u_x(0, 0)$.

[35] Indeed, on the smaller domain $\mathbb{C}\backslash\{0\}$, u is the imaginary part of $\frac{1}{z^2}$, but there is no way to extend this function even continuously to 0.

[36] Restricted to $\mathbb{C}\backslash(-\infty, 0]$, we know $u(x, y)$ does have a harmonic conjugate $\operatorname{Arg} z$. Suppose $v(x, y)$ is a harmonic conjugate for $u(x, y)$ on $\mathbb{C}\backslash\{0\}$. Then restricted to $\mathbb{C}\backslash(-\infty, 0]$, $v(x, y) = \operatorname{Arg} z + C$ by uniqueness of harmonic conjugates. So there is not much freedom for $v(x, y)$: it is the angle function up to a constant. As $\operatorname{Arg} z$ jumps across $(-\infty, 0]$, so does $v(x, y)$. Thus, $v(x, y)$ can not be continuous on $\mathbb{C}\backslash\{0\}$, but harmonic functions have to be at least continuous on its domain. Oops! It looks like this is almost the whole proof, which has been spelled out for free.

Lesson 9 Summary

Harmonic functions make sense in all dimensions, and in all kinds of spaces. In the flat 2-dimensional space, $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is called harmonic if u is C^2, and $u_{xx} + u_{yy} = 0$. Denoting the second order partial differential operator $\frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y}$ by Δ, the previous equation can also be written $\Delta u = 0$, called the Laplace equation, and Δ is called the Laplacian. Laplace is a person, who is, simply, amazing. Though, engineering students hate him, for that the Laplace transforms, which are useful when solving linear differential equations, were fed into their mouths without explanation. That's because they didn't learn the elegant complex analysis behind the scene, as I can attest by my experience. So, you, students of 345, will be protected from this feeling.

It will turn out that u merely being C^2 actually implies C^∞, i.e., its partial derivatives of all orders exist and are continuous. But the assumption that u is C^2 has to be checked carefully, as we saw in Problem 4 that, there are functions which satisfy the Laplace equation, but the function still is not harmonic, as their derivatives are not C^2. The requirement of sufficient continuous differentiability is important for making a connection between real harmonicity and complex analyticity.

Indeed, if $f = u + iv$ is analytic, then both u and v are harmonic. In some sense, we can regard an analytic function as storing two harmonic functions. Conversely, if both u and v are harmonic, then $f = u + iv$ is not necessarily analytic. However, if we start from a harmonic u, then we can always find another harmonic function v such that $u + iv$ is analytic, at least over a smaller domain. Such v is called a harmonic conjugate of u, which is unique up to adding a constant over their domain D. So harmonic conjugate is essentially unique. In Lesson 12, we will use line integrals to find harmonic conjugate over simply-connected domains. Today, it's enough to do antiderivative.

Harmonic functions appear in almost all second-order partial differential equations, which govern many physical, chemical, and geometrical phenomena. Many books are devoted to this subject.

$\pi\alpha\theta\epsilon\tilde{\iota}\nu$

$\mu\alpha\theta\epsilon\tilde{\iota}\nu$

"These two verbs mean 'to suffer' and 'to learn'. Do you see how they're almost identical? What Socrates is doing here is punning on these words to remark on the similarity of the two actions."

Hello, Math 345 students again!

It's been one month since August 28.

The above was taken from page 77 of Han Kang's *Greek Lessons*, which I happen to order and read since this past Sunday, after I saw it at the town library in Northampton. I'm sending this letter as I would like to help you with one homework problem, Problem 1 of Lesson 9, which asks us to prove that the angle function $\operatorname{Arg} z$ on $\mathbb{C}\backslash(-\infty, 0]$ is harmonic by doing concrete calculations, i.e., showing that all its four second order partial derivatives are continuous and it satisfies the Laplace equation $\Delta \operatorname{Arg} z = 0$. Surely, as $\operatorname{Arg} z$ is the imaginary part of the analytic function $\operatorname{Log} z$, we know it is automatically harmonic. The point is: we would not appreciate how wonderful this theorem is until we go through the pain of doing calculations. To suffer is to learn. So true it is!

But I don't fully agree. At least, we can make it more enjoyable, or less painful.

First of all, this problem is necessarily complicated, as our aim is to write $\operatorname{Arg} z$ using x and y so that we can do derivatives with respect to x and y, and using x and y means we are using coordinates, but one set of coordinates which works for the entire domain $\mathbb{C}\backslash(-\infty, 0]$ does not necessarily exist. For $\operatorname{Arg} z$, as it is an angle, inverse trigonometric functions would provide concrete expressions, but the output of such a function only covers an angular range less than π. So if we want to cover the angular range from $-\pi$ to π, we need at least three (two won't work) such functions. Now, I'll go through (b), as (c) is almost identical, and (a) is simpler.

In (b), $\operatorname{Arg} z = \operatorname{Arccos}(\frac{x}{\sqrt{x^2+y^2}})$ where $y > 0$ and x can be any number. Recall (or you can find using google, or ChatGPT, or prove using the 1D real Inverse Function Theorem) that $\operatorname{Arccos}'(t) = \frac{-1}{\sqrt{1-t^2}}$. Thus, if we let $v(x, y) = \operatorname{Arccos}(\frac{x}{\sqrt{x^2+y^2}})$ so that we don't need to write the latter many times, then by the chain rule and also the quotient rule applied to the inner function,

$$v_x = \frac{-1}{\sqrt{1 - (\frac{x}{\sqrt{x^2+y^2}})^2}} \frac{1 \cdot \sqrt{x^2 + y^2} - x\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}}{(\sqrt{x^2 + y^2})^2}.$$

Never, ever, take another derivative of this expression directly. Instead, simplify it as much as possible before anything else. You see, something your former teacher told you could be important: *simplify your work*. There is truth in "wisdom" and value in "tradition". Some movie director used to say so. Therefore, by multiplying the numerator and denominator of the second term by $\sqrt{x^2 + y^2}$, and also simplifying the first term, we have

$$v_x = \frac{-1}{\sqrt{\frac{y^2}{x^2+y^2}}} \frac{(x^2 + y^2) - x^2}{(\sqrt{x^2 + y^2})^3} = \frac{-\sqrt{x^2 + y^2}}{|y|} \frac{y^2}{(\sqrt{x^2 + y^2})^3} = \frac{-y}{x^2 + y^2},$$

where in the last step, $|y|$ was replaced by y, as $y > 0$. (So in (c), you should change $|y|$ to $-y$ as $y < 0$.) Compare this result $v_x = \frac{-y}{x^2+y^2}$ with the previous one. How simple it is!

Similarly,

$$v_y = \cdots = \frac{-1}{\sqrt{1 - \frac{x^2}{x^2+y^2}}} \frac{-xy}{(\sqrt{x^2+y^2})^3} = \cdots = \frac{x}{x^2+y^2}.$$

Then you can also check that

$$v_{xx} = \frac{2xy}{(x^2+y^2)^2},$$

$$v_{xy} = v_{yx} = \frac{-x^2+y^2}{(x^2+y^2)^2},$$

and

$$v_{yy} = \frac{-2xy}{(x^2+y^2)^2},$$

which are continuous. Furthermore,

$$\Delta v = v_{xx} + v_{yy} = 0.$$

Therefore, v is harmonic on $\operatorname{Im} z > 0$.

Finally, some of you said you don't like any of these, as they feel like engineering. First of all, I can assure you the math engineers do are 100 times more complicated than this, and they have to do it right, as it would be directly used in systems supporting humans and even saving lives. Every time you took the elevator in SMUD, imagine what exactly made it work and how much care could have possibly been put into it. If you keep thinking the same about objects surrounding us, you would marvel at the "miracles" nuanced math has produced. The devils are in the details. Secondly, retreating to math itself, we would eventually realize that most new math are discovered by doing computations. Computations for mathematicians are to experiments for scientists. Through tedious and persistent work are truth eventually unveiled. In concrete terms, most mathematics are eventually reduced to algebra (broadly defined) and calculus. Training yourself on how to do them and how to do them well will empower you when tackling what the future will bring forth.

Then, with great power comes great responsibility.

Ok. So much for today. I'm typing this letter with lights off as my sons have fallen asleep by my side. If I keep staring at the screen in darkness like I sometimes do with a cellphone, I would really become the blind Greek lecturer in the book. I'll schedule this letter to be sent to you at 8:20am on Friday morning, and I can't wait to see you in class. By you, I mean all of you, and all of u!

> P.S.: Page 78: "However, we cannot see the twinning of these verbs simply as a play on words. Since, for Socrates, learning literally meant suffering. Even granting that Socrates himself did not think this in so many words, the thought was at least formulated as such by the young Plato."

September 28, 2023
Yongheng Zhang (your Math 345 instructor who cares about you)

Lesson 10 Conformal mappings and analytic functions

1. In class, we used complex number multiplication to show that an analytic function with non-vanishing derivative is conformal. In this problem, we will use linear algebra instead to show the same result. Recall from Lesson 1 that a complex number can be written as a 2×2 matrix. This proof is very much in the same spirit. Here we begin: let $f(x, y) = u(x, y) + iv(x, y)$ be an analytic function. Let $z(t) = x(t) + iy(t)$ be the parametrization of a C^1 curve such that $z(0) = z_0 \in D$ and $f'(z_0) \neq 0$. Writing both $f(x, y)$ and $z(t)$ as column matrices, use the chain rule and Cauchy-Riemann equations to show that[37]
$$(f(z(t)))' = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}.$$

2. Consider the familiar polar coordinate transformation in calculus from the right-half plane $\{(r, \theta) \big| r > 0\}$ to the xy-plane. Prove that this map is not conformal. [38] [39]
$$x = r\cos\theta \text{ and } y = r\sin\theta$$

3. Consider the function $f(z) = z^2$. Find a domain D such that $f : D \to \mathbb{C}$ is a conformal mapping (one-to-one and conformal everywhere on the domain D). The answer of course is not unique.

4. Consider the Jourkowsky map $J(z) = z + \frac{1}{z}$ from Lesson 5 which we used to understand the trigonometric functions in Lesson 6. Prove that $J : D \to \mathbb{C}$ is a conformal mapping[40] if D is (1) the punctured open unit disk: $0 < |z| < 1$, (2) the outside of the unit disk: $|z| > 1$.

5. Find a conformal mapping from the vertical strip
$$-\pi/2 < \operatorname{Re} z < \pi/2$$
to the sector
$$0 < \operatorname{Arg} z < \pi/2$$
by composing familiar conformal mappings. [41]

   Drawing successive pictures helps us to think.

[37]This proves that f is conformal at z_0 because the 2×2 matrix is very special: it is an overall scaling composed with a pure rotation and thus preserves oriented angles. To see these: note that the two columns are perpendicular as their dot product is 0; the two columns have equal length; the determinant of this matrix > 0 and thus preserves orientation. Therefore, we can factor this matrix as an overall scalar multiplied to a matrix in $SO(2)$.

[38]There are many ways to prove it. For example, you can calculate its Jacobian, and show that it's not a scalar times a matrix in $SO(2)$. Or show the map is not analytic by disproving the Cauchy-Riemann equations.

[39]Note that this is similar but different from the exponential function $e^{r+i\theta} = e^r \cos\theta + ie^r \sin\theta$ because of the r and e^r difference. The former does map the horizontal and vertical rectangular grid to orthogonal circular and radial web, though in general, it doesn't preserve angle for other grids.

[40]In addition to showing the $J'(z) \neq 0$ condition, we need to show J is 1-1 on the given D. A way to do this is to assume $J(z_1) = J(z_2)$ and then prove $z_1 = z_2$.

[41]Some conformal mapping building blocks to consider are e^z, Log$z$, z^n, $z^{1/n}$ and multiplication by a complex number with modulus 1 for rotation.

Lesson 10 Summary

A C^1 function $f : D \to \mathbb{C}$ is conformal if it preserves angles, i.e., if the angle from the tangent vector v_1 of the curve γ_1 to the tangent vector v_2 of the curve γ_2 at the common point z_0 is α, then the angle from the tangent vector u_1 of the curve $f(\gamma_1)$ to the tangent vector u_2 of the curve $f(\gamma_2)$ at the common point $f(z_0)$ is also α.

Apparently, the geometric property of conformality bears no direct relations with analyticity, which is what we have been studying. Surprisingly, under a mild assumption, they mean the same thing. Indeed, we have the following theorem:

Under the further assumption that $\det(Jf(z_0)) \neq 0$, f *is analytic if and only if* f *is conformal.*

The forward direction was proved in class. Problem 1 you just did proved it again, using another method. The textbook has a third proof, which is the fastest, using a process like that in the proof of the chain rule.

The backward direction is proved here now: Suppose $f = u + iv$ is conformal. Let a curve be parametrized by $z(t) = x(t) + iy(t) = z_0 + te^{i\theta}$. So the curve is simply a straight line through z_0 when $t = 0$ that can point along any angular direction θ. Furthermore, $z'(t) = e^{i\theta}$ and thus $x'(t) = \cos\theta$ and $y'(t) = \sin\theta$. Then $(f(z(t)))' = (u_x x'(t) + u_y y'(t)) + i(v_x x'(t) + v_y y'(t)) = (u_x \cos\theta + u_y \sin\theta) + i(v_x \cos\theta + v_y \sin\theta)$. As f is conformal, $(f(z(t)))'/z'(t)$ should not depend on θ. This imposes a very strong condition on $(f(z(t)))'/z'(t)$ from which we will get the Cauchy-Riemann equations. To see the details, let's do concrete calculuations:

$$(f(z(t)))'/z'(t) = (f(z(t)))'/e^{i\theta} = (f(z(t)))'e^{-i\theta}$$
$$= (u_x \cos^2\theta + u_y \cos\theta\sin\theta + v_x \cos\theta\sin\theta + v_y \sin^2\theta) + i(-u_x \sin\theta\cos\theta - u_y \sin^2\theta + v_x \cos^2\theta + v_y \sin\theta\cos\theta)$$
$$= (u_x + iv_x)\cos^2\theta + (u_y + v_x + i(v_y - u_x))\sin\theta\cos\theta + (v_y - iu_y)\sin^2\theta.$$

Now, writing $\cos^2\theta = 1 - \sin^2\theta$, this expressions is

$$= u_x + iv_x + ((u_y + v_x + i(v_y - u_x))\cos\theta + (v_y - u_x - i(u_y + v_x))\sin\theta)\sin\theta.$$

Note that, the two vectors $u_y + v_x + i(v_y - u_x)$ and $v_y - u_x - i(u_y + v_x)$ are of equal length and orthogonal to each other, thus $(u_y + v_x + i(v_y - u_x))\cos\theta + (v_y - u_x - i(u_y + v_x))\sin\theta$ is a vector obtained by turning $u_y + v_x + i(v_y - u_x)$ with respect to θ. As $(f(z(t)))'/z'(t)$ doesn't depend on θ, this vector should be zero. Thus, we get $u_y = -v_x$ and $v_y = u_x$, the Cauchy-Riemann equations. As f is also C^1, by the Cauchy-Riemann analyticity criterion[42], f is analytic.

Later, after we learn the differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\bar{z}}$, we will see that the above calculation can be written in a very concise way.

A conformal mapping is a 1-1 map which is conformal at all points in its domain. Typically, we can get a conformal mapping by restricting an analytic function to a subset of its domain where $f'(z) \neq 0$, or by composing several such mappings.

Conformal mappings are used often in generating graphical arts. Conformal mappings also make sense in higher-dimensions. Perhaps some of you can tell us about it in your project.

[42]You see, the Cauchy-Riemann analyticity criterion, the powerful translator bridging multivariable calculus and complex analysis is used again! Last time, it was when we proved the 1D complex Inverse Function Theorem from the 2D real Inverse Function Theorem. This time, we use it to go from the conformal property expressed using multivariable calculus to analyticity in complex analysis.

Lesson 11 Important conformal mappings: the fractional linear transformations (FLI)

1. Given a fractional linear transformation $f(z) = \frac{az+b}{cz+d}$, its matrix M is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Recall $\det(M)$ is defined to be nonzero.

   (a) Compute the inverse matrix M^{-1}.
   (b) Find a formula for $f^{-1}(z)$ by solving for w from $z = f(w)$.
   (c) Have you noticed any relationship between your answers in (a) and (b)?

2. Fractional linear transformations map circles in $\mathbb{C}^*$ to circles in $\mathbb{C}^*$. This can be proved by decomposing such a map into dilation $z \mapsto az$, $a \neq 0$, translation $z \mapsto z + b$, and possibly an inversion $z \mapsto \frac{1}{z}$ and show each of the three simpler maps do. In this problem, we check this for inversion through four examples. Show that $f(z) = \frac{1}{z}$ maps [43]:

   (a) the circle $|z - 1| = 2$ to another circle,
   (b) the circle $|z - 1| = 1$ to a line,
   (c) the line $y = x$ to another line,
   (d) the line $y = x + 1$ to a circle.

3. Read the proof of the theorem that there is a unique fractional linear transformation mapping three different points $z_1, z_2, z_3 \in \mathbb{C}^*$ to each of the three different points $w_1, w_2, w_3 \in \mathbb{C}^*$, respectively, and then fill in the following two little details.

   (a) Construct a FLT f mapping z_0, ∞, z_2 to $0, 1, \infty$, where $z_0, z_2 \in \mathbb{C}$.
   (b) Construct a FLT f mapping z_0, z_1, ∞ to $0, 1, \infty$, where $z_0, z_1 \in \mathbb{C}$.

4. Consider the function $g(z) = \frac{z-a}{1-\bar{a}z}$ where $|a| < 1$. In Lesson 1, you proved that it maps the unit circle $|z| = 1$ to itself.

   (a) Prove that g is a FLT.
   (b) Prove that if $a = Re^{i\theta}$, then $g(e^{i\theta}) = e^{i\theta}$. Let $a \neq 0$, then show that g maps the straight line through 0 and a to the straight line through $-a$ and 0.[44]

5. A FLT f maps $1 + i$ to 0, 2 to ∞ and 0 to $i - 1$.

   (a) Without finding $f(z)$ explicitly, sketch the images of the circle $|z - 1| = 1$ and the line segment $[0, 2]$ under f by using the conformal property and the fact that f maps "circles" to "circles".
   (b) Find an explicit expression of $f(z)$.

[43] See the textbook for how to do this. Alternatively, letting $u + iv = w = \frac{1}{z}$, we have $x + iy = z = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$. We can then plug $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$ into each of the four equations to get an equality for u and v.

[44] Note that g maps 0, a and $e^{i\theta}$ to $-a$, 0 and $e^{i\theta}$. The former three points are on a line. The latter three points are also on a line. FTL maps line to line.

Lesson 11 Summary

After learning conformal mappings in Lesson 10, today we studied an important class of such functions, called Möbius transformations $f(z) = \frac{az+b}{cz+d}$. In general, when z takes values in other fields, or algebraic structures, such maps are called fractional linear transformations (FLT) as our textbook did, or linear fractional transformations (LFT) as the Internet webpages do.

That $ad - bc \neq 0$ is part of the definition of FLT suggests us to associate a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to the map. Indeed, we can build a group homomorphism from $GL(2, \mathbb{C})$, the group of all 2×2 invertible matrices with complex entries to the group of FLTs. This homomorphism has a nontrivial kernel, consisting of all invertible matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Thus, we can identify FLTs with elements in $GL(2, \mathbb{C})$ up to a nonzero multiplicative constant. This surely has to be the case, as if we multiply a, b, c, d by the same nonzero constant, the FLT doesn't change.

FLT maps circles on $\mathbb{C}^*$ to circle on $\mathbb{C}^*$, i.e., it maps circles and lines in $\mathbb{C}$ to circles and lines in $\mathbb{C}$. Furthermore, if you tell to which three different locations w_1, w_2, w_3 to send three different points z_1, z_2, z_3 on $\mathbb{C}^*$, a FLT exists to do that, and only this FLT can do it. Combined with the good conformal property of FLT, we can tell a lot about what FLT do to lines and circles on the plane, even without knowing the formula. Of course, a formula can be easily found if we wanted to.

Möbius transformation is a hot topic in the very early days of YouTube. This short and old one[45], for example, `https://www.youtube.com/watch?v=JX3VmDgiFnY`, has more than 2 millions views, which was quite rare for a video in mathematics. Indeed, it's a Top Favorite in the categories of Education and earlier in Film and Animation. You see, Möbius transformations have such simple and physical explanations if we pull everything back to the Riemann sphere.

[45] Be aware that there are four basic transformations in the video: dilation, rotation, translation, and inversion, but the first two together is just multiplication by a complex number, which was called dilation by us.

Lesson 12 Real line integrals and harmonic conjugates

1. Consider the line integral $\int_C ydx + xdy$, where C is a smooth curve from $(0,0)$ to $(2,0)$.

   (a) Evaluate this line integral by plugging in a parametrization if C is the straight line segment from $(0,0)$ to $(2,0)$.

   (b) Evaluate this line integral by plugging in a parametrization if C first goes from $(0,0)$ to $(1,1)$ along a straight line and then goes from $(1,1)$ to $(2,0)$ along another straight line.

   (c) Show that if C_1 and C_2 are any two smooth curves, both from $(0,0)$ to $(2,0)$, then $\int_{C_1} ydx + xdy = \int_{C_2} ydx + xdy$. (So your answers to (a) and (b) should be the same.)

2. It's been some time since you proved Green's Theorem in Math 211. Here is an opportunity to review part of it. [46] All functions below are C^1.

   (a) If $D = \{(x,y) \in \mathbb{R}^2 \big| a \le x \le b, g_1(x) \le y \le g_2(x)\}$, show that $\int_{\partial D} Pdx = -\iint_D P_y dA$.

   (b) If $D = \{(x,y) \in \mathbb{R}^2 \big| a \le y \le b, g_1(y) \le x \le g_2(y)\}$, show that $\int_{\partial D} Qdy = \iint_D Q_x dA$.

3. The domains D of the following harmonic functions $u(x,y)$ are all simply-connected. Practice writing their harmonic conjugates as line integrals using the formula we proved in class today. [47]

   (a) $u(x,y) = x^2 - y^2$
   (b) $u(x,y) = \frac{x}{x^2+y^2}$ where $D = \mathbb{C} \backslash (-\infty, 0]$
   (c) $u(x,y) = e^x \cos y$

4. Prove that the harmonic conjugate $v(x,y) = \int_C \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ of $u(x,y) = \ln |z|$, where C is the curve from 1 to $|z|$ on the x-axis, and then from $|z|$ to z along a circular arc, is the θ of z, by plugging in parametrization of C.

5. Prove that if $h(z) = u_1(x,y) + iu_2(x,y)$ is harmonic (which means both real functions u_1 and u_2 are harmonic) on their simply-connected domain D [48], then there are two analytic functions $f(z)$ and $g(z)$ on D such that
$$h(z) = f(z) + \overline{g(z)}.$$

[46] (a) also holds for general regions D we typically draw on paper, which can be cut into simpler regions of the type described. Apply the result to each simple region and then add up the answer. Similarly, (b) also holds for general regions D. The full Green's Theorem is obtained by adding up the equalities in (a) and (b). From this process, we see that Green's Theorem also holds for regions with holes.

[47] The formula is $v(x,y) = \int_C -u_y dx + u_x dy$, where C is any curve in D from a chosen point $z_0 \in D$ to the variable point $(x,y) \in D$.

[48] So we know there are $v_1(x,y)$ and $v_2(x,y)$ such that $a(z) := u_1 + iv_1$ and $b(z) := u_2 + iv_2$ are analytic on D. Note that $u_1 = 1/2(a(z) + \overline{a(z)})$ and $u_2 = 1/2(b(z) + \overline{b(z)})$.

Lesson 12 Summary

We knew what a harmonic conjugate is back in Lesson 9. However, we left one question open: does it always exist? Today, we give an answer which depends on the shape of the domain: yes, it does, if the domain is simply-connected.

Simple-connectedness is a topological notion. The rigorous definition has things to do with shrinkability of loops in the domain. Intuitively, simple-connectivity means the region doesn't have any holes or punctures. Alternatively, it means you can deform your curve freely however you want within the domain (but if your region is not simply-connected, and if you draw your loop around a hole or a puncture, then you can not deform it past this hole or puncture). Why do we require this condition? Because we want to show the harmonic conjugate defined as a line integral does not depend on the curve by moving any curve to any other through successive little regions over which we can apply Green's Theorem.

Here is the BIG theorem we proved today:

Suppose (1) domain D is simply-connected, (2) P and Q are C^1 over D, (3) $Q_x - P_y = 0$ (people say $Pdx + Qdy$ is a *closed differential form*), then if we fix a point $A \in D$ and let C be any curve from A to the variable point (x, y) in D, then (1) $f(x, y) := \int_C Pdx + Qdy$ is well-defined (does not depend on the curve), and (2) $f_x = P$ and $f_y = Q$ (thus $Pdx + Qdy = f_x dx + f_y dy$, which people call an *exact differential form*).

Its corollary gives us the a way to construct harmonic conjugate:

If u is harmonic on simply-connected domain D, then its harmonic conjugate exists on D, and it can be given by

$$v(x, y) := \int_C -u_y dx + u_x dy,$$

where C is any path in D connecting a chosen point A and the variable point (x, y).

The above holds because we can let $P = -u_y$ and $Q = u_x$ in $v(x, y) := \int_C Pdx + Qdy$, where P and Q are C^1 as u is C^2 and $Q_x - P_y = u_{xx} + u_{yy} = 0$. Thus, $v_x = P = -u_y$ and $v_y = Q = u_x$, which are continuous. Therefore, by the Cauchy-Riemann analyticity criterion, $v(x, y)$ as this well-defined line integral is a harmonic conjugate of $u(x, y)$ as $u(x, y) + iv(x, y)$ is analytic.

However, keep in that simple-connectedness is not a necessary condition. In fact, in Problem 3(b), we can change D from the simply-connected $\mathbb{C}\backslash(-\infty, 0]$ to the region $\mathbb{C}\backslash\{0\}$ with a puncture. The harmonic conjugate exists in this bigger domain. In fact, $v = \frac{-y}{x^2+y^2}$, which is defined on $\mathbb{C}\backslash\{0\}$. You may worry not being able to deform any two curves in $\mathbb{C}\backslash\{0\}$. Indeed, we cannot: a curve wrapping around 0 once cannot be deformed to a curve wrapping around 0 twice. However, once a curve goes around 0 a complete turn, the line integral $\int_C -u_y dx + u_x dy$ is 0, as you can verify (which was a homework problem in Math 211, or you can find it in the textbook by Stewart.) So even though we can't deform any two curves, after cancelling extra turning of one curve compared to the other, the two line integrals are still the same, demonstrating the well-definedness of $v(x, y)$ as a line integral.

Lesson 13 The mean value property of harmonic functions

1. Find the value of the following integrals. Note that both integrals are average values of harmonic functions over circles, so there is no point in actually evaluating these integrals, if it is even doable. Using the Mean Value Property is so much faster.

   (a) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\frac{d\theta}{2\pi}$
   (b) $\int_0^{2\pi} e^{\cos\theta}\sin(1+\sin\theta)\frac{d\theta}{2\pi}$

2. Now it's your turn. Write down a really complicated integral with respect to θ for which those who doesn't know the mean value property of harmonic functions would be intimidated and have no idea how to evaluate, but you know its answer.

3. Now, let's use what we learned in dimension 2 to do dimensionality reduction strike on the 1-dimensional case in this and the next problem. Consider a continuous function $u : [a, b] \to \mathbb{R}$. Let $(x_0 - \rho, x_0 + \rho) \subseteq (a, b)$. For any $0 < r < \rho$, we define the average value $A(r)$ of u over the boundary of the interval $[x_0 - r, x_0 + r]$ as follows
$$A(r) = \frac{u(x_0 + r) + u(x_0 - r)}{2}.$$
Prove that
$$\lim_{r\to 0} A(r) = u(x_0).$$

4. Consider the same function above, but we also assume that u is harmonic, which simply means u is C^2 and $u_{xx} = 0$ over an open interval containing $[a, b]$. Prove that $\frac{d}{dr}A(r) = 0$ so as to conclude that $A(r) = u(x_0)$, just like what we did in class[49]. Thus, the mean value property also holds for 1-dimensional harmonic functions.

5. In this last problem, we will see that the mean value property also holds in dimension 3. Let $u : \mathbb{R}^3 \to \mathbb{R}$ be harmonic, i.e., u is C^2 and $u_{xx} + u_{yy} + u_{zz} = 0$. Without loss of generality, for any $r > 0$, consider the sphere centered at the origin with radius r: ∂B_r. Define the average value of u over ∂B_r as
$$A(r) = \frac{1}{4\pi r^2}\iint_{\partial B_r} u\,dS,$$
where $4\pi r^2$ is the total area of ∂B_r and this integral is the surface integral of scalar function u over the sphere. Using spherical coordinates $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$ and $z = r\cos\phi$, where r is fixed, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$, we see that $A(r) = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(x, y, z)r^2\sin\phi\,d\phi\,d\theta = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(x, y, z)\sin\phi\,d\phi\,d\theta$. It can be seen again as we did in lower dimensions that $\lim_{r\to 0} A(r) = u(0, 0, 0)$. Show that $\frac{d}{dr}A(r) = 0$ so as to conclude that $A(r) = u(0, 0, 0)$ for any $r > 0$.[50]

[49] $\frac{d}{dr}A(r) = \frac{1}{2}(u_x(x_0 + r) - u_x(x_0 - r))$ using interchanging finite sum with derivative and the chain rule. Then by the Fundamental Theorem of calculus (the 1D version of Green's Theorem), this expression is $\frac{1}{2}\int_{x_0-r}^{x_0+r} u_{xx}dx$.

[50] By interchanging derivative with integral, we have $\frac{d}{dr}A(r) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u_r(x, y, z)\sin\phi\,d\phi\,d\theta$, which by the chain rule is $\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(u_x\sin\phi\cos\theta + u_y\sin\phi\sin\theta + u_z\cos\phi)\sin\phi\,d\phi\,d\theta$. Adding r^2 in front of $\sin\phi\,d\phi\,d\theta$ to get back dS and noticing that $\langle\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$ is the unit normal vector $\vec{n}$ to the surface, we see that this integral is $\frac{1}{4\pi r^2}\iint_{\partial B_r}\langle u_x, u_y, u_z\rangle\cdot d\vec{S}$ (recall $d\vec{S} = \vec{n}dS$), which by the divergence theorem, is $\frac{1}{4\pi r^2}\iint_{\partial B_r} u_{xx} + u_{yy} + u_{zz}dV$.

Lesson 13 Summary

The average value of a continuous function $u : D \to \mathbb{R}$ over a circle contained in any open disk in its domain is the line integral of the function over this circle with respect to arc length divided by the length of this circle. After parametrizing the circle using angle θ, it can also be written as an integral with respect to θ, divided by the total angle turn 2π. As u is continuous, the average value function is continuous with respect to the radius of the circle, and it converges to the function's value at the center of the circle.

Surprisingly, for a harmonic function, this average value is independent of the radius of the circle. Thus, by the last sentence in the previous paragraph, this common average value is the function's value at the center of the circle. This is called the mean value property. In class, we proved that harmonic functions have this mean value property by showing that the derivative of the average value with respect to radius is 0. Things we used are the chain rule, the rewriting of the integral with respect to θ as a line integral of a vector field, and then lastly, the Green's Theorem at which point the Laplace equation was finally used to show the value 0.

When we proved the mean value property for harmonic functions in class, we interchanged derivative with integral. Now let's justify it: prove that if $f(r,\theta)$ is continuous and $f_r(r,\theta)$ exists and is continuous so that the following Riemann integrals make sense, then

$$\frac{d}{dr}\int_a^b f(r,\theta)d\theta = \int_a^b f_r(r,\theta)d\theta.$$

Proof. We will show that $\lim\limits_{s\to r}\dfrac{\int_a^b f(s,\theta)d\theta - \int_a^b f(r,\theta)d\theta}{s-r} = \displaystyle\int_a^b f_r(r,\theta)d\theta$ and we focus on a closed rectangle $[r-l, r+l]\times[a,b]$ contained in the domain of $f_r(r,\theta)$. Let $\epsilon > 0$. As f_r is continuous, and $[r-l,r+l]\times[a,b]$ is compact, f_r is uniformly continuous, which means, responding to $\frac{\epsilon}{b-a}$, there is $\delta > 0$ such that whenever $|r'-r| < \delta$ and for any $\theta \in [a,b]$, we have $|f_r(r',\theta) - f_r(r,\theta)| < \frac{\epsilon}{b-a}$. Now, for all $s \in [r-l, r+l]$ and $s \neq r$, $\dfrac{\int_a^b f(s,\theta)d\theta - \int_a^b f(r,\theta)d\theta}{s-r} = \displaystyle\int_a^b \frac{f(s,\theta)-f(r,\theta)}{s-r}d\theta$, which is $\int_a^b f(s(\theta),\theta)d\theta$ for some $s(\theta)$ between s and r by the Mean Value Theorem. Thus, if $0 < |s-r| < \delta$, then $|s(\theta) - r| < \delta$, whence $\left|\dfrac{\int_a^b f(s,\theta)d\theta - \int_a^b f(r,\theta)d\theta}{s-r} - \displaystyle\int_a^b f_r(r,\theta)d\theta\right| = \left|\displaystyle\int_a^b f_r(s(\theta),\theta) - f_r(r,\theta)d\theta\right| \leq \displaystyle\int_a^b |f_r(s(\theta),\theta) - f_r(r,\theta)|d\theta < \int_a^b \frac{\epsilon}{b-a}d\theta = \epsilon.$ ☐

It's even more surprising that the converse is also true: if a continuous function on a domain satisfies the mean value property, without further assumption, it turns out that the function is harmonic. In the next lesson, we will sketch a proof after we learn a consequence of the mean value property: the maximum principle.

Lesson 14 The maximum principle from the mean value property

1. Let's prove the following corollary to the real version of the maximum principle again: Let $u : D \cup \partial D \to \mathbb{R}$ be continuous on $D \cup \partial D$ and harmonic on the bounded D. Then if $a \leq u \leq b$ on ∂D, then $a \leq u \leq b$ on D as well.

2. Suppose $u : D \cup \partial D \to \mathbb{R}$ is continuous and satisfies the mean value property on D. On the other hand, someone found a continuous function $g : D \cup \partial D \to \mathbb{R}$ which is harmonic on the bounded D and agrees with u on ∂D. Show that u is actually g on D. [51]. This is the method used to show the the mean value property implies harmonicity. See the lesson summary for more details.

3. Consider the function $f(z) = z^{14} + 1$ over the closed disk $|z| \leq 1$. Find the maximal value of $|f|$. [52] At what points is this maximum value attained?

4. Let $f(z)$ be analytic on its domain D. Assume that f is never zero on D. [53]

   (a) Prove that if $|f(z)|$ attains its minimum on D, then $f(z)$ is a constant on D.

   (b) Prove that if D is bounded and f extends to be a continuous function on $D \cup \partial D$, then $|f(z)|$ attains its minimum on ∂D.

5. The celebrated *Fundamental Theorem of Algebra* states: Let $p(z)$ be a polynomial of degree $n \geq 1$, i.e., $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, for some $a_n, \cdots, a_0 \in \mathbb{C}$ where $n \geq 1$ and $a_n \neq 0$. Then there must be $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$. Now we can prove it as follows:

   For the sake of contradiction, suppose $p(z) \neq 0$ everywhere on $\mathbb{C}$. Thus, $\frac{1}{p(z)}$ is analytic on $\mathbb{C}$ and note that $|\frac{1}{p(0)}| > 0$. On the other hand, $\lim_{z\to\infty} \frac{1}{p(z)} = \lim_{z\to\infty} \frac{\frac{1}{z^n}}{a_n + a_{n-1}\frac{1}{z} + \cdots + \frac{a_0}{z^n}} = \frac{0}{a_n + 0 + \cdots + 0} = 0$.

   (a) Show that[54], there is $R > 0$, such that for any z on the circle $|z| = R$, $|\frac{1}{p(z)}| < |\frac{1}{p(0)}|$.

   (b) Show that the above result contradicts the maximum principle for $\frac{1}{p(z)}$ over the disk $|z| \leq R$.

   Remark: We will prove the Fundamental Theorem of Algebra again (and again), whenever we learn any relevant new technique.

[51] Consider $u - g$, which satisfies the mean value property, and thus the maximum principle. Note that, over ∂D, $0 \leq u - g = \leq 0$. We can use Problem 1 now.

[52] Note that as f is analytic, it's harmonic. By the maximum principle, the maximum modulus occurs on the boundary of the domain. Also draw a picture to help us think.

[53] Consider the function $\frac{1}{f(z)}$ instead, and note that the max of $|\frac{1}{f(z)}|$ is attained when the min of $|f(z)|$ happens at the same point. So we can use the maximum principles.

[54] As in calculus, $\lim_{z\to\infty} f(z) = L$ means for any $\epsilon > 0$, there is $R > 0$ such that whenever $|z| \geq R$, $|f(z) - L| < \epsilon$. In this problem, we can let $\epsilon = |\frac{1}{p(0)}|$, $L = 0$ and only consider $|z| = R$ and never mind $|z| > R$.

Lesson 14 Summary

Today's class is a direct follow-up to the last one. Last time, we learned that harmonic functions satisfy the mean value property (MVP). Today, we show that the MVP implies the maximum principle (MP).

We proved four MPs in class. Two for real, two for complex; two for open domain, two for bounded domain with its boundary.

The open versions say, if a harmonic function (or its modulus) attains its maximum somewhere in D, then this function is constant. The complex version of this follows from the real version by doing a rotation through angle $e^{i\theta}$ so that at the maximum value of $|f(z_0)| = M$, $e^{-i\theta}f(z_0) = |f(z_0)| = M$ and then showing the real part of $e^{-i\theta}f(z)$ is M while the imaginary part is 0. The proof of the real version uses the connectedness of D. Write $D = \{z \in D \big| f(z) = M\} \cup \{z \in D \big| f(z) < M\}$. The latter set is automatically open as it can be written as the inverse image of the open set $(-\infty, M)$ by the continuous function f. The former set can also be shown to be open by using the mean value theorem and continuity of f to show it contains open disks (on which the value of f is M). As the former set is nonempty, it has to be the whole of D, and thus f is constant.

The version for bounded domain with its boundary follows from the above open version, using the fact that continuous functions attains maxima over its closed and bounded (and thus compact) domain $D \cup \partial D$. If the maximum occurs on ∂D, then we are done. If the maximum occurs in D, then the function (or its modulus) is a constant over D and thus over $D \cup \partial D$, so the maximum occurs on the boundary. In both cases, maximum occurs on ∂D.

The maximum principle is very useful in that we can say something about the function over the entire domain if we only know some information over its boundary. As one application, we sketch a proof of the following, which is the converse of what we proved in Lesson 13:

If $u : \mathbb{D} \to \mathbb{R}$ is continuous and satisfies the mean value property, then u is harmonic on D.

Proof. As D is open, at any point, there is a little open disk centered at this point which is contained in D. We can choose a smaller disk centered at the same point but with a smaller radius such that the closed disk is contained in D. Thus, D is the union of the interior of these closed disks. Over each such closed disk $\mathbb{D}$, it's a fact that one can construct a continuous function g (to be described below) which is harmonic inside $\mathbb{D}$ such that $g = u$ over $\partial\mathbb{D}$. As g is harmonic, it satisfies the mean value property. Thus, $u - g$ satisfies the mean value property, and thus the maximum principle. By Problem 2, as $u - g = 0$ on $\partial\mathbb{D}$, $u - g = 0$ on D. Therefore, u is the harmonic g inside the open disk D. So u is harmonic on D.

Such g can be realized as an integral with the Poisson kernel (See Problem 5 of Lesson 1). Over $|z| < 1$, let $P_r(\theta) = 1 + \frac{z}{1-z} + \frac{\overline{z}}{1-\overline{z}} = \frac{1-|z|^2}{|1-z|^2} = \frac{1-r^2}{1+r^2-2r\cos\theta}$, where $z = re^{i\theta}$. Then define

$$g(z) = \int_{-\pi}^{\pi} u(e^{i\varphi})P_r(\theta - \varphi)\frac{d\varphi}{2\pi}.$$

It can be shown (and it is in a later chapter of the textbook, which we will not have the energy to cover) that g is harmonic on $|z| < 1$ and extends to be the continuous u on $|z| = 1$. □

In dimension 1, such a $g(x)$ can be taken to be a linear function $g(x) = ax + b$. Indeed, any $1D$ harmonic function is a linear function.

Lesson 15 Complex Line Integrals

1. Let γ be the counterclockwise oriented piecewise smooth boundary curve of the compact region D. Prove[55] that
$$\int_\gamma \overline{z}dz = 2i\text{Area}(D).$$

2. For the same γ above, prove that
$$\int_\gamma zdz = 0.$$

3. Use the parameterization $C_R : z(t) = Re^{it}$, $t : 0 \to 2\pi$ to show that[56] [57]
$$\int_{C_R} z^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

4. Let $P(z) = a_m z^m + \cdots + a_1 z + a_0$ be a polynomial of degree m and $Q(z) = b_n z^n + \cdots + b_1 z + b_0$ a polynomial of degree n. Let Γ_R be the upper half of the circle centered at 0 with radius R. Use the ML-estimate to how that[58]
$$\left| \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz \right| < \pi^2 \frac{|a_m|}{|b_n|} R^{m-n+1}$$
if R is big enough. Feel free to use the fact[59] that there is some $R > 0$, such that whenever $|z| \geq R$, we have $|P(z)| < \frac{3}{2}|a_m||z|^m$ and $|Q(z)| > \frac{1}{2}|b_n||z|^n$.

5. For the same Γ_R above, it is quite straightforward to see that $\int_{\Gamma_R} |e^{iz}||dz| < \pi R$. However, this is not very useful. Instead, the following sharper version can be used to show $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$, something which was perhaps left unanswered in single variable calculus, but we will get to later in the course. Prove this inequality. [60]
$$\int_{\Gamma_R} |e^{iz}||dz| < \pi$$

[55] Write the complex line integral $\int_\gamma (u+iv)dz$ in the form $\int_\gamma udx - vdy + i\int_\gamma vdx + udy$ and then use Green's Theorem to both the real part and imaginary part.

[56] After studying Lesson 16, you will get this result within a split second.

[57] The whole theory of complex residue is partially built on this result.

[58] Note that if $n \geq m+2$, i.e., the degree of the denominator is at least 2 higher than that of the numerator, we see that $\lim_{R\to\infty} \int_{\Gamma_R} \frac{P(z)}{Q(z)} dz = 0$. This is quite a useful result, as we shall see in future.

[59] Here is a proof of these two facts: As $\lim_{z\to\infty} \frac{P(z)}{z^m} = \lim_{z\to\infty} a_m + a_{m-1}/z + \cdots + a_0/z^m = a_m$, letting $\epsilon = |a_m|/2$, we know that there is $R_1 > 0$ such that if $|z| \geq R_1$, then $|\frac{P(z)}{z^m} - a_m| < \frac{|a_m|}{2}$. Whence, $|\frac{P(z)}{z^m}| = |\frac{P(z)}{z^m} - a_m + a_m| \leq |\frac{P(z)}{z^m} - a_m| + |a_m| < \frac{3}{2}|a_m|$ and thus $|P(z)| < \frac{3}{2}|a_m||z|^m$. Similarly, there is $R_2 > 0$, such that if $|z| \geq R_2$, then $|\frac{Q(z)}{z^n} - b_n| < \frac{|b_n|}{2}$, whence $|\frac{Q(z)}{z^n}| = |b_n + \frac{Q(z)}{z^n} - b_n| \geq |b_n| - |\frac{Q(z)}{z^n} - b_n| > \frac{|b_n|}{2}$, and thus $|Q(z)| > \frac{1}{2}|b_n||z|^n$. To make both equalities hold, we can let $R = \max\{R_1, R_2\}$.

[60] Its whole proof is on Page 216 of the textbook. Feel free to lift it from there. This inequality even has a name: Jordan's Lemma.

Lesson 15 Summary

Complex line integral $\int_\gamma f(z)dz$ is also defined as the limit of a Riemann sum

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(z_i^*)\Delta z,$$

for any continuous $f(z)$ over piecewise smooth curve γ on the complex plane. Using $\Delta z = \Delta x + i\Delta y$, and $f = u + iv$, we can write the the complex line integral as two real line integrals in the form of a complex number:

$$\int_\gamma udx - vdy + i\int_\gamma vdx + udy,$$

which is useful for proving theorems, as we saw in Problems 1 and 2, and will see in Cauchy's Theorem starting from Lesson 17. On the other hand, if we parameterize the curve γ by $z(t) = x(t) + iy(t)$, $t : a \to b$, then the line integral becomes

$$\int_a^b f(z(t))z'(t)dt,$$

which is useful in calculations. In future applications, it is quite useful to have an estimate of how big (in terms of modulus/length) the complex line integral is. This is where the ML-estimate comes handy:

We proved the ML-estimate formula for complex line integrals:

$$|\int_\gamma f(z)dz| \le \int_\gamma |f(z)||dz| \le ML,$$

if $|f(z)| \le M$ for all z on γ and $L = \int_\gamma |dz|$ is the length of γ. This is very useful to show certain integrals are zero when we study contour integrals in Chapter VII. The first inequality is called the triangle inequality for complex line integrals. The following was used to show its validity, where we can let $w_n = \sum_{i=1}^n f(z_i^*)\Delta z$ and $b_n = \sum_{i=1}^n |f(z_i^*)||\Delta z|$.

Proposition. Let b_n, $n \ge 1$ be a sequence of real numbers converging to b_0. [61] Let w_n, $n \ge 1$ be a sequence of complex numbers converging to w_0. [62] Then if $|w_n| \le b_n$ for all n, then $|w_0| \le b_0$. [63]

Proof. Let $\epsilon > 0$, however small it is. We will show that $|w_0| < \epsilon + b_0$ so as to demonstrate $|w_0| \le b_0$. (If $|w_0| > b_0$, then $\epsilon := \frac{|w_0| - b_0}{2}$ satisfies $|w_0| > \epsilon + b_0$ for this particular ϵ, contradicting $|w_0| < \epsilon + b_0$ for all ϵ.)

To see why $|w_0| < \epsilon + b_0$ is true, we use the two convergence assumptions:

(1) As $w_n \to w_0$, there is $N_1 \in \mathbb{N}$ responding to $\frac{\epsilon}{2}$ such that $|w_n - w_0| < \frac{\epsilon}{2}$ if $n \ge N_1$.

(2) As $b_n \to b_0$, there is $N_2 \in \mathbb{N}$ responding to $\frac{\epsilon}{2}$ such that $|b_n - b_0| < \frac{\epsilon}{2}$ if $n \ge N_2$.

Therefore, letting $N = \max\{N_1, N_2\}$, if $n \ge N$, then $n \ge N_1$ and $n \ge N_2$. Thus, $|w_0| = |w_0 - w_n + w_n| \le |w_0 - w_n| + |w_n| \le |w_0 - w_n| + b_n \le |w_n - w_0| + |b_n| = |w_n - w_0| + |b_n - b_0 + b_0| \le |w_n - w_0| + |b_n - b_0| + |b_0| < \frac{\epsilon}{2} + \frac{\epsilon}{2} + |b_0| = \epsilon + |b_0|$. $\square$

[61] This means for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \ge N$, $|b_n - b_0| < \epsilon$.

[62] The definition of convergence of a sequence of complex numbers is the same as that for the real: simply replace absolute value by length. So this means for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \ge N$, $|w_n - w_0| < \epsilon$.

[63] Geometrically, this means if each w_n is inside the disk centered at the origin with radius b_n, then the limit of w_n is inside the disk with the limit radius.

Lesson 16 Fundamental Theorem of Calculus for analytic functions

1. In Multivariable Calculus, the 2D Fundamental Theorem of Line Integral says that for any piecewise smooth curve γ from point A to point B and real continuous conservative vector field $\nabla f = \langle f_x, f_y \rangle$, we have $\int_\gamma \nabla f \cdot d\vec{r} = f(B) - f(A)$, which can also be written
$$\int_\gamma f_x dx + f_y dy = f(B) - f(A).$$
In class, we used the following vector version of this theorem for complex functions $F(z)$. However straightforward it it, prove[64] this vector version from the real version.
$$\int_\gamma F_x dx + F_y dy = F(B) - F(A).$$

2. Calculate[65] the following complex line integrals, where γ is any piecewise smooth curve on the right-half plane from $-i$ to i.

   (a) $\displaystyle\int_\gamma z e^{z^2} dz$

   (b) $\displaystyle\int_\gamma \cos z dz$

   (c) $\displaystyle\int_\gamma \mathrm{Log} z dz$

3. In (c) of Problem 2, what would the answer be if the branch cut of Log$z$ is along the positive x axis and γ still goes from $-i$ to i, but on the left-half plane.

4. Consider the following complex line integral, where γ is the counterclockwise oriented arc from $-i\sqrt{3}$ on $i\sqrt{3}$ on the circle $|z-1| = 2$, and the square root is the principal branch.
$$\int_\gamma \frac{1}{\sqrt{z-1}} dz$$

   (a) Evaluate this line integral by plugging in a parametrization of γ. [66]

   (b) Evaluate this line integral by using the fundamental theorem of calculus.

5. Let $f = u + iv$ and $F = U + iV$ be analytic functions on their common simply-connected domain. Assume $F'(z) = f(z)$. Write both U and V as real line integrals in terms of u and v. [67]

[64] Write $F(z) = U + iV$, and then apply the FTLI to the real and imaginary parts of $\int_\gamma F_x dx + F_y dy$.

[65] Note that the antiderivatives of the integrands are $\frac{1}{2}e^{z^2}$, $\sin z$ and $z\mathrm{Log}z - z$, respectively.

[66] Let $z = 1 + 2e^{i\theta}$, where $\theta : -\frac{2\pi}{3} \to \frac{2\pi}{3}$.

[67] The answer is in the proof of the second part of FTC we did in class.

Lesson 16 Summary

The Fundamental Theorem of Calculus (FTC) also holds for complex (analytic) functions. Same as the real version, there are also two parts. The first part says, if F is an antiderivative of the continuous $f = u + iv$, i.e., $F'(z) = f(z)$, then $\int_\gamma f(z)dz = F(B) - F(A)$, where γ is any piecewise smooth curve in the domain of $f(z)$ from point A to point B. The proof is quite straightforward, relying on the Fundamental Theorem of Line Integrals (FTLI). Indeed, writing $dz = dx + idy$, and using $f(z) = F'(z) = F_x = \frac{1}{i}F_y$ (taking the complex derivative along the x and iy directions, respectively), $\int_\gamma f(z)dz = \int_\gamma f dx + \int_\gamma if dy = \int_\gamma F_x dx + F_y dy$, which by the complex version of the FTLI, that you proved Problem 1, is $F(B) - F(A)$.

This part of the fundamental theorem is very useful, as the answer only depends on the end points B and A of the curve γ, which we substitute into the antiderivative $F(z)$. However, a natural question is if $F(z)$ always exists, and if it does, how to find it. Even though $F(z)$ doesn't always exist (for example, consider $1/z$ over $\mathbb{C}\backslash\{0\}$, whose antiderivative is a branch of $\log z$, that is only defined on a slit plane.), if we only consider domain that is simply-connected, then $F = U + iV$ always exists. This is the second part of the FTC. In class, fixing z_0 in the domain, we defined $U := \int_\gamma udx - vdy$ for any curve γ from z_0 to the variable point z, which is well-defined by Green's Theorem: if we deform γ to another curve γ', then the line integral over the region bounded by $\gamma - \gamma'$ is of $(-v)_x - u_y$, which by a Cauchy-Riemann equation, is 0, and thus this integral is independent of the curve. Furthermore, we have $U_x = u$ and $U_y = -v$. Then it follows that $U_{xx} + U_{yy} = u_x - v_y = 0$, by the other equation of the Cauchy-Riemann equations. So U is harmonic. By what we learned in Lesson 12, as the domain is simply-connected, its harmonic conjugate V exists and is given by $V = \int_\gamma -U_y dx + U_x dy$, which is $\int_\gamma vdx + udy$, and we have $V_x = v$, $V_y = u$. Therefore, $F := U + iV$ is analytic, and we see that $F'(z) = F_x = U_x + iV_x = u + iv = f$. This F is unique up to a constant. This can be proved using a theorem we saw in Lesson 7: Suppose G is another antiderivative, then $(F - G)'(z) = F'(z) - G'(z) = f - f = 0$. As the domain is connected, we know $F - G$ has to be a constant C. Thus, $G = F + C$.

Writing $U + iV = \int_\gamma udx - vdy + i\int_\gamma vdx + udy$, we see that $F(z) = \int_\gamma (u + iv)d(x + iy) = \int_\gamma f(z)dz$. So while both the real and imaginary parts of F can be expressed as real line integrals, F itself can be written as a complex line integral, whose form is easier to remember. In fact, we can start from this complex line integral, and then recover the formula for U and V by multiplying all the terms out. This is what we will do in Lesson 17, to prove Cauchy's Theorem, the cornerstone of complex analysis, out of which almost all important results grow.

Lesson 17 Cauchy's Theorem

1. Use Cauchy's Theorem, but not the Cauchy's Integral Formula to prove that
$$\frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}dw = 1$$
as long as γ is any smooth simple closed curve winding around the point z counterclockwise once. [68]

2. From Cauchy's Integral Formula
$$f(z_0) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-z_0}dz,$$
prove[69] the Mean Value Property for Analytic Functions[70]: Let the disk $|z-z_0| \le r$ be contained in the domain of the analytic function $f(z)$. Then
$$f(z_0) = \int_0^{2\pi} f(z_0 + re^{i\theta})\frac{d\theta}{2\pi}.$$

3. Find the value of the following complex line integrals, using either the Cauchy's Integral Formula (carefully identify $f(z)$ and $z - z_0$ of the integrand.) or Cauchy's Theorem (some of the integrals is just zero).

   (a) $\displaystyle\int_{|z|=2}\frac{z^{2023}}{z-i}dz$ (b) $\displaystyle\int_{|z|=\frac{1}{2}}\frac{z^{2023}}{z-i}dz$ (c) $\displaystyle\int_{|z|=1}\frac{\cos z}{z}dz$

4. Now let's prove the Fundamental Theorem of Algebra a second time: Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \ge 1$. Then there is $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

   (a) As $p(z)$ is of degree at least 1, we know $p(z) = zq(z) + a_0$ where $q(z) = a_n z^{n-1} + \cdots + a_1$ is also a polynomial. Verify that $\frac{1}{z} = \frac{q(z)}{p(z)} + \frac{a_0}{zp(z)}$.

   (b) Prove that if $p(z)$ is never 0 on $\mathbb{C}$, and thus $\frac{q(z)}{p(z)}$, as a rational function with non-vanishing denominator, is analytic on $\mathbb{C}$, then by integrating both sides of the above equality over a circle of radius R and then letting $R \to \infty$, we have $2\pi i = 0$. [71]

5. Let D be a bounded domain with piecewise smooth ∂D. Suppose $f(z)$ is analytic on a region containing $D \cup \partial D$. Prove[72] that
$$\max_{z\in\partial D}|\overline{z} - f(z)| \ge \frac{2\text{Area}(D)}{\text{Length}(\partial D)}.$$

[68] Apply Cauchy's Theorem to the region bounded by γ and a small circle centered at z to show that the above integral is the same as that over this small circle, which can be evaluated by substituting a parametrization, just like Problem 3 of Lesson 15.

[69] Let your curve be the boundary circle of this disk.

[70] Taking the real or the imaginary part, we get the Mean Value Property for real harmonic functions.

[71] Footnote 4 of Lesson 15 is useful.

[72] Consider applying the ML-estimate to the integral $\int_{\partial D}\overline{z} - f(z)dz$. Problem 1 of Lesson 15 is also useful.

Lesson 17 Summary

Cauchy's Theorem in complex analysis is the cornerstone of complex analysis. However, its proof is surprisingly simple: just use Green's Theorem twice, once to the real part of $\int_{\partial D} f(z)dz$ and once to its imaginary part. Nonetheless, we need to check the assumption carefully: $f(z)$ is analytic in D and extends smoothly to the boundary ∂D, i.e., f is C^1 over $D \cup \partial D$ in addition to being analytic inside the open D. The C^1 part is to make sure after taking the partial derivatives in the application of Green's Theorem, those partial derivatives are continuous on $D \cup \partial D$, and thus as we get close to the edge ∂D in the double integral, the integral is guaranteed to make sense. Note that Cauchy's Theorem is actually quite subtle, due to such continuity issues and some unexpected wildness of ∂D. Our textbook handled it well: our analytic function $f(z)$ are defined to have continuous $f'(z)$ and ∂D is assumed to be piecewise smooth. Later, we will see that by Goursat's Theorem, continuity actually follows from existence of derivative, but without the continuity assumption, the treatment in the proof of Cauchy's Theorem would be quite discouraging to newcomers. Cauchy's Theorem also holds for D with rather strange ∂D, but such exotic situation is better left to your future endeavor.

Anyway, the proof of Cauchy's Theorem is here: $\int_{\partial D} f(z)dz = \int_{\partial D}(u+iv)(dx+idy) = \int_{\partial D} udx - vdy + i\int_{\partial D} vdx + udy = \iint_D -v_x - u_y dA + i\iint_D u_x - v_y dA = 0$, by the two Cauchy-Riemann equations.

Cauchy's Theorem has many important consequences. The most direct one is the Cauchy's Integral Formula: $f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-z_0}dz$, which is proved by excising a small disk B centered at z_0 contained in D and then applying Cauchy's Theorem to $\frac{f(z)}{z-z_0}$ over the region with a new hole. Such things were done merely to show that the integral $\int_{\partial D} \frac{f(z)}{z-z_0}dz$ can be changed to $\int_{\partial B} \frac{f(z)}{z-z_0}dz$ over the boundary of B, which is something we are able to handle concretely as ∂B can be described precisely using a parametrization. Here, the textbook used the Mean Value Property to show that this integral is $2\pi i f(z_0)$ and thus concluded the proof. During class, we used a characterization of analytic function: $f(z) = f(z_0) + f'(z_0)(z-z_0) + \epsilon(z)(z-z_0)$ instead, just to present an alternative treatment, and also to preview a technique to be used in the proof of Goursat's Theorem in Lesson 20.

Conversely, you proved in your homework that the Mean Value Property can actually be proved using the Cauchy's Integral Formula.

Lesson 18 Cauchy's Integral Formulas

1. We relied on the following fact when we proved Cauchy's Integral Formulas for $f^{(m)}(z_0)$ in class today:
If $\lim_{z\to z_0} g(w,z) = g(w,z_0)$ uniformly for $w \in \partial D$ (i.e., for any $\epsilon > 0$, there is $\delta > 0$ such that if $0 < |z-z_0| < \delta$, then $|g(w,z) - g(w,z_0)| < \epsilon$ for all $w \in \partial D$), then $\lim_{z\to z_0} \int_{\partial D} g(w,z)dw = \int_{\partial D} g(w,z_0)dw$.
Prove it. [73]

2. As a review of the techniques used in the proof of the Cauchy's Integral Formulas, let's prove this special version when $m = 1$. Show that if f is analytic on D and C^1 on $D \cup \partial D$, then
$$f'(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z_0)^2} dw,$$
where $f'(z_0)$ is defined as $\lim_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ and $f(z)$ and $f(z_0)$ satisfy the 0th order Cauchy's Integral Formula $f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw$.

3. In this problem, let's practice using the Cauchy's Integral Formulas by calculating the following integrals.

(a) $\displaystyle\int_{|z|=2} \frac{e^z}{(z-1)^{2023}} dz$ (b) $\displaystyle\int_{|z|=2} \frac{e^z}{(z-1)(z-2023)} dz$ (c) $\displaystyle\int_{|z|=2} \frac{e^z}{(z-1)^2(z-2023)} dz$

4. Use Cauchy's Integral Formula to calculate the following integral. Note that, as the circle encloses two zeros of the denominator, we can draw two little circles inside the big one and then use Cauchy's Theorem to show this integral is the same as the sum of the two over these new circles.
$$\int_{|z-3|=2} \frac{\mathrm{Log} z}{(z-2)^2(z-4)^2} dz.$$

5. Calculate the following integral. Of course, its answer depends on where a and b are. So we consider the following cases. [74]
$$\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{(z-a)(z-b)} dz$$
   (a) if both a and b are outside of $|z| = 1$.
   (b) if a is outside of $|z| = 1$ and b is inside of $|z| = 1$.
   (c) if a is inside of $|z| = 1$ and b is outside of $|z| = 1$.
   (d) if both a and b are inside of $|z| = 1$.

[73] We are proving a limit statement. So let $\epsilon > 0$. And consider $\frac{\epsilon}{L}$ where L is the length of ∂D. Then produce a δ responding to $\frac{\epsilon}{L}$, such that if $0 < |z - z_0| < \delta$, then $|g(w,z) - g(w,z_0)| < \frac{\epsilon}{L}$. After that, use the ML-estimate to show $|\int_{\partial D} g(w,z)dw - \int_{\partial D} g(w,z_0)dw| < \epsilon$. In the summary, we will show that why the convergence is uniform in the proof of Cauchy's Integral Formulas.

[74] It is out of the question to consider the cases when a or b is on $|z| = 1$, as in this case, the integral diverges.

Lesson 18 Summary

The Cauchy's Integral Formula we proved in Lesson 17 expresses $f(z)$ as an integral of a related function over a curve (of one or more components) circling around z:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)}{w-z}dw$$

OK. We just opened the Pandora's box. It turns out that once we can express the 0th order derivative of $f(z)$ as an integral, we can do the same for all orders. We just knocked down the first block of a domino chain. Then blocks fall down forever:

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D}\frac{f(w)}{(w-z)^{m+1}}dw$$

You see. Derivative is integral. It only holds in the fairy tale of complex analysis.

These integral formulas show that once f is analytic, its derivatives of all orders exist. (Wow.) As $f^{(m)}(z)$ is continuous if $f^{(m+1)}(z)$ exists, $f^{(m-1)}(z)$ is analytic. So the derivatives of f of all orders are analytic. It also follows that if u is real harmonic, then u is C^∞ by finding a local harmonic conjugate v of u and then pass the C^∞ property of $f = u + iv$ to u. In Lesson 19, we will see a class of applications, which boils down to an estimation of $|f^{(m)}(z)|$ using the ML-estimate. (So we are estimating the modulus of derivative actually using the good estimation property of integrals. This is what makes complex analysis so powerful and special.)

When we proved the above formula, we used Problem 1 and the fact that the function $g(w, z)$ in the integral does converge uniformly for $w \in \partial D$. The latter indeed is the case because $g(w, z)$ is jointly continuous in w and z and w is over ∂D, which is both closed and bounded, and thus compact. Using this continuity and compactness condition, let's prove that $\lim_{z\to z_0} g(w, z) = g(w, z_0)$ uniformly given that $\lim_{z\to z_0} g(w, z) = g(w, z_0)$ pointwise.

Proof. Let $\epsilon > 0$. As $g(w, z)$ is jointly continuous in w and z, for any $w \in \partial D$, there are $\delta_{1w} > 0$ and $\delta_{2w} > 0$ such that if $|w' - w| < \delta_{1w}$ and $|z - z_0| < \delta_{2w}$, we have $|g(w', z) - g(w, z_0)| < \epsilon/2$. In particular, $|g(w', z) - g(w', z_0)| = |g(w', z) - g(w, z_0) + g(w, z_0) - g(w', z_0)| \le |g(w', z) - g(w, z_0)| + |g(w, z_0) - g(w', z_0)| < \epsilon/2 + \epsilon/2 = \epsilon$ if $|w' - w| < \delta_{1w}$ and $|z - z_0| < \delta_{2w}$. Consider the open disks $B_w = \{w' \in \mathbb{C} \big| |w' - w| < \delta_{1w}\}$, whose union as w exhausts points in ∂D covers ∂D, which means $\cup_{w\in\partial D} B_w \supseteq \partial D$. As ∂D is compact, finitely many of these disks cover ∂D. Say these disks are $B_{w_1}, B_{w_2}, \cdots, B_{w_N}$. Then let $\delta := \min\{\delta_{2w_1}, \delta_{2w_2}, \cdots, \delta_{2w_N}\}$, which is > 0. Finally, if $0 < |z - z_0| < \delta$, then for any $w \in \partial D$, w is contained in some B_{w_i}, and we have $|z - z_0| < \delta_{2w_i}$. Therefore, $|g(w, z) - g(w, z_0)| < \epsilon$. $\square$

In the induction proof of the formula for $f^{(m)}(z)$,

$$g(w, z) = \frac{1}{z - z_0}\left(\frac{f(w)}{(w-z)^m} - \frac{f(w)}{(w-z_0)^m}\right) = \frac{f(w)\sum_{i=0}^{m-1}(w-z)^i(w-z_0)^{m-1-i}}{(w-z)^m(w-z_0)^m},$$

which is jointly continuous in w and z, as this is the product of a rational function and a continuous function.

Lesson 19 Cauchy's Estimate and Liouville's Theorem

1. In class, we proved the Fundamental Theorem of Algebra (FTA) a third time, using Liouville's Theorem. In this problem, as a review, state and prove FTA using Liouville again.

In the next three problems. We will apply Liouville's Theorem to (entire) functions which are not necessarily bounded, but turn out to be constant anyway.

2. Prove that if u is real harmonic on $\mathbb{C}$ and u is bounded above, then u is a constant. [75]

3. Prove that if $f : \mathbb{C} \to \mathbb{C}$ is analytic and $\mathrm{Im} f(z) > 0$ for all $z \in \mathbb{C}$, then f is a constant. [76]

4. Prove that if $f : \mathbb{C} \to \mathbb{C}$ is analytic and there is a disk $|z - z_0| < r$ such that f maps $\mathbb{C}$ to the outside of this disk, then f is a constant. [77]

Now we come back to (a variant of) Cauchy's Estimate itself.

5. The reason that a bounded entire function $f(z)$ is constant is $f'(z) = 0$ for all $z \in \mathbb{C}$, because it leads to $f(z) = C$, a polynomial of degree 0, as we did in Lesson 7. Similar result holds when higher order derivatives vanish.

   (a) Let f be an entire function (the domain of f is the entire $\mathbb{C}$ and f is analytic on it). Prove that if $f^{(n+1)}(z) = 0$ for all $z \in \mathbb{C}$, then $f(z)$ is a polynomial of degree less than or equal to n. [78]

   (b) Suppose f is an entire function and there is $M > 0$ such that $|f(z)| \leq M|z^n|$ on $\mathbb{C}$, show that for any $z \in \mathbb{C}$ and any $R > 0$ satisfying $R \geq |z|$ (so that if $|w - z| = R$, then $|w| = |w - z + z| \leq |w - z| + |z| = R + |z| \leq 2R$),

$$|f^{(n+1)}(z)| = \left| \frac{(n+1)!}{2\pi i} \int_{|w-z|=R} \frac{f(w)}{(w-z)^{n+2}} dw \right| \leq \frac{(n+1)! M (2R)^n}{R^{n+1}}.$$

   (c) For the same f in (b), conclude that f is a polynomial with degree at most n.

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[75] Find a harmonic conjugate v of u and let $g = u + iv$. Then consider $f(z) = e^{g(z)}$.

[76] Compose f with the Cayley map $g(z) = \frac{z-i}{z+i}$. Recall that g is a conformal mapping from the open upper half plane to open the unit disk.

[77] What function maps things outside this disk into a disk centered at the origin?

[78] Start by something like this: as $(f^{(n)})'(z) = f^{(n+1)}(z) = 0$ on $\mathbb{C}$, we know $f^{(n)}(z) = a_n$ for some $a_n \in \mathbb{C}$. Then, as $(f^{(n-1)}(z) - a_n z)' = f^{(n)}(z) - a_n = 0$, we know $f^{(n-1)}(z) - a_n z = a_{n-1}$, i.e., $f^{(n-1)}(z) = a_n z + a_{n-1}$ for some $a_{n-1} \in \mathbb{C}$. Then as $((f^{(n-2)})'(z) - \frac{a_n}{2} z^2 - a_{n-1} z)' = f^{(n-1)}(z) - a_n z - a_{n-1} = 0$, we know there is $a_{n-2} \in \mathbb{C}$ such that $(f^{(n-2)})'(z) - \frac{a_n}{2} z^2 - a_{n-1} z = a_{n-2}$... Eventually, $f(z) = \frac{a_n}{n!} z^n + \frac{a_{n-1}}{(n-1)!} z^{n-1} + \cdots + a_1 z + a_0$ for some $a_0, a_1, \cdots, a_n \in \mathbb{C}$.

Lesson 19 Summary

This lesson is a further application of the Cauchy's Integral Formulas for the any-order derivatives of analytic functions as complex line integrals, where we simply let the region be a disk:

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{|w-z|=R} \frac{f(w)}{(w-z)^{m+1}} dw$$

The Cauchy's Estimate for $|f^{(m)}(z)|$ is obtained by evaluating such formulas over a circle centered at z by plugging in the usual parameterization $w(t) = z_0 + Re^{it}$ where $t : 0 \to 2\pi$, and then using the triangle inequality after taking the modulus of this integral. One pair of R is cancelled from the numerator and denominator, then we get

$$|f^{(m)}(z)| \leq \frac{m!M}{R^m},$$

where M is an upper bound for $|f(w)|$ over $|w - z| = R$.

With the Cauchy's Estimate in hand, when $m = 1$, we can use it to bounded analytic functions over the entire complex plane to show such functions, called bounded entire functions, must be constant. This is because as f has a global bound M, then over a circle $|w - z| = R$ with arbitrarily large radius R, we have $|f'(z)| \leq \frac{M}{R}$. Taking the limit by sending R to ∞, we se that $|f'(z)| \leq 0$. Therefore, $f'(z) = 0$ on $\mathbb{C}$. Thus, by what we learned in Lesson 7, f is a constant.

In practice, many entire functions are not bounded, but missing some points on the complex plane, e.g., it maps $\mathbb{C}$ to the upper half plane, or outside of some disk. In this case, we can compose our entire function with another function to get a new entire function which is bounded. By Liouville's Theorem, this new function is bounded, and then so is our original function. Problems 2, 3, and 4 are of this type.

The method for proving Cauchy's Estimate can be used to give other variant versions of Cauchy's Estimate, which are more useful in certain applications. Problem 5 (b) is such an example.

Lesson 20 Morera's Theorem and Goursat's Theorem

1. State Morera's Theorem (and read through its proof at least twice).

2. In this problem, we will prove that Morera's Theorem is quite useful. Let f be analytic on $|z| < 1$ and satisfies $f(0) = 0$. Consider the function

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } 0 < |z| < 1 \\ f'(0) & \text{if } z = 0. \end{cases}$$

Prove[79] that g is analytic on $|z| < 1$. Feel free to use the following version[80] of Cauchy's Theorem: If f is analytic on rectangle R and is continuous (we didn't say C^1) on $R \cup \partial R$, then $\int_{\partial R} f(z)dz = 0$.

3. State Goursat's Theorem (and read through its proof at least three times).

4. The following Cantor's Intersection Theorem is essential when proving Goursat's Theorem: if R_n, $n \in \mathbb{N}$ is a sequence of closed rectangles such that $R_{n+1} \subseteq R_n$ for all n, and diameter$(R_n) \to 0$ as $n \to \infty$, then there is a single point $z_0 \in \mathbb{C}$ such that

$$\bigcap_{n=1}^{\infty} R_n = \{z_0\}.$$

The "closedness" condition is necessary here. Show that if $R_n = (0, \frac{2}{n}) \times (0, \frac{1}{n})$, i.e., the Cartesian product of the open intervals $(0, \frac{2}{n})$ and $(0, \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} R_n$ is empty, i.e., there is no point that is in all these rectangles.

5. Now that we finally see the continuity condition in the definition of analytic functions can be removed. It took us quite a while. To recall this long journey, write a very short essay explaining why it is the case that if f' exists everywhere on D, then f' is continuous on D. You can start by reviewing the definition of analytic functions, then try to go through the theorems we learned, before arriving at Goursat's Theorem.

[79] On $0 < |z| < 1$, $f(z)/z$ is analytic and thus continuous. At 0, you can show that $g(z)$ is continuous by demonstrating $\lim_{z\to 0} g(z) = g(0)$, i.e., $\lim_{z\to 0} f(z)/z = f'(0)$, which is just the definition of derivative. So $g(z)$ is continuous on the entire $|z| < 1$. To show $g(z)$ is analytic, let R be any rectangle with sides parallel to the coordinate axes in $|z| < 1$, and consider $\int_{\partial R} g(z)dz$. If 0 is not in the interior of the rectangle, as $g(z) = f(z)/z$ which is analytic on the closed R, this integral must be zero. If 0 is on ∂R, $g(z)$ is still $f(z)/z$ in the interior of R and thus analytic there and g is continuous over the closed R. Thus, by the above version of Cauchy's Theorem, the integral is again 0. Lastly, if 0 is in the interior of R, then we can cut R into four rectangles sharing 0 as one of their vertices. Then you finish up the proof.

[80] Here is the idea why this version is true: in the interior of this rectangle R, draw another parallel rectangle R' whose sides are very close to those of R. As f is analytic over $R' \cup \partial R'$, $\int_{\partial R'} f(z)dz = 0$. On the other hand, the original integral $\int_{\partial R} f(z)dz$ is very close to his integral. Actually, $\int_{\partial R} f(z)dz$ can be made arbitrarily close to $\int_{\partial R'} f(z)dz = 0$ by making R' sufficiently close to R and also using the continuity of f over the edge of R. This can be made into a rigorous proof.

Lesson 20 Summary

Morera's Theorem was used to prove Goursat's Theorem. This is the relationship between the two.

Morera's Theorem can be viewed as a converse to Cauchy's Theorem. A corollary to Cauchy's Theorem says, if f is analytic on D, then for any closed rectangular region $R \subseteq D$, we have $\int_{\partial R} f(z)dz = 0$. This holds because f is analytic on the closed R and of course also C^1 on $R \cup \partial R$. Morera's Theorem states that if f is only assumed to be continuous on D, then if $\int_{\partial R} f(z)dz = 0$ for any closed rectangle contained in D, then f is analytic on D. Note that it suffices to consider only those R with sides parallel to the coordinate axes. Some textbooks use triangles, which work equally fine.

Note that analyticity is a local criteron: existence of derivative and its continuity at any point only depends on the function's behavior at a neighborhood of this point. The idea of the proof is that at each open disk centered at z_0 and contained in D, construct another function $F(z)$ on this disk which satisfies $F'(z) = f(z)$. Since $f(z)$ is assumed to be continuous, then by definition, $F(z)$ is analytic on this disk. From Lesson 18, we learned that an analytic function's derivative is also analytic. So the proof is completed. To construct this $F(z)$, we do $F(z) := \int_\gamma f(\zeta)d\zeta$, where ζ is an L-shaped path first from z_0 along the horizontal line to the shadow of z on this line, and then from this point vertically to z. (Surely, we can also first go vertically, and then horizontally. Just pick one.) The condition $\int_{\partial R} f(z)dz = 0$ is then used to show $F(z + \Delta z) - F(z) = \int_\gamma f(\zeta)d\zeta$ where γ is again an L-shaped path, this time from z to $z + \Delta z$. The continuity of f is then used to show $F'(z) = f(z)$.

Goursat's Theorem says: if f' exists everywhere on D (without assuming f' is continuous on D), then f is analytic. To prove it, note that as f' exists, f is continuous. Thus, if we can show $\int_{\partial R} f(z)dz = 0$ for each rectangle with sides parallel to the axes, then it follows that f is analytic. This is where the ingenious part comes in: first, we can use a horizontal line and a vertical line to cut R into four equals parts, and consider the four line integrals over each of the four parts. Then one of them has the largest modulus. Thus, $|\int_{\partial R} f(z)dz|$ is less than or equal to 4 times this modulus. Then we cut this quarter rectangle further into four parts, and consider the resultant four line integrals, among which one has the largest modulus. Thus, $|\int_{\partial R} f(z)dz|$ is less than or equal to 4^2 times this modulus. We continue this n times, and have that $|\int_{\partial R} f(z)dz| \leq 4^n |\int_{\partial R_n} f(z)dz|$, where R_n is the very small rectangle in the nth iteration. This nested sequence of decreasing closed rectangles contains one and only one point z_0 in common. (Google Cantor's Intersection Theorem.) After changing $|\int_{\partial R_n} f(z)dz|$ to $|\int_{\partial R_n} f(z) - f(z_0) - f'(z_0)(z - z_0)dz|$ due to the vanishing of the integral of the analytic linear function $f(z_0) + f'(z_0)(z - z_0)$, and applying the ML-estimate, we finally see that $|\int_{\partial R} f(z)dz| \leq 4^n \max_{z \in \partial R_n} |\epsilon(z)(z - z_0)| \frac{L}{2^n} \leq 4^n \max_{z \in \partial R_n} |\epsilon(z)| \frac{L}{2^n} \frac{L}{2^n} = \max_{z \in \partial R_n} |\epsilon(z)| L^2$, where L is the length of ∂R, $|\epsilon(z)(z - z_0)| = |f(z) - f(z_0) - f'(z_0)(z - z_0)|$ and $\epsilon(z)$ goes to 0 as z goes to z_0. Thus, $|\int_{\partial R} f(z)dz| \leq 0$ and so $\int_{\partial R} f(z)dz = 0$.

To recap, two ingredients were essential here. (1) The subdivision of the rectangle to create a nested sequence of decreasing rectangles over which the integral satisfies an inequality. (2) The use of the reformulation of the existence of derivative: $f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon(z)(z - z_0)$ where $\lim_{z \to z_0} \epsilon(z) = 0$. This holds because $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$ can be rewritten as $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = 0$ where we let $\epsilon(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$.

It's interesting to note that, even though this theorem is named after Édouard Goursat, it was actually discovered by Alfred Pringsheim, who criticized Goursat's original treatment of this proof.

Lesson 21 An elegant notation and Pompeiu's formula

1. Let $g(z) = f(z, \overline{z})$ be a C^1 function of $z = x + iy$ where $f : \mathbb{C}^2 \to \mathbb{C}$ is analytic in each of its variables. Show that
$$\frac{\partial}{\partial z} g(z) = D_1 f(z, \overline{z}) \text{ and } \frac{\partial}{\partial \overline{z}} g(z) = D_2 f(z, \overline{z}),$$
where D_1 means taking partial derivative with respect to its first input and D_2 means taking partial derivative with respect to its second input. Thus, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ indeed behave like partial derivatives, as their notations suggest. [81]

2. Let $f(z) = z^{2024} + \overline{z}^{2023} + 1$. Using the above interpretation, calculate $\frac{\partial f(z)}{\partial \overline{z}}$. Is $f(z)$ analytic? Explain your answer.

3. Show that $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) f = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} f = 4 \frac{\partial}{\partial \overline{z}} \frac{\partial}{\partial z} f$ if f is C^2. Use this formula to verify that $\ln |z|$ is harmonic. Note that $\ln |z| = \frac{1}{2} \ln z\overline{z}$.

4. Suppose f is C^1 on D and $z_0 = x_0 + iy_0 \in D$. Verify that for $z = x + iy$,
$$f_x(z_0)(x - x_0) + f_y(z_0)(y - y_0) = \frac{\partial f}{\partial z}(z_0)(z - z_0) + \frac{\partial f}{\partial \overline{z}}(z_0)\overline{(z - z_0)}.$$
Now you can read the proof that if f is C^1 on D, $f'(z)$ is non-vanishing, and f is conformal, then f is analytic on D.[82] It's on Page 126 of our textbook.

5. Let $z_0 \in \mathbb{C}$. Show that
$$\overline{z_0} = \frac{1}{2\pi i} \int_{\partial D} \frac{\overline{z}}{z - z_0} dz - \frac{1}{\pi} \iint_D \frac{1}{z - z_0} dA,$$
where D is a disk containing z_0 in its interior.

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[81] I'll prove the first. You do the second. $\frac{\partial}{\partial z} g(z) = \frac{1}{2} \frac{\partial}{\partial x} g(z) + \frac{1}{2i} \frac{\partial}{\partial y} g(z)$, which by the usual chain rule is $\frac{1}{2}(D_1 f(z, \overline{z}) \frac{\partial z}{\partial x} + D_2 f(z, \overline{z}) \frac{\partial \overline{z}}{\partial x}) + \frac{1}{2i}(D_1 f(z, \overline{z}) \frac{\partial z}{\partial y} + D_2 f(z, \overline{z}) \frac{\partial \overline{z}}{\partial y}) = \frac{1}{2}(D_1 f(z, \overline{z}) + D_2 f(z, \overline{z})) + \frac{1}{2i}(D_1 f(z, \overline{z}) i + D_2 f(z, \overline{z})(-i)) = D_1 f(z, \overline{z})$.

[82] This was proved in the summary of Lesson 10, without using $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \overline{z}}$, but the presentation in the book is more elegant.

Lesson 21 Summary

Cauchy's Theorem and Cauchy's Integral Formula are for analytic functions. What if the functions are not analytic, but just C^1? It turned out that there are the extended Cauchy's Theorem

$$\int_{\partial D} f(z)dz = 2i \iint_D \frac{\partial f}{\partial \overline{z}} dA$$

and extended Cauchy's Integral Formula (after Dimitrie Pompeiu):

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz - \frac{1}{\pi} \iint_D \frac{\partial f}{\partial \overline{z}} \frac{1}{z - z_0} dA.$$

If f is analytic, then $\frac{\partial f}{\partial \overline{z}} = 0$. Then these two formulas recover Cauchy's Theorem and Cauchy's Integral Formula.

But wait a second? What is $\frac{\partial}{\partial \overline{z}}$? Let's go back to the complex derivative. We know if $f(z)$ is analytic, then $f'(z) = \frac{\partial}{\partial x} f = \frac{1}{i} \frac{\partial}{\partial y} f$, because it doesn't matter along which direction we take the derivative (so specifically, the derivatives along x and iy are both $f'(z)$).

Then taking the average, we have $' = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y})$.

Now we also want to take the derivative of functions which are only C^1. Then we define

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}),$$

which is just the above complex derivative if f is analytic. Symmetrically, we also define

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}).$$

If f is analytic, as the two parts of $\frac{\partial f}{\partial \overline{z}}$ cancel each other, we know $\frac{\partial f}{\partial \overline{z}} = 0$. This can also be proved by writing $f = u + iv$ and using the Cauchy-Riemann equations. Indeed, for C^1 functions, by Cauchy's Analyticity Criterion, f is analytic if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

It is so simple!

These two new first order differential operators satisfy all properties you expect them to satisfy. In particular, as $f(z) = f(x+iy) = f(\frac{z+\overline{z}}{2} + i\frac{z-\overline{z}}{2i})$, we see f can be viewed as a function of the two variables z and $\overline{z}$. As you discovered in Problem 1, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ behave just like partial derivatives, which is really convenient.

Lastly, back to Pompeiu's formula. It can be written in the following more symmetric way, which perhaps is the way you will find it in research papers.

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz + \frac{1}{2\pi i} \iint_D \frac{\partial f}{\partial \overline{z}} \frac{1}{z - z_0} dz \wedge d\overline{z}.$$

Here is what $dz \wedge d\overline{z}$ means without digging much into it. First of all, $dz = dx + idy$. $d\overline{z} = dx - idy$. Then $\wedge$ is a thing which is distributive and kills anything of the form $dw \wedge dw$, i.e., $dw \wedge dw = 0$ from which it also follows that $-dy \wedge dx = dx \wedge dy$. (Proof: $0 = (dx + dy) \wedge (dx + dy) = dx \wedge dx + dx \wedge dy + dy \wedge dx + dy \wedge dy = dx \wedge dy + dy \wedge dx$.) Thus, $dz \wedge d\overline{z} = (dx + idy) \wedge (dx - idy) = dx \wedge dx + idy \wedge dx - idx \wedge dy - i^2 dy \wedge dy = -2i dx \wedge dy$. You can take it as a definition that $dx \wedge dy = dxdy = dA$, even though in fact, there is a coordinate transformation process here, which is hidden as the Jacobian determinant of the transformation from x and y to x and y is 1. Now we see how $\frac{1}{2\pi i}$ would become $-\frac{1}{\pi}$ in the original formula.

Lesson 22 Sequence and series of functions

1. This problem is about the geometric series $\sum_{k=0}^{\infty} z^k$.

   (a) Show that the sequence of terms $z^n \to 0$ pointwise on $|z| < 1$. [83]

   (b) Even though we did this in class, prove again that $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ on $|z| < 1$, and that the convergence is uniform[84] on $|z| \le R$ for any $0 < R < 1$.

   (c) Does the above series converge at $z = i$, which is on the boundary of $|z| \le 1$? Justify your claim.

2. This problem is about the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$, where p is real.

   (a) Use the comparison test[85] to show that the harmonic series (when $p = 1$) diverges.

   (b) Use the comparison test[86] to show that the p-series diverges when $p < 1$.

   (c) Use the comparison test[87] to show that the p-series converges when $p > 1$.

3. Show that if $f_n \to f$ uniformly and each f_n is continuous, then f is also continuous. [88]

4. Show that if $f_n \to f$ uniformly on the piecewise smooth $\gamma = \partial D$ where D is bounded, then $\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$. [89]

5. Consider the famous series[90] $\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$, where the variable is denoted by $s = \sigma + i\tau$ for historical reasons, and we take the principal branch of Log in the definition of n^s.

   (a) Recall how n^s was defined and show that $\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma}$.

   (b) Show that for each $\sigma_0 > 1$, $\zeta(s)$ converges uniformly on the half plane $\mathrm{Re}s = \sigma > \sigma_0$.

   (c) Show that for each $\sigma_0 > 1$, $\zeta(s)$ is analytic on the half plane $\mathrm{Re}s = \sigma > \sigma_0$.

   (d) Conclude that $\zeta(s)$ is analytic on the half plane $\mathrm{Re}s = \sigma > 1$.

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[83] You may find a useful hint on page 34 of the textbook. Also note that $\frac{\ln a}{\ln b} = \log_b a$, where $a, b > 0$ and $b \neq 1$.
[84] Use the Weierstrass M-test.
[85] Note that $\frac{1}{k} = \int_k^{k+1} \frac{1}{k} dx \ge \int_k^{k+1} \frac{1}{x} dx$ and $\sum_{k=1}^{\infty} \int_k^{k+1} \frac{1}{x} dx = \int_1^{\infty} \frac{1}{x} dx = \ln x\big|_1^{\infty}$.
[86] Compare with the harmonic series.
[87] Note that in this case, for $k \ge 2$, $\frac{1}{k^p} = \int_{k-1}^{k} \frac{1}{k^p} dx \le \int_{k-1}^{k} \frac{1}{x^p} dx$ and $\sum_{k=2}^{\infty} \int_{k-1}^{k} \frac{1}{x^p} dx = \int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{1-p} \frac{1}{x^{p-1}}\Big|_1^{\infty}$.
[88] This is the classic $\epsilon/3$ proof. Its solution is on the back.
[89] The ML-estimate produces a short proof.
[90] This is called the Riemann's zeta function.

Lesson 22 Summary

A sequence of functions f_n is just infinitely many functions, one for each $n \in \mathbb{N}$. These functions share a common domain, say D. We say f_n converges to the limit f (pointwise), if for each $z \in D$, the sequence of points $f_n(z)$ converges to the point $f(z)$, and write $f_n(z) \to f(z)$, or $\lim_{n\to\infty} f_n(z) = f(z)$. Using the formal language, this means at any $z \in D$, for any $\epsilon > 0$, there is $N \in \mathbb{N}$ (which depends on both z and ϵ), such that if $n \geq N$, then $|f_n(z) - f(z)| < \epsilon$. Note that each N depends on z.

However, this notion of convergence is not very useful, as there is no way to pass certain good properties of f_n to f. The following is more useful:

We say the sequence f_n converges to f uniformly (versus pointwise), if for any $\epsilon > 0$, there is $N \in \mathbb{N}$, which only depends on ϵ, such that if $n \geq N$, then $|f_n(z) - f(z)| < \epsilon$ for all $z \in D$. Thus, the same N works for all z! This is why it's called uniformly convergence. Now we pass good properties of f_n to their limit f:
Theorem 1. If $f_n \to f$ uniformly and each f_n is continuous, then f is also continuous.
Theorem 2. If $f_n \to f$ uniformly on γ with finite length, then $\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz$.
Theorem 3. If $f_n \to f$ uniformly on D and each f_n is analytic, then so is f.
Theorem 1 is proved in the end. Theorem 2 can be handled using ML-estimate (that's why we require γ has finite length). Now let's prove Theorem 3.

Proof. As each f_n is analytic, it is also continuous. By Theorem 1, their uniform limit f is also continuous. Now let R be a closed rectangle in D with sides parallel to the axes. As f_n is analytic, by Cauchy's Theorem, $\int_{\partial R} f_n(z)dz = 0$. Now by Theorem 2, $\int_{\partial R} f(z)dz = \int_{\partial R} \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_{\partial R} f_n(z)dz = \lim_{n\to\infty} 0dz = 0$. Therefore, by Morera's Theorem, f is analytic. $\square$

Recall that a series is just the limit of its sequence of partial sums, so we say $\sum_{k=1}^\infty z_k$ converges if the sequence $s_n := \sum_{k=1}^n z_k = z_1 + z_2 + \cdots + z_n$ converges. When the series $\sum_{k=1}^\infty a_k$ and $\sum_{k=1}^\infty b_k$ consist of real terms, the following test is very useful. (For example, you used it in Problem 2.)

The Comparison Test. Suppose $0 \leq a_k \leq b_k$. (1) If $\sum_{k=1}^\infty b_k$ converges, then so does $\sum_{k=1}^\infty a_k$. (2) Thus, if $\sum_{k=1}^\infty a_k$ diverges, then so does $\sum_{k=1}^\infty b_k$.

Series of functions are defined similarly. We say $\sum_{k=1}^\infty f_k(z)$ converges pointwise, if for each z, the series of numbers $\sum_{k=1}^\infty f_k(z)$ converges. Uniform convergence is defined similarly. There is an extremely useful test (as in Problem 1, Problem 5 and many times in future), called the Weierstrass M-test, that can be used to show a series converges uniformly. Its proof depends on the reformulation of convergence of sequence as a Cauchy sequence, in which the limit is not specified, because we don't know what it is, which is typically the case.

The Weierstrass M-Test. Suppose M_k are real and $\sum_{k=1}^\infty M_k$ converges. If for each k, $|f_k(z)| \leq M_k$ for all z, then $\sum_{k=1}^\infty f_k(z)$ converges uniformly.

As promised, now we prove Theorem 1.

Proof. Let $\epsilon > 0$. Let z_0 be any point in the common domain of these functions. We will show f is continuous at z_0. First of all, as $f_k \to f$ uniformly, responding to $\frac{\epsilon}{3}$, there is $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(z) - f(z)| < \frac{\epsilon}{3}$ for any z in the domain. Fix such an n. Then as f_n is continuous at z_0, there is $\delta > 0$ responding to $\frac{\epsilon}{3}$, such that for all z in the domain and $|z - z_0| < \delta$, we have $|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$. Therefore, for such z, $|f(z) - f(z_0)| = |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. $\square$

Lesson 23 Power series are analytic

1. State the Ratio Test for power series $\sum a_k z^k$. Read through its proof in the textbook (page 141). And use them to calculate the radius of convergence of the following two series.

$$\text{(a)} \sum_{k=1}^{\infty} k z^k \qquad\qquad \text{(b)} \sum_{k=1}^{\infty} \frac{z^k}{k+1}$$

2. State the Root Test for power series $\sum a_k z^k$. Read through its proof in the textbook (page 142). And use them to calculate the radius of convergence of the above two series again.

3. We are back to the two series in Problem 1 again.

   (a) Prove that on $|z| < 1$, $\displaystyle\sum_{k=1}^{\infty} k z^k = \frac{z}{(1-z)^2}$.[91]

   (b) Prove that on $0 < |z| < 1$, $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k+1} = -1 - \frac{1}{z}\mathrm{Log}(1-z)$.[92]

   (c) Let $f(z) = \frac{z}{(1-z)^2}$. What is $f^{(2023)}(0)$?

4. In this problem, we review the proof of the uniform convergence of derivatives by proving a special case of it: Suppose each f_n is analytic on $|z - z_0| \le R_0$, and $f_n \to f$ uniformly on $|z - z_0| \le R_0$. Show that for any r with $0 < r < R_0$, $f_n' \to f'$ uniformly on $|z - z_0| \le r$.

5. Finally, let's show that power series converges within the radius of convergence R and this convergence is uniform if we stay away from the boundary by a positive distance. R is defined as follows: Let $T_r = \{|a_k| r^k : k \in \mathbb{N}\}$ be the set of the lengths of the terms in the series. Then R is defined to be the supremum of $\{r : T_r \text{ is bounded.}\}$, the "largest" r such that T_r is bounded. So if $s < R$, then T_s is bounded. This means, there is $C > 0$ such that $|a_k| s^k \le C$ for all $k \in \mathbb{N}$.

   (a) Let $0 < r < R$. Choose s such that $r < s < R$. Show that the series $\sum_{k=0}^{\infty} a_k z^k$ converges uniformly on $|z| \le r$ by using the Weierstrass M-test where $M_k = C(\frac{r}{s})^k$.[93]

   (b) Conclude that $\sum_{k=0}^{\infty} a_k z^k$ converges on $|z| < R$.

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[91] Start from the geometric series $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$. Use term-by-term differentiation. Also feel free to multiply or divide z on both sides. We can call this term-by-term multiplication/division. This can be done due to the algebraic limit theorems.

[92] Start from the geometric series $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$. Use term-by-term integration. Also feel free to multiply or divide z on both sides.

[93] The proof in on page 139 of the textbook.

Lesson 23 Summary

Today, we added one more property of uniformly convergent sequence of functions to the three in Lesson 22.

Theorem 4. If $f_n \to f$ uniformly on $|z - z_0| \le R_0$ and each f_n is analytic on $|z - z_0| \le R_0$ (so by Theorem 3, f is also analytic). Then for each $m \in \mathbb{N}$ and $0 < r < R_0$, $f_n^{(m)} \to f^{(m)}$ uniformly on $|z - z_0| \le r$.

This theorem will permit us to differentiate power series term-by-term, as we will see soon. Power series are series of the form $\sum_{k=0}^{\infty} a_k (z - z_0)^k$. Here, k doesn't have to start from 0. It can actually start from any integer, as the first few terms doesn't influence the convergence property of the series. Power series are special in that they are analytic inside a perfectly circular disk, diverge outside this disk, though it's hard to say anything about its behavior on the rim of this disk. To be more precise, let $T_r = \{|a_k| r^k : k \in \mathbb{N}\}$, and then define R to be the supremum of $\{r : T_r \text{ is bounded.}\}$, where supremum is the smallest upper bound of this set, which is guaranteed to exist for this nonempty set (note $0 \in T_r$) by the completeness property of real numbers. Then we have the following key theorem for power series.

Key Theorem for Power Series. Let $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$. Then

(1) f diverges in $|z - z_0| > R$.
(2) f converges in $|z - z_0| < R$.
(3) f converges uniformly in $|z - z_0| < R_0$ for any $0 < R_0 < R$.
(4) It's inconclusive on $|z - z_0| = R$.

(2) and (3) were proved in Problem 5. For (1), as $|a_k(z - z_0)^k|$ is not bounded, then there is no way for $a_k(z - z_0)^k$ to converge to 0, which is necessary for a series to converge. (For a convergent series $\sum_{k=0}^{\infty} b_k = s$ in general, let $s_n = b_0 + b_1 + \cdots + b_n$ be its partial sum. Then $b_n = s_n - s_{n-1}$. So $\lim_{n\to\infty} b_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$.)

Using the Key Theorem, and Theorems 4 and 3 for uniformly convergent sequence of analytic functions, we have the following useful corollaries for $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$.

Corollary 1. On $|z - z_0| < R$, $f(z)$ is analytic, and $f(z)$ can be differentiated termwise: $f^{(m)}(z) = \sum_{k=0}^{\infty} a_k ((z - z_0)^k)^{(m)}$. Furthermore, we have $a_k = \frac{f^{(k)}(z_0)}{k!}$.

Corollary 2. On $|z - z_0| < R$, $f(z)$ can also be integrated termwise: $\int_{z_0}^{z} f(\zeta)d\zeta = \sum_{k=0}^{\infty} a_k \int_{z_0}^{z} (\zeta - z_0)^k d\zeta = \sum_{k=0}^{\infty} a_k \frac{(z-z_0)^{k+1}}{k+1}$.

The proofs of both also depend on the fact that partial sums for power series are simply polynomials (finite sum of power functions), which can be differentiated and integrated termwise. For example, $f'(z) = \lim_{n\to\infty} f_n'(z)$, where $f_n(z)$ is a polynomial, and thus $f_n'(z)$ is just $\sum_{k=0}^{n} a_k ((z - z_0)^k)'$. Similarly, $\int_{z_0}^{z} f(\zeta)d\zeta = \lim_{n\to\infty} \int_{z_0}^{z} f_n(\zeta)d\zeta = \lim_{n\to\infty} \int_{z_0}^{z} \sum_{k=0}^{n} a_k (\zeta - z_0)^k d\zeta =$
$\lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_{z_0}^{z} (\zeta - z_0)^k d\zeta$.

There are formulas for finding the radius of convergence R of power series. In addition to the ratio test and the root test, there is also a formula named after Hadamard, in case the limit of the constructed sequence do not exist. Hadamard's formula uses $\limsup$, which always exists.

Lesson 24 Analytic functions as power series

1. In class, we showed that if $f(z)$ is analytic on $|z-z_0|<\rho$, then $f(z)=\sum_{k=0}^{\infty} a_k(z-z_0)^k$. Prove that if $|f(z)|\le M$ on $|z-z_0|<\rho$, then for any $0<r<\rho$, $|a_k|\le \frac{M}{r^k}$. [94]

2. In Lesson 1, we defined $e^{i\theta}=\cos\theta+i\sin\theta$. In Lesson 5, we defined $\cos z$ and $\sin z$, from which we get
$$e^{iz}=\cos z+i\sin z.$$
Using power series expansion at $z_0=0$, verify the above equality.

3. Suppose $f(z)$ is analytic on $|z|<\rho$ and $f(z)$ has been expanded as a power series $\sum_{k=0}^{\infty} a_k z^k$.

   (a) Show that if $f(z)$ is odd, i.e., $f(-z)=-f(z)$ for all z, then the power series only contain odd terms, i.e., $a_k=0$ if k is even. [95]

   (b) Show that if $f(z)$ is even, i.e., $f(-z)=f(z)$ for all z, then the power series only contain even terms, i.e., $a_k=0$ if k is odd.

4. Consider the functions $f(z)=\frac{z-2}{z^2-4}$ and $g(z)=\frac{\mathrm{Log} z}{z-1}$.

   (a) Find the radius of convergence of the power series expansion of $f(z)$ at $z_0=1$.

   (b) Find the power series expansion of $f(z)$ at $z_0=1$. [96]

   (c) What is the radius of convergence of the power series expansion of $g(z)$ at $z_0=5$? [97]

5. L'Hosptial's rule also holds for complex analytic functions. Here is a simple version. Suppose both f and g are analytic, $f(z_0)=g(z_0)=0$ and $g'(z_0)\neq 0$. Show that[98]
$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{f'(z)}{g'(z)}.$$

---

[94] This is just the Cauchy Estimate we discussed in Lesson 19. Recall that $a_k=\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta$.

[95] Take derivatives of both sides of $f(-z)=-f(z)$ and then show $\frac{f^{(k)}(0)}{k!}=0$ if k is even.

[96] We can use $a_k=\frac{f^{(k)}(1)}{k!}$ or notice that $\frac{1}{z+2}=\frac{1}{3}\frac{1}{1-(-\frac{z-1}{3})}$, which can then be expanded using the geometric series where $-\frac{z-1}{3}$ is substituted. The latter method will be used again when we study Laurent series, and it is preferred to the former method, which is extremely tedious in general.

[97] The solution in on page 147 of our textbook. Problem 5 below may provide another way to explain why $g(z)$ can be extended to $z=1$ and thus $z=1$ is not really a singularity. As further hints, recall that on $|w|<1$, we have $-\mathrm{Log}(1-w)=\sum_{k=0}^{\infty}\frac{w^{k+1}}{k+1}$ by integrating $\frac{1}{1-w}=\sum_{k=0}^{\infty} w^k$. Using $z=1-w$, we have $\mathrm{Log}(z)=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}(z-1)^m$, which converges on $|z-1|<1$. Thus, on $0<|z-1|<1$, $g(z)=\frac{\mathrm{Log} z}{z-1}=1-(z-1)/2+(z-1)^2/3-(z-1)^3/4\cdots$, which is analytic on $|z-1|<1$. So $z=1$ is really not a singularity, even though $z=0$ still is.

[98] Use the power series expansions of f and g at z_0: $f(z)=\sum_{k=1}^{\infty} a_k(z-z_0)^k$ and $g(z)=\sum_{k=1}^{\infty} b_k(z-z_0)^k$. The series start at $k=1$ because $f(z_0)=g(z_0)=0$. When we calculate the left-hand-side, write $f(z)=(z-z_0)\sum_{k=1}^{\infty} a_k(z-z_0)^{k-1}$ and $g(z)=(z-z_0)\sum_{k=1}^{\infty} b_k(z-z_0)^{k-1}$, and then use the fact that limit of a quotient is the quotient of limits, provided each limit exists. For the right-hand-side, the series of f' and g' can be found by term-by-term differentiation. Then use limit of quotient is the quotient of limits again. You will see that the two sides are equal.

Lesson 24 Summary

In Lesson 23, we learned that power series are analytic functions on an open disk with radius called the radius of convergence. Today we learned the opposite story, that analytic functions are power series on an open disk, and we can also say something about how big this disk is.

The reason that analytic functions are power series is due to the Cauchy's integral formula, the expressions of $\frac{1}{\zeta - z}$ as a power series whose convergence is uniform, and the interchange of integral with series by the uniform convergence property. In more details, if $f(z)$ is analytic on the disk $|z-z_0| < \rho$, then for any z in this disk, there is $r > 0$ such that $|z-z_0| < r < \rho$, and by the Cauchy's Integral formula, we have $f(z) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta - z_0 - (z-z_0)} d\zeta = \frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta - z_0}\frac{1}{1-\frac{z-z_0}{\zeta-z_0}} d\zeta$, which as $|\frac{z-z_0}{\zeta-z_0}| < 1$, is $\frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{f(\zeta)}{\zeta - z_0}\sum_{k=0}^{\infty}(\frac{z-z_0}{\zeta-z_0})^k d\zeta$, i.e., $\frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \sum_{k=0}^{\infty}\frac{f(\zeta)}{(\zeta - z_0)^{k+1}}(z-z_0)^k d\zeta$. As the convergence of the series is uniform on the circle $|\zeta - z_0| = r$ contained in the disk $|z-z_0| < \rho$, we can switch limit with integral. Thus, the previous integral becomes $\sum_{k=0}^{\infty}\frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}(z-z_0)^k d\zeta$, which finally is

$$\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|\zeta-z_0|=r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta\right)(z-z_0)^k.$$

The integral inside the parentheses is a_k, which we saw is $f^{(k)}(z_0)$ divided by $k!$ according to the higher-oder Cauchy's Integral Formulas. This can also be deduced from what we learned in Lesson 23: term by term differentiation, to see $a_k = \frac{f^{(k)}(z_0)}{k!}$. Once we have this formula, we can also do Cauchy's Estimate on $|a_k|$, which you did Problem 1. Note that the radius of convergence of this series is at least ρ.

We have two immediate corollaries to this theorem, which are very useful.

Corollary 1. Suppose f and g are both analytic on the disk $|z - z_0| < \rho$. If the derivatives of all orders of f and g agree at the single point z_0, then f and g are the same function.

Corollary 2. If $f(z)$ has been expanded as a power series over a disk centered at z_0, then the radius of convergence equals the largest ρ such that $f(z)$ can be extended to be an analytic function on $|z - z_0| < \rho$.

Corollary 1 holds simply because $f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \sum_{k=0}^{\infty}\frac{g^{(k)}(z_0)}{k!}(z-z_0)^k = g(z)$ on $|z-z_0| < \rho$. To prove Corollary 2, Let R be the radius of convergence of the series and L the largest radius such that $f(z)$ can be extended to an analytic function on $|z-z_0| < L$. Since $f(z)$ is (extended to be) analytic on $|z-z_0| < L$, its power series, which is the same as the original power series by the uniqueness of Corollary 1, has $R \geq L$ by the Theorem. Now if $R > L$, then $f(z)$ is extended to be this power series, which is analytic on this larger disk $|z - z_0| < R$, contradicting to the maximum of L. Thus, $R = L$.

Therefore, if $f(z)$ has a singularity at z_0 where $\lim_{z\to z_0} f(z) = \infty$, and we are considering expanding $f(z)$ at z_1, which is different from z_0, then the radius of convergence is $|z_0 - z_1|$.

Lesson 25 Analyticity and power series at infinity

1. Consider the function $f(z) = \dfrac{1}{1+z^3}$.

   (a) Find its power series at 0 on $|z| < 1$.

   (b) Show that $f(z)$ is analytic at ∞, and also find its power series expansion on $|z| > 1$.

2. Show that $f(z) = e^{1/z}$ is analytic at ∞ and also find its power series expansion on $|z| > 0$.

3. In this problem, let's prove the coefficient formula for power series at infinity again. Suppose $f(z) = \sum_{k=0}^{\infty} \frac{b_k}{z^k}$ which converges uniformly on $|z| \geq r$. Show that
$$b_k = \frac{1}{2\pi i} \int_{|z|=r} f(z) z^{k-1} dz.$$ [99]

4. Stereographic projection can also be done using the south pole $S = (0, 0, -1)$. Let (X, Y, Z) be any point but S on the sphere $X^2 + Y^2 + Z^2 = 1$. Let the $w = u + iv$ complex plane be the xy-plane, whose u-axis is the x-axis and the v-axis is the $-y$-axis. Draw a line through S and (X, Y, Z). This line intersects the w complex plane at a point. Show that this point is given by
$$w = \frac{X}{1+Z} - i\frac{Y}{1+Z}.$$

5. Recall from Lesson 1 Problem 6 that the stereographic projection of (X, Y, Z) onto the $z = x + iy$ plane from the north pole $N = (0, 0, 1)$ is given by
$$z = \frac{X}{1-Z} + i\frac{Y}{1-Z}.$$
Show that if $z \neq 0$, then
$$\frac{1}{z} = w,$$
the analytic change of variable we used today to study power series at infinity. [100] [101]

[99] The proof is on the top of page 150 of our textbook. It shows a technique for finding residues which we will come back to after learning Laurent series.

[100] Don't forget to use $X^2 + Y^2 + Z^2 = 1$.

[101] $w = \dfrac{1}{z}$ can also be seen by simple high-school geometry.

Lesson 25 Summary

We have been discussing analytic functions and (their) power series on subsets of $\mathbb{C}$, but recall in Lesson 1 that ∞ is an important point which can be joined with $\mathbb{C}$ to form the Riemann sphere. What does it mean to say f is analytic at ∞, and what about power series expansion at ∞?

The trouble with ∞ is that it is not expressed as numbers familiar to us, i.e., in the form of $x + iy$ where x and y are real. After all, ∞ is not on $\mathbb{C}$.

Here is the rationale to get around this difficulty, if we really want to discuss matters at ∞, which is on the Riemann sphere, we can think of the sphere as the union of two copies of complex planes. To be more concrete, consider the unit sphere $S^2 : X^2 + Y^2 + Z^2 = 1$ in the three dimensional Euclidean space. In Lesson 1, we learned that we can identify $\mathbb{C}$ with $S^2\backslash\{N\}$, the unit sphere with the north pole $N = (0, 0, 1)$ removed, by doing stereographic projection using rays from N. So complex analysis on $S^2\backslash\{N\}$ can be handled by doing the usual thing on $\mathbb{C}$. However, ∞, which is identified with N, cannot be mapped to $\mathbb{C}$ by this stereographic projection. If we want to discuss matters about N, we can switch our point of view, and consider $S^2\backslash\{S\}$, the unit sphere with the south pole $S = (0, 0, -1)$ removed. This set contains N, and we can do a stereographic projection, this time using rays from the south pole S to the point on the sphere as described in Problem 4. In particular, N is projected to 0 on the complex plane. One subtlety is that this complex plane labelled by $w = u + iv$ has u along x but v along $-y$ so that the transformation formula will be analytic.

So we can view the sphere S^2 as the union of two open sets $U_1 = S^2\backslash\{N\}$ and $U_2 = S^2\backslash\{S\}$. The former U_1 is identified with $\mathbb{C}$ labelled by $z = x + iy$ using the stereographic projection from the north pole, and the latter U_2 is identified with another copy of $\mathbb{C}$ labelled by $w = u + iv$ using the stereographic projection from the south pole. If we started from using z, but would like to consider $\infty = N$, then we can switch to w, as in this case, N can written as the concrete number $w = 0 + i0$, and is surrounded by other complex numbers described by $w = u + iv$ whereas in the former case, not only that N cannot be expressed using z, it is isolated from its neighbors expressed using $z = x + iy$. The switch, as you proved in Problem 5, is given by $w = \frac{1}{z}$, or $z = \frac{1}{w}$.

This issue being resolved, we have the following definition: $f(z)$ is analytic at $z = \infty$ if $g(w) := f(\frac{1}{w})$ when $w \neq 0$ and $g(0) := \lim_{w\to 0} f(\frac{1}{w})$ is analytic at 0. So we have made the above change of variable and are considering ∞ as the concrete $w = 0$.

If f is analytic at infinity, then using the power series expansion of $f(\frac{1}{w}) = \sum_{k=0}^{\infty} b_k w^k$ at $w = 0$, we get the power series expansion of $f(z)$ by substituting $\frac{1}{z}$ into w: $\sum_{k=0}^{\infty} \frac{b_k}{z^k}$. Its coefficient b_k can be calculated by doing line integral over circles $|w| = r$ inside the disk of convergence for the power series in w:

$$b_k = \frac{1}{2\pi i}\int_{|w|=r} \frac{f(\frac{1}{w})}{w^{k+1}}dw.$$

After making the change of variable $w = \frac{1}{z}$, and noticing that $dw = -\frac{1}{z^2}dz$ and changing w to z changes the counterclockwise oriented circle $|w| = r$ to the clockwise oriented circle $|w| = \frac{1}{r}$, we have

$$b_k = \frac{1}{2\pi i}\int_{|z|=\frac{1}{r}} f(z)z^{k-1}dz,$$

which you proved using another method in Problem 3.

Lesson 26 Manipulating power series

1. Using the equality $e^{2z} = e^z e^z$, prove[102] that
$$\frac{2^n}{n!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!}$$

2. Prove that if $f_n(z) \to f(z)$, $g_n(z) \to g(z)$ uniformly on their common domain D, and there is $M > 0$ such that $|f(z)|, |g(z)| < M$ for all $z \in D$, then $f_n(z)g_n(z) \to f(z)g(z)$ also uniformly. [103]

3. In this problem, we finish the computation of the first two Bernoulli numbers.

   (a) Calculate $\dfrac{z}{\sin z}$ up to z^4. (Answer: $= 1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots$)

   (b) Show that $\dfrac{z}{2}\cot(\dfrac{z}{2}) := \dfrac{z/2}{\sin(z/2)}\cos(z/2) = 1 - \dfrac{1}{6}\dfrac{z^2}{2!} - \dfrac{1}{30}\dfrac{z^4}{4!} + \mathcal{O}(z^6)$. [104]

4. Consider the function $f(z) = \dfrac{e^z}{1+z}$ and its power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

   (a) Show that $a_k = (-1)^k(\dfrac{1}{0!} + \dfrac{-1}{1!} + \dfrac{1}{2!} + \cdots + \dfrac{(-1)^k}{k!})$.

   (b) What is the radius of convergence of this series?

5. In this problem, we study the series expansion of $f(z) = z^{1/2}$ at $z = 1$.

   (a) Writing $f(z) = z^{1/2} = e^{\frac{1}{2}\mathrm{Log} z}$ and using the series expansions of $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ and $\mathrm{Log} z = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z-1)^k}{k}$, calculate the series expansion of $f(z)$ at $z = 1$ up to $(z-1)^3$.

   (b) Calculate the series expansion of $f(z)$ up to $(z-1)^3$ using $a_k = \frac{f^{(k)}(1)}{k!}$. [105]

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[102] You see, you get the binomial formula using complex analysis. Even though this is not a good example showing the power of power series in combinatorics, as the binomial formula has much simpler proofs, in general, complex analysis is quite useful in the latter subject.

[103] Some hint: Consider $|f_n(z)g_n(z) - f(z)g(z)| = |f_n(z)g_n(z) - f(z)g_n(z) + f(z)g_n(z) - f(z)g(z)| \leq |f_n(z) - f(z)||g_n(z)| + |f(z)||g_n(z) - g(z)|$. As $f_n(z) \to f(z)$ uniformly, there is $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|f_n(z) - f(z)| < \frac{\epsilon}{4M}$. As $g_n(z) \to g(z)$ uniformly, there is $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|g_n(z) - g(z)| < \frac{\epsilon}{2M}$ and $|g_n(z) - g(z)| < M$ and thus $|g_n(z)| = |g_n(z) - g(z) + g(z)| \leq |g_n(z) - g(z)| + |g(z)| < M + M = 2M$ for all z. Then if $N = \max\{N_1, N_2\}$, then if $n \geq N$, then $n \geq N_1$ and $n \geq N_2$. Thus, all the previous results hold. Then you can finish by showing $|f_n(z)g_n(z) - f(z)g(z)| < \epsilon$.

[104] From (a), we have $\frac{z/2}{\sin(z/2)} = 1 + \frac{1}{6}(z/2)^2 + \frac{7}{360}(z/2)^4 + \cdots = 1 + \frac{1}{24}z^2 + \frac{7}{4!\cdot 240}z^4 + \cdots$ Also recall that $\cos(z/2) = 1 - \frac{1}{2!}(z/2)^2 + \frac{1}{4!}(z/2)^4 + \cdots = 1 - \frac{1}{8}z^2 + \frac{1}{16\cdot 4!}z^4 + \cdots$ Thus, $\frac{z}{2}\cot(\frac{z}{2}) := \frac{z/2}{\sin(z/2)}\cos(z/2) = 1 + (\frac{1}{24} - \frac{1}{8})z^2 + (-\frac{1}{4!\cdot 8} + \frac{7}{4!\cdot 240} + \frac{1}{4!\cdot 16})z^4 + \mathcal{O}(z^6) = 1 - \frac{2}{24}z^2 + \frac{1}{4!}\frac{-30+7+15}{240}z^4 + \mathcal{O}(z^6) = 1 - \frac{1}{6}\frac{z^2}{2!} - \frac{1}{30}\frac{z^4}{4!} + \mathcal{O}(z^6)$.

[105] In fact, we can go further to find the formula for all terms, which is pretty straightforward, but we stop here, as we have seen that the above two methods have yielded the same result for enough number of terms.

Lesson 26 Summary

Even though we can always find a_k in $f(z) = \sum_{k=0}^{\infty} a_k z^k$ by doing derivatives $a_k = \frac{f^{(k)}(0)}{k!}$, in general, calculating $f^{(k)}(0)$ is tedious, if not extremely so.

Another way to find the series expansion is to decompose an analytic function into simpler components, and then combine the familiar series of these simpler components. The reason we can do this is due to the good property of power series: they behave as polynomials, i.e., we can manipulate them as if they are polynomials. We can add, subtract, and scalar multiply them as we do for polynomials. We can also multiply them as multiplying polynomials. More precisely, we have the following:

Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ (over some disk at 0), then

(1) $f(z) \pm g(z) = \sum_{k=0}^{\infty} (a_k + b_k) z^k$.

(2) $cf(z) = \sum_{k=0}^{\infty} c a_k z^k$.

(3) $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

(4) $f(z)^m = f(z)f(z)\cdots f(z)$, which can be done by using (3) $m-1$ times. If $f(z)^m = \sum_{n=1}^{\infty} c_n z^n$, then $c_n = \sum_{k_1+\cdots+k_m=n} a_{k_1} \cdots a_{k_m}$.

(5) If $g(0) = b_0 \neq 0$, then

$$\frac{1}{g(z)} = \frac{1}{b_0} \frac{1}{1 - (\sum_{k=1}^{\infty} -\frac{b_k}{b_0} z^k)} = \frac{1}{b_0} \sum_{m=0}^{\infty} (\sum_{k=1}^{\infty} -\frac{b_k}{b_0} z^k)^m,$$

where the last step holds because $|\sum_{k=1}^{\infty} -\frac{b_k}{b_0} z^k| < 1$ if $|z|$ is small by the continuity of $g(z)$ at $z = 0$, and each $(\sum_{k=1}^{\infty} -\frac{b_k}{b_0} z^k)^m$ can be expanded as in (4).

(6) $\frac{f(z)}{g(z)} = f(z)\frac{1}{g(z)}$, which can be expanded using (3) and (5).

(1) and (2) holds because of the algebraic limit theorems for limits (series). (4), (5) and (6) follows from (1), (2) and (3). Thus, it suffices to prove (3).

Proof. Let $f_n(z) = \sum_{k=0}^{n} a_k z^k$ and $g_n(z) = \sum_{k=0}^{n} b_k z^k$ be the partial sums, which are polynomials. As f_n and g_n converge uniformly to f and g, respectively, over a closed disk at 0, on which f and g are continuous and thus also bounded, $f_n(z)g_n(z)$ converges to $f(z)g(z)$ uniformly over this closed disk by Problem 2. As all functions involved are analytic, by the Theorem we proved in Lesson 23, the mth derivative of $f_n(z)g_n(z)$ converges uniformly to the mth derivative of $f(z)g(z)$. When $n \geq m$, the mth derivative of $f_n(z)g_n(z)$ is the coefficient in front of z^m, which is the c_m above. Thus, its limit as $n \to \infty$ is also this c_m. □

Problem 5 shows we can also get the series of $f(g(z))$ by composing those of $f(z)$ and $g(z)$. In this problem, the series of $f(z)$ and $g(z)$ are not expanded at the same point.

Lesson 27 Zeros of analytic functions and their magic

1. Find the zeros and their orders of: (a) $\sin z$ (b) $\cos z$ (c) $\frac{(z-1)\mathrm{Log} z}{z}$

2. State and prove the Principle of Permanence of Functional Equations. Use it to show that $e^z e^w = e^{z+w}$ from $e^s e^t = e^{s+t}$ where s and t are real. [106]

3. In this problem, we show that each non-constant analytic function is a conformal mapping composed with an Nth power function for some $N \in \mathbb{N}$: [107] Show that if $f(z)$ is analytic and has a zero of order N at z_0, then there is another function $g(z)$ analytic near z_0 such that $g'(z_0) \neq 0$ and $f(z) = g(z)^N$. [108]

4. Show that the Open Mapping Theorem (see the footnote to the previous problem) implies the maximum principle: If f is analytic on the domain D and $|f|$ attains its maximal value at some point $z_0 \in D$, then f is constant. [109]

5. Before we leave Chapter V on power series, let's get some taste of analytic continuation. The basic idea is that we can extend the definition of an analytic function beyond the zones of comfort by varying its power series along a path trespassing into a previously forbidden place. The identity principle we learned today guarantees the uniqueness of this extension.

   (a) If you are not sure what the answer to Problem 5 of Lesson 26 is, it is $f(z) := z^{1/2} = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{3}{48}(z-1)^3 + \cdots$ In fact, the coefficient $a_k = \frac{f^{(k)}(1)}{k!} = \frac{(-1)^{k-1} 1\cdot 3\cdot 5 \cdots (2k-3)}{k! 2^k}$ for $k \geq 2$ and $a_0 = 1$, $a_1 = \frac{1}{2}$. Now consider the twice traversed circular path e^{it}, where $t : 0 \to 4\pi$, and we define $f_t(z) = \sum_{k=1}^{\infty} a_k(t)(z - e^{it})^k$, where $a_0 = 1$, $a_1 = \frac{1}{2} e^{-\frac{it}{2}}$, and for $k \geq 2$, $a_k(t) = \frac{(-1)^{k-1} 1\cdot 3\cdot 5 \cdots (2k-3)}{k! 2^k} e^{-\frac{i(2k-1)t}{2}}$, i.e.,
   $$f_t(z) = 1 + \frac{e^{-\frac{it}{2}}}{2}(z - e^{it}) - \frac{e^{-\frac{i3t}{2}}}{8}(z - e^{it})^2 + \frac{3e^{-\frac{i5t}{2}}}{48}(z - e^{it})^3 + \cdots$$
   Show that the radius of convergence of $f_t(z)$ is 1. [110]

   (b) Show that $f_{2\pi}(z) = -f(z)$.

   (c) Show that $f_{4\pi}(z) = f(z)$. [111]

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[106] It's on page 157 of our textbook.

[107] As conformal mapping simply mildly deforms the domain, it maps open set to open set. As the Nth power function simply wraps angles around and stretches radial lines, it also maps open set to open set. Therefore, non-constant function maps open set to open set. This is called the Open Mapping Theorem, which does not hold for real analytic functions. For example, the square power functions folds the real line into the interval $[0, \infty)$, which reveals the sharp 0 as a boundary point.

[108] Since $f(z)$ has a zero of order N, we know $f(z) = (z - z_0)^N h(z)$ where $h(z_0) \neq 0$. Let Log be a branch of $\log z$ whose domain includes $h(z_0)$ and let $g(z) = (z - z_0)e^{\frac{1}{N}\mathrm{Log} h(z)}$.

[109] If f is not constant, then by the Open Mapping Theorem, an open disk centered at $f(z_0)$ is contained in $f(D)$. On this disk, more than half of the points $f(z)$ has length larger than $|f(z_0)|$.

[110] Note that the radius of convergence of $f(z)$ is 1 and if you use the ratio test, the two series $f_t(z)$ and $f(z)$ have the same $|a_k|/|a_{k+1}|$ sequence. Or just use the ratio test.

[111] In fact, $f_t(z)$ as we vary t is defined on the part of the Riemann surface of $z^{1/2}$, each point of which has distance less than 2 from the origin.

Lesson 27 Summary

For an analytic function f, z_0 is called a zero of f if $f(z_0) = 0$. If we expand $f(z)$ as a power series $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$, this means $a_0 = 0$. But what if more a_k's are zero?

First of all, if all $a_k = 0$, then $f(z) = 0$ for all z. This is not very interesting, but it is often a case not to forget. Now let's assume not all a_k's are 0, and let a_N be the first coefficient which is not 0, i.e., $a_0 = a_1 = \cdots = a_{N-1} = 0$, but $a_N \neq 0$. Then $f(z) = a_N(z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + \cdots$, which can be factored as

$$f(z) = (z - z_0)^N h(z),$$

where $h(z) = a_N + a_{N+1}(z - z_0)^{N+1} + \cdots$ is analytic and $h(z_0) = a_N \neq 0$.

In this case, z_0 is called a zero (of f) of order N. Equivalently, if $f(z_0) = f'(z_0) = \cdots = f^{(N-1)}(z_0) = 0$ but $f^{(N)}(z_0) \neq 0$, then z_0 is a zero of order N. When $N = 1$, we say z_0 is a simple zero. When $N = 2$, a double zero... The above expression permits us to write $f(z) = g(z)^N$, where $g(z)$ is analytic over a disk centered at z_0, and $g'(z_0) \neq 0$. See Problem 3. Thus, locally, $f(z)$ is the composition of the conformal map $g(z)$ with the N's power function. As both maps open sets to open sets, f maps open sets to open sets. This is called the Open Mapping Theorem. The maximum principle also follows from it. See Problem 4.

The zeros of a nonconstant analytic function are isolated, meaning each zero stays a positive distance away from all the other zeros. The proof consists of two parts. Part I: if f is nonconstant, then the zeros of f are of finite order. This is proved by considering U: the set of all $z \in D$ such that $f^{(m)}(z) = 0$ for all m and its complement $D \backslash U$: the set of all $z \in D$ such that $f^{(m)}(z) \neq 0$ for some m. It can be shown that both U and $D \backslash U$ are open. U is open because the power series with all coefficients 0 is locally 0 and thus an open disk's z are in U. $D \backslash U$ is open because if $f^{(m)}(z_0) \neq 0$ for some m at some z_0. By the continuity of $f^{(m)}$, there is a disk worth of z around z_0 such that $f^{(m)}(z) \neq 0$. Thus, this disk in contained in $D \backslash U$. As D is connected, either $U = D$ or $D \backslash U = D$. As f is nonconstant, it can only be the case that $D \backslash U = D$, i.e., no zeros are of infinite order, i.e., all zeros are of finite order. Part II: Let z_0 be a zero of f. So $f(z) = (z - z_0)^N h(z)$ and $h(z_0) \neq 0$. As h is continuous, there is a disk centered at z_0 over which $h(z) \neq 0$. Thus, z_0 is the only zero over this disk, and thus z_0 stays a finite distance from the other zeros.

The above theorem is so strong that it forces a uniqueness property of analytic functions: if $f(z) = g(z)$ over a subset E of D and E contains a nonisolated point, then $f(z) = g(z)$ for all $z \in D$. The proof of this theorem simply reduces to showing $f(z) - g(z)$ has to be the constant 0 on D if it is 0 on the set E with nonisolated point by quoting the previous theorem. As an application, if there is another definition $g(z)$ of e^z such that both agree with e^x on the real axis. As every point of $E = \mathbb{R}$ is nonisolated, then we must have $g(z) = e^z$.

In a similar vein, we also have the Principle of Permanence of Functional Equations, which allows us to imply $e^z e^w = e^{z+w}$ simply from its real version. This theorem is proved by using the above uniqueness property of analytic functions twice. See Problem 2.

Lesson 28 Laurent decomposition and Laurent series

1. In this problem, we prove the uniqueness of Laurent decomposition. [112]

   (a) Suppose $h(z)$ is entire and $\lim_{z\to\infty} h(z) = 0$. Show that $h(z)$ is bounded on $\mathbb{C}$.

   (b) Suppose $f(z)$ is analytic on $\rho < |z - z_0| < \sigma$, and $f(z) = f_0(z) + f_1(z)$ where $f_0(z)$ is analytic on $|z - z_0| < \sigma$ and $f_1(z)$ is analytic on $|z - z_0| > \rho$ with $\lim_{z\to\infty} f_1(z) = 0$. Show that f_0 and f_1 are unique.

2. Consider the function $f(z) = \frac{1}{z^2-z} = \frac{1}{(z-1)z}$ defined on $\mathbb{C}\backslash\{0, 1\}$. Find its Laurent decomposition and its Laurent series expansion over each of the following regions.

   (a) $0 < |z| < 1$

   (b) $|z| > 1$

3. Consider the same function $f(z) = \frac{1}{(z-1)z}$ again. Expand it as a Laurent series centered at -1 such that it converges at $1/2$. (Note that $1 < |z - (-1)| < 2$ is the largest annulus including $1/2$ over which $f(z)$ is analytic). [113]

4. Suppose $f(z)$ has been decomposed as $f_0(z) + f_1(z)$ in the usual sense and expanded as $\sum_{k=-\infty}^{\infty} a_k z^k$.

   (a) If $f(z)$ is even[114], i.e., $f(-z) = f(z)$, then $a_k = 0$ for all odd k and both f_0 and $f_1$are even.

   (b) If $f(z)$ is odd, i.e., $f(-z) = -f(z)$, then $a_k = 0$ for all even k and both f_0 and $f_1$are odd.

5. Suppose $f(z)$ is analytic on the non-simply connected $\mathbb{C}\backslash\{0\}$. Show that for some $c \in \mathbb{C}$, there is $F(z)$ analytic on $\mathbb{C}\backslash\{0\}$ such that $f(z) - \dfrac{c}{z} = F'(z)$. [115]

[112] The proof is on the top of page 166 in the textbook.

[113] Note that $f(z) = \frac{1}{z-1} - \frac{1}{z} = \frac{1}{z+1-2} - \frac{1}{z+1-1}$. The first function is analytic on $|z + 1| < 2$ and the second function is analytic on $|z + 1| > 1$. Then we expand both using power series in terms of some a satisfying $|a| < 1$.

[114] Note that if $f(z) = \sum a_k z^k$, then $f(-z) = \sum a_k(-z)^k = \sum(-1)^k a_k z^k$. If $f(-z) = f(z)$, then corresponding coefficients in the series expansions of $f(-z)$ and $f(z)$ should be equal, by the uniqueness of coefficients. Alternatively, we can use the integral representation of a_k to show $a_k = (-1)^k a_k$.

[115] Consider the Laurent series expansion of $f(z)$, separate the z^{-1} term. Then each of the rest terms has antiderivative on $\mathbb{C}\backslash\{0\}$ and $F(z)$ is just the sum of all of them.

Lesson 28 Summary

Analytic functions somewhere on $\mathbb{C}$ can be expanded as a power series of positive powers over an open disk. Analytic functions at ∞ can be expanded as a power series of negative powers over the outside of a closed disk centered at the origin. Today, each region we consider are a combination of these two types: it is inside a bigger disk, and outside a smaller disk, which share the center. This is called an annulus, described by $0 \leq \rho < |z - z_0| < \sigma \leq \infty$, which also includes the punctured disks, punctured planes and the exterior of any closed disk.

If $f(z)$ is analytic over any such annulus, then there are analytic functions on larger domains: f_0 on $|z - z_0| < \sigma$ and f_1 on $|z - z_0| > \rho$ with $\lim_{z\to\infty} f_1(z) = 0$ such that $f(z) = f_0(z) + f_1(z)$ over the annulus. This is called the *Laurent Decomposition* of $f(z)$ over this annulus.

Existence of this decomposition follows from Cauchy's Integral Formula applied to a closed annulus $r \leq |z - z_0| \leq s$ contained in this open annulus, whose boundary consists of two circles $|z-z_0| = r$ and $|z-z_0| = s$, where the former inside circle originally is clockwise oriented and thus need a negative sign:

$$f(z) = \frac{1}{2\pi i}\int_{|z-z_0|=s} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{-1}{2\pi i}\int_{|z-z_0|=r} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We then simply let the first integral be $f_0(z)$, and the second be $f_1(z)$. Note that indeed $\lim_{z\to\infty} f_1(z) = 0$ by doing an ML-estimate of $|f_1(z)|$ first.

Uniqueness of this decomposition can be shown by assuming that there are two other analytic functions g_0 and g_1 with the same property and $f(z) = g_0(z) + g_1(z)$. Then by constructing an entire function $h(z)$ which is defined as $h(z) = g_0(z) - f_0(z)$ over $|z-z_0| < \sigma$ and $h(z) = f_1(z) - g_1(z)$ over $|z-z_0| > \rho$, and noticing that $\lim_{z\to\infty} h(z) = 0$, we see $h(z) = 0$ by Liouville's Theorem. Then it follows that $g_0 = f_0$ and $g_1 = f_1$ over their individual domains. You proved this in Problem 1.

We know each of f_0 and f_1 over their domains can be expanded as a series, the former consisting of nonnegative powers, and the latter consisting of negative powers. Thus, over the annulus, f can be expanded as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

where a_n can be calculated by first dividing the above equality by $(z-z_0)^{n+1}$, and then integrating both sides of the result over a circle contained in the annulus with center z_0: $\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz = \int_{|z-z_0|=r} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k-n-1} dz$. As the convergence of the series is uniform, we can switch integral with series. Thus, the previous expressions is $\sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=r} (z-z_0)^{k-n-1} dz$, which consists of only one term for $k = n$: $\int_{|z-z_0|=r} \frac{1}{z-z_0} dz = 2\pi i$. Thus,

$$a_n = \frac{1}{2\pi i}\int_{|z_0-z|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

This formula includes the series formula for analytic functions on $\mathbb{C}$ and analytic functions at ∞ as special cases.

Lesson 29 Classification of isolated singularities

1. Find and classify the isolated singularities of each of the following functions.

   (a) $\dfrac{ze^z}{z^2-4}$ (b) $\dfrac{e^{2z}-1}{z}$ (c) $\tan z = \dfrac{\sin z}{\cos z}$ (d) $z^2 \sin(\dfrac{1}{z})$ (e) $\dfrac{\text{Log} z}{(z-1)^3}$

2. Prove that $e^{\frac{1}{1+z^2}}$ has essential singularities at $\pm i$ by showing they are neither removable nor poles. [116]

In the following problems, let's prove the characterization of each of the three types of isolated singularities. Almost all details of proofs can be found in the textbook from page 172 to 175.

3. **Riemann's Theorem on Removable Singularities.** Le z_0 be an isolated singularity of $f(z)$. Then $f(z)$ has a removable singularity at z_0 if and only if $f(z)$ is bounded near z_0.

4. **MATH 345 Students' Theorem on Poles.** Le z_0 be an isolated singularity of $f(z)$. Then $f(z)$ has a pole at z_0 if and only if $\lim_{z\to z_0} |f(z)| = \infty$.

5. **Casorati-Weierstrass Theorem on Essential Singularities.** Le z_0 be an isolated singularity of $f(z)$. Then $f(z)$ has an essential singularity at z_0 if and only if for every complex number w_0, there is a sequence $z_n \to z_0$ such that $f(z_n) \to w_0$.

[116] We can consider special paths $z = x + i$ and $z = yi$ through i, and $z = x - i$ and $z = yi$ through $-i$, where x and y are real.

Lesson 29 Summary

In lesson 28, we studied Laurent decomposition and Laurent series expansion over annulus, which includes punctured disk as a special case. Today, we only consider punctured disk over which we are given an analytic function. So the question is: what happens at z_0, or when z gets close to z_0?

First of all, $f(z)$ may be not defined at all at z_0. Then we call z_0 an isolated singularity of $f(z)$ over this punctured disk. There are nonisolated singularities. For example, a slit for the Log$z$ function is a line of singularities, and Log$z$ is not even defined along this half line.

Over this punctured disk, we can expand $f(z)$ into its Laurent series

$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Then we classify the isolated singularities into three types:

(1) If $a_k = 0$ for all $k < 0$, then z_0 is called a removable singularity of $f(z)$.
(2) If $a_{-N} \neq 0$ for some $N > 0$, and $a_k = 0$ for all $k < -N$, then z_0 is called a pole of $f(z)$ and N is called the order of the pole.
(3) If $a_k \neq 0$ for infinitely many negative k, then z_0 is called an essential singularity of $f(z)$.

In (1), we can define $f(z)$ to be over the entire disk, including the puncture z_0 by letting $f(z)$ to be its power series. Thus, the singularity can be removed.

In (2), as there are only finitely many negative powers of $z - z_0$, we can write $f(z) = \frac{a_{-N}}{(z-z_0)^N} + \frac{a_{-(N-1)}}{(z-z_0)^{N-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots = f_1(z) + f_0(z)$, where $f_1(z)$ is the sum of the negative powers and $f_0(z)$ is the analytic part. $f_1(z)$ is called the principal part. This is **one** way to view poles. We can also factor out $\frac{1}{(z-z_0)^N}$ to have $f(z) = \frac{g(z)}{(z-z)^N}$, where $g(z)$ as a power series with $g(z_0) = a_{-N} \neq 0$. This is the **second** way to view poles. Lastly, $\frac{1}{f(z)} = (z-z_0)^N \frac{1}{g(z)} = (z-z_0)^N h(z)$, where $h(z) = \frac{1}{g(z)}$ is analytic as $g(z)$ is and $g(z_0) \neq 0$. Furthermore, we also have $h(z_0) \neq 0$. Thus, $f(z)$ has a pole of order N at 0 if and only if $\frac{1}{f(z)}$ has a zero of order N at z_0. This is a **third** way to view poles.

(3) is the most weird case, as any small region around z_0, no matter how small it is, can be mapped to get arbitrarily close to an arbitrary point on the complex plane. This is summarized in the Casorati-Weierstrass Theorem you proved in Problem 5. In contrast, for a pole, as z gets close to z_0, $f(z)$ are sent to approach ∞ and for a removable singularity, as z gets close to z_0, $f(z)$ predictably are sent to a finite value to make $f(z)$ analytic also at the puncture. These were proved in Problems 4 and 3.

In practice, to prove z_0 is an essential singularity, we just show it's neither removable nor a pole. This is done by showing $|f(z)|$ is not bounded near z_0 (so z_0 is not removable) and $f(z)$ does not go to ∞ when z goes to z_0 along some path (so z_0 is not a pole). This is a strategy that can be used in Problem 2.

Problem 2 is a special case of the following fact: if z_0 is a non-removable isolated singularity of $f(z)$, then z_0 is an essential singularity of $e^{f(z)}$. In problem 2, $\pm i$ are poles of $\frac{1}{1+z^2}$, which are not removable. Thus, $\pm i$ are essential singularities of $e^{\frac{1}{1+z^2}}$.

Lesson 30 Meromorphic functions and partial fractions decomposition on $\mathbb{C}^*$

1. Find the principal part of
$$f(z) = \frac{1}{\sin z}$$
at each of its poles on the Riemann sphere $\mathbb{C}^*$. Note that ∞ is not an isolated singularity of $f(z)$ and thus it is not a pole. [117]

2. Show that
$$f(z) = \frac{z^3}{z-1}$$
has a pole at ∞ and find the principal part $P_\infty(z)$ by finding the series expansion of $f(z)$ on $|z| > 1$.

3. From the proof of the theorem of meromorphic functions on $\mathbb{C}^*$, we know
$$f(z) = \frac{z^3}{(z-1)(z-2)}$$
is the sum of the principal parts P_∞, P_1 and P_2. Find these principal parts. [118]

4. Consider the function
$$f(z) = \frac{\sin z}{1+z}.$$
Show that it is meromorphic on $\mathbb{C}$ but not meromorphic on $\mathbb{C}^*$.

5. Let V be the complex vector space of meromorphic functions $\mathbb{C}^*$ with possible poles at 0, 1, ∞, each with orders at most 2. Find a basis of this vector space. What is the dimension of V?

[117] Find the power series expansion of $\sin z$ at $k\pi$ and then manipulate series to find the principal parts for these simple poles.

[118] We can do a long division to find $P_\infty(z)$ and then split the rest to find $P_1(z)$ and $P_2(z)$. The answers are $P_\infty(z) = z + 3$, $P_1(z) = \frac{-1}{z-1}$ and $P_2(z) = \frac{8}{z-2}$.

Lesson 30 Summary

In Lesson 29, we studied isolated singularities of analytic functions over punctured disk $0 < |z-z_0| < r$. They were classified into removable, pole, and essential, depending on how many negative powers of $z - z_0$ are present in the Laurent series expansion of $f(z)$ at z_0: none for the removable, finitely many for pole, and infinitely many for the essential.

Today, we started by considering isolated singularity at ∞: $f(z)$ has an isolated singularity at ∞ if $f(z)$ is analytic on some $|z| > R$, but the behavior at ∞ is left open. This infinite annulus $|z| > R$ can be thought as a punctured disk centered at ∞, which makes sense if you use the usual $w = \frac{1}{z}$ change of variable, corresponding to the switching from the north-pole to the south-pole stereographic projections. Then $z = \infty$ is classified into removable, pole, and essential singularities if $w = 0$ is removable, a pole, or essential. Therefore, in the Laurent series expansion of $f(z)$ over $|z| > R$, ∞ is (1) removable, (2) a pole, or (3) essential, if there are (1) no positive powers of z, (2) finitely many positive powers, or (3) infinitely many positive powers, respectively.

This lesson's focus is on functions with possible poles: a function is meromorphic on a domain D of $\mathbb{C}^*$ if f is analytic on D expect at possible isolated singularities each of which is a pole. For a pole $z_0 \in \mathbb{C}$, the principal part $P(z)$ is defined to be the sum of all the negative powers of $z - z_0$ in the Laurent series expansion of $f(z)$ on $0 < |z - z_0| < r$; At ∞, the principal part $P_\infty(z)$ is defined to be the sum of all nonnegative powers of z in the Laurent series expansion of $f(z)$ over $|z| > R$. So if ∞ is removable, $P_\infty(z)$ is simply a constant, and if ∞ is a pole, then $P_\infty(z)$ is a polynomial of degree at least 1.

If $f(z)$ is a rational function, i.e., $f(z) = \frac{P(z)}{Q(z)}$ where both $P(z)$ and $Q(z)$ are polynomials, then $f(z)$ is meromorphic on $\mathbb{C}^*$. What is surprising is that the converse is also true:

Theorem. If $f(z)$ is meromorphic on $\mathbb{C}^*$, then $f(z)$ is rational.

First of all, note that there can only be finitely many poles on $\mathbb{C}^*$. This is because if there were infinitely many, then as $\mathbb{C}^*$ is compact, then there is a limit point of these poles, which itself has to be a singularity, because any disk around it contains nearby approaching poles and thus $f(z)$ can not be analytic over any disk around this point. Furthermore, this singularity is not isolated. These contradict to the assumption that all singularities of meromorphic functions are isolated. After resolving this impossible infinitude of the number of poles, the proof relies on the construction of the function $g(z) = f(z) - P_\infty(z) - P_1(z) - \cdots - P_M(z)$, where $P_i(z)$ is the principal part of $f(z)$ at a pole $z_i \in \mathbb{C}$. So we are simply subtracting the principal parts of all poles from the function $f(z)$ itself, including that of ∞, if it is a pole. Now on $\mathbb{C}$ minus the poles, as the sum of analytic functions, $g(z)$ of course is analytic. But $g(z)$ is also analytic at each of the poles. This is because at z_i, $f(z) - P_{z_i}(z)$ is a power series and thus is analytic at z_0, and the sum of the rest of the principal parts are also analytic at z_0. Being an entire function, $g(z)$ is also bounded, as $\lim_{z\to\infty} g(z) = \lim_{z\to\infty}(f(z) - P_\infty(z)) - \sum_{i=1}^{M} \lim_{z\to\infty} P_i(z) = 0 - 0 = 0$, where each of the limit is of a finite sum of negative powers of a linear function. Hence, by Liouville's Theorem, $g(z)$ is a constant, which has to be 0. So $f(z) = P_\infty(z) + P_1(z) + \cdots + P_M(z)$, which is rational by expressing it as a single fraction.

For meromorphic functions on $\mathbb{C}^*$, the powers of z and $\frac{1}{z-z_0}$ are important, as they serve as basis elements for the vector space of meromorphic functions. In a future course of complex analysis, you will learn that over a donut surface, the Weierstrass functions, all denoted by $\wp(z)$, which are meromorphic functions with a double pole, and their derivatives, will serve analogous roles.

LESSON 31 THE RESIDUE THEOREM

1. Find the residue of the following functions at the given points.

$$\text{(a) } \frac{1}{z^2-1} \text{ at } 1 \text{ (b) } \tan z = \frac{\sin z}{\cos z} \text{ at } \pi/2 \text{ (c) } \frac{z}{\text{Log}z} \text{ at } 1 \text{ (d) } \frac{z}{(z^2+1)^2} \text{ at } i \text{ (e) } \frac{e^z}{z^{2023}} \text{ at } 0$$

2. Evaluate the following integrals.
   (a) $\displaystyle\int_{|z|=2} \tan z dz$
   (b) $\displaystyle\int_{|z-1|=1} \frac{1}{z^8-1} dz$ *Hint*: Three of the simple zeros of z^8-1 are inside the circle $|z-1|=1$: 1, $e^{i\frac{\pi}{4}}$ and $e^{-i\frac{\pi}{4}}$.

3. We should also have Rule 2.5 for finding residues: If $f(z)$ is analytic and z_0 is a pole of order $N \geq 1$, then $\text{Res}[f(z), z_0] = \dfrac{1}{(N-1)!} \lim\limits_{z \to z_0} \dfrac{d^{N-1}}{dz^{N-1}}((z-z_0)^N f(z))$.
   (a) Prove this rule.
   (b) Calculate $\displaystyle\int_{|z-1-i|=2} \frac{1}{(z^2+1)^5} dz$.

4. Now we prove a formula which you could have seen in Calculus II. Suppose $P(z)$ and $Q(z)$ are polynomials such that the zeros $z_1, \cdots, z_m$ of $Q(z)$ are all simple and the degree of $Q(z)$ is bigger than the degree of $P(z)$. Show that $\dfrac{P(z)}{Q(z)} = \displaystyle\sum_{i=1}^{m} \frac{P(z_i)}{Q'(z_i)} \frac{1}{z-z_i}$. [119]

5. A function is doubly periodic if there are complex numbers ω_1 and ω_2 which do not lie on the same line such that $f(z+\omega_1) = f(z)$ and $f(z+\omega_2) = f(z)$. Thus, if we know what f does over the parallelogram $D = \{z \big| z = t\omega_1 + s\omega_2, 0 < t, s < 1\}$ and any of its two adjacent sides, then we know what f does on $\mathbb{C}$ by translation. Liouville proved three theorems for doubly periodic functions. We consider the first two today, and leave the last to Chapter 8. These proofs are short and elegant.

   (a) Prove **Liouville's First Theorem:** Suppose f is analytic and doubly periodic on $\mathbb{C}$, then f is a constant. [120]

   (b) Prove **Liouville's Second Theorem:** Suppose f is meromorphic and doubly periodic on $\mathbb{C}$, and none of its poles are on ∂D, then the sum of its residues inside D is 0. [121] Also conclude that doubly periodic meromorphic functions with only a simple pole in D does not exist. [122]

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[119] By the theorem we proved for meromorphic functions over $\mathbb{C}^*$, $P(z)/Q(z) = P_\infty(z) + \sum_{i=1}^{m} P_{z_i}(z)$, where $P_\infty(z) = 0$ by long division, and the $P_{z_i}(z)$ is of first order. Then the rest is mere calculations of the residues.

[120] Note that f is continuous over the compact parallelogram $D \cup \partial D$, so it's bounded. Hence, f is bounded on the entire $\mathbb{C}$. Now we can use the usual Liouville's Theorem, which we might call **Liouville's Theorem 0**.

[121] Note that integral over opposite edges cancel as their directions are opposite and the functions are the same due to double periodicity.

[122] The Weierstrass $\wp(z)$ functions have a double pole inside its fundamental domain. These are the simplest doubly periodic meromorphic functions, which look pretty complicated.

Lesson 31 Summary

If $f(z)$ has an isolated singularity at z_0, then $\text{Res}[f(z), z_0]$, the residue of $f(z)$ at z_0, is defined to be the coefficient a_{-1} of the $\frac{1}{z-z_0}$ term in the Laurent series expansion of $f(z)$ over $0 < |z - z_0| < \rho$. The residue theorem relates this complex number to the line integral of this function.

The residue theorem.

(1) If $0 < r < \rho$, then $\displaystyle\int_{|z-z_0|=r} f(z)dz = 2\pi i \text{Res}[f(z), z_0]$.

(2) If domain D is bounded with piecewise smooth ∂D and f is analytic on $D \cup \partial D$ except at isolated singularities at $z_1, \cdots, z_m \in D$. Then $\displaystyle\int_{\partial D} f(z)dz = 2\pi i \sum_{i=1}^{m} \text{Res}[f(z), z_i]$.

We have seen the proof of (1) before. Simply expand $f(z)$ into its Laurent series. Switch integral with the infinite sum by uniform convergence, and then integrate each power $(z - z_0)^k$. We learned this when we first studied complex line integrals: only the integral of $(z - z_0)^{-1}$ is nonzero, and it is $2\pi i$. (2) can be proved by drawing little disks around each z_i and then consider the new region D' formed by taking off these disks and then apply Cauchy's Theorem to D', as f is analytic on D'. Thus, the integral over ∂D is the sum of the integrals over each of the little circles, which by (1), is the sum of $2\pi i$ times the residue at z_i.

So if we know the residues of $f(z)$ inside D, then we can evaluate $\int_{\partial} f(z)dz$ by using (2). But how do we find the residues? Of course, we can expand $f(z)$ into its Laurent series (or just look at a few terms containing the $\frac{1}{z-z_0}$ term) by manipulating series. However, if z_0 is a pole, then we have the following four rules:

(1) If $f(z)$ has a simple pole at z_0, then $\text{Res}[f(z), z_0] = \lim_{z \to z_0} (z - z_0)f(z)$.

(2) If $f(z)$ has a double pole at z_0, then $\text{Res}[f(z), z_0] = \lim_{z \to z_0} \dfrac{d}{dz}((z - z_0)^2 f(z))$.

(3) If $f(z)$ and $g(z)$ are analytic at z_0, and $g(z)$ has a simple zero at z_0, then $\text{Res}[f(z), z_0] = \dfrac{f(z_0)}{g'(z_0)}$.

(4) If $g(z)$ is analytic at z_0 and has a simple zero at z_0, then $\text{Res}[f(z), z_0] = \dfrac{1}{g'(z_0)}$.

(1) and (2) and Problem 3(a) are proved by writing out the Laurent series and then multiply by $(z - z_0)^{\text{order of } z_0}$ to get a power series. Then we find a_{-1} by doing a Taylor series coefficient computation. (3) follows from (1) by writing $(z - z_0)\frac{f(z)}{g(z)} = \frac{f(z)}{\frac{g(z)-g(z_0)}{z-z_0}}$ before taking the limit. (4) is a special case of (3).

In Lessons 32 and 33, we will use the Residue Theorem to help us evaluate some real integrals which may look impossible to calculus students. Afterwards, we will use the Residue Theorem to count the sum of the number of zeros and poles inside a region. Liouville's Third Theorem is related to this last point. Liouville's First and Second Theorems are Problem 5.

Lesson 32 The residue calculus – contour integrals

1. Consider the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+9}$.

   (a) Find its value using method from Calculus I.

   (b) Find its value using contour integral.

2. Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+9}dx = \frac{\pi}{3e^6}$. Consider integrating $\frac{e^{i2z}}{z^2+9}$.

3. Find the value of $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+9}dx$. Note that $\sin^2 x = \frac{1-\cos 2x}{2}$. So you can find the answer to this problem simply by using the answers to Problems 1 and 2.

4. Find the values of the following integrals.

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^5}dx$. [123]

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx$.

5. Some real variable integrals can be converted to complex contour integrals directly without a limiting process. This is particular so for integrals of rational functions of trigonometric entries. The simplest nontrivial case is the following
$$\int_0^{2\pi} \frac{d\theta}{a \pm \cos\theta},$$
where $a > 1$.

   (a) Show that the value of this integral is $\frac{2\pi}{\sqrt{a^2-1}}$. The whole proof for the "+" case is on pages 203 and 204 of our textbook. The conversion from the real to the complex is through $z = e^{i\theta}$ and note that $dz = iz d\theta$ and $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2}$.

   (b) Consider the Poisson kernel we saw in Lesson 1, which played a role when we studied harmonic functions in Lesson 14,
$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2},$$
which came from $P_r(\theta) = 1 + \frac{z}{1-z} + \frac{\overline{z}}{1-\overline{z}} = \frac{1-|z|^2}{|1-z|^2}$, where we then substitute $z = re^{i\theta}$. Use the result from (a) to show that for all $0 \le r < 1$,
$$\int_0^{2\pi} P_r(\theta)\frac{d\theta}{2\pi} = 1.$$

[123] Recall that you had already calculated the residue of the integrand at i. It's $35/(256i)$.

Lesson 32 Summary

Whenever we learn something new, we can check it against something we encountered before, and see if we can solve any puzzles we couldn't do or whose solution is cumbersome before. Someone said this before. I don't remember who. Perhaps Richard Feynman.

This applies to the residue theorem particularly: The residue theorem allows us to solve some real integrals over the entire number line or half of it by expressing it using complex line integral, which in turn can be solved by quoting the residue theorem. Today, we considered the situation $\int_{-\infty}^{\infty} f(x)dx$, where $f(x)$ is a rational function or a trigonometric function times a rational function which does not vanish at any real number. The strategy is to find a complex function $g(z)$, and then consider a domain D, which for today is the upper half disk centered at 0 with radius R, that could contain some isolated singularities of $g(z)$. Then on the one hand, $\int_{\partial D} g(z)dz$ can be calculated by using the residue theorem. On the other hand, $\int_{\partial D} g(z)dz$ is the sum of $\int_{-R}^{R} g(x)dx$ and $\int_{\Gamma_R} g(z)dz$, where Γ_R is the semi-circle with radius R. Typically, using the LM-estimate, the modulus of the second integral can be shown to be smaller than a constant times $\frac{1}{R^a}$, where $a > 0$. Then after taking the limit $R \to \infty$, this integral goes to 0. The first integral, containing $\int_{-R}^{R} f(x)dx$, which in the limit is what we want to find: $\int_{-\infty}^{\infty} f(x)dx$. Therefore, we have connected what we want to find to $2\pi i$ times the sum of the residues.

For example, when the integrand is a rational function $f(x) = \frac{P(x)}{Q(x)}$ where the degree of $Q(x)$ is at least 2 larger than the degree of $P(x)$, and $Q(x)$ is not zero on the real line, then to find $\int_{-\infty}^{\infty} f(x)dx$, we just let $g(z) = f(z)$, i.e, consider $\int_{\partial D} \frac{P(z)}{Q(z)}dz$, which can be evaluated using residue's theorem. The $\int_{\Gamma_R} \frac{P(z)}{Q(z)}dz$ part has modulus smaller than $\frac{\frac{3}{2}|a_m|R^m}{\frac{1}{2}|b_n|R^n}\pi R = \frac{3|a_m|\pi}{|b_n|}\frac{1}{R^{n-m-1}}$, where $P(z) = a_m z^m + \cdots + a_0$ and $Q(z) = b_n z^n + \cdots + b_0$, and by assumption $n \geq m+1$. Thus, once we let $R \to \infty$, $\int_{\Gamma_R} \frac{P(z)}{Q(z)}dz$ goes to 0. The other part of the line integral becomes $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}dx$, whose value is the residue result.

We also considered integral of the form $\int_{-\infty}^{\infty} \cos ax \frac{P(x)}{Q(x)}dx$ for some $a > 0$, where the degree of $Q(z)$ is again at least 2 larger than that of $P(z)$. In this case, using $\cos az \frac{P(z)}{Q(z)}$, would not be a good idea, because along the imaginary axis $z = iy$, $\cos az = \frac{e^{iaiy}+e^{-iaiy}}{2} = \frac{e^{-ay}+e^{ay}}{2}$, which grows unbounded as we enlarge the radius of the half circle. Instead, we use $e^{iaz}\frac{P(z)}{Q(z)}$. Then ML-estimate can be used to show that the integral over Γ_R goes to 0 as $R \to \infty$ as $|e^{iaz}| = |e^{ia(x+iy)}| = e^{-ay} \leq 1$. The other part $\int_{-R}^{R} e^{iax}\frac{P(x)}{Q(x)}dx$ then becomes $\int_{-\infty}^{\infty} \cos ax \frac{P(x)}{Q(x)}dx + i\int_{-\infty}^{\infty} \sin ax \frac{P(x)}{Q(x)}dx$. After equating with the residue result, we have both $\int_{-\infty}^{\infty} \cos ax \frac{P(x)}{Q(x)}dx$ and $\int_{-\infty}^{\infty} \sin ax \frac{P(x)}{Q(x)}dx$.

Sometimes, a real variable integral can be directly written as a complex line integral without doing ML-estimation. Problem 5 is such an example.

Lesson 33 The residue calculus cont'd

1. This is a review question: in class, we calculated the integral $\int_0^\infty \frac{x^{-a}}{1+x}dx$, $0 < a < 1$, for which we used the following two facts. Prove them by using ML-estimates.

   (a) $|\int_{\Gamma_R} \frac{z^{-a}}{1+z}dz| \le \frac{R^{-a}}{R-1} \cdot 2\pi R$, where Γ_R is a big circle centered at 0 with radius R.

   (b) $|\int_{\gamma_\epsilon} \frac{z^{-a}}{1+z}dz| \le \frac{\epsilon^{-a}}{1-\epsilon}2\pi\epsilon$, where γ_ϵ is a small circle centered at 0 with radius ϵ.

2. Prove that $\int_0^\infty \frac{x^a}{(1+x)^2}dx = \frac{\pi a}{\sin(\pi a)}$, where $0 < a < 1$. [124]

3. This problem concerns the fractional residue theorem and its application.

   (a) State and prove the fractional residue theorem.

   (b) Why doesn't the fractional residue theorem hold for poles of orders other than 1?

4. Use an indented upper half disk, the residue theorem, and the fractional residue theorem to show that PV $\int_{-\infty}^\infty \frac{1}{x^5-1}dx = -\frac{2\pi}{5}(\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5})$. [125]

5. Prove that $\int_{-\infty}^\infty \frac{\sin x}{x}dx = \pi$ by integrating $f(z) = \frac{e^{iz}}{z}$ along an indented upper half disk. You have solved the most difficult part in Problem 5 of Lesson 15. [126]

[124] The solution is on page 206-207 of our textbook.

[125] After applying the residue theorem and the fractional residue theorem to $f(z) = \frac{1}{z^5-1}$ and taking the limits, we have $\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz + I - \frac{\pi i}{5} = \frac{2\pi i}{5}(e^{\frac{2\pi i}{5}} + e^{\frac{4\pi i}{5}})$, where I denotes this principal value integral, $-\frac{\pi i}{5}$ is the fractional residue of $f(z)$ at $z = 1$ with negative angle. The right-hand-side is $2\pi i$ times the sum of the residues of $f(z)$ at the two fifth roots of unity inside the indented upper half disk.

[126] The full solution is on page 217-218 of our textbook.

Lesson 33 Summary

Today, we continued our study of contour integrals by focusing on two topics: (1) integrals with branch points, and (2) fractional residues.

When we evaluate integrals like $\int_0^\infty \frac{x^{-a}}{x+1}dx$, where $0 < a < 1$, by doing contour integral of $f(z) := \frac{z^{-a}}{z+1}$, we have to be careful about the numerator z^{-a}, which is defined as $e^{-a\mathrm{Log}z}$, for a suitable definition of a continuous branch of Log$z$. So whatever the region D we choose, it should lie in a slit plane. For this integral, we choose $[0, \infty)$ as the slit, and the region D is the keyhole region bounded inside by the small circle γ_ϵ with radius ϵ, outside by the large circle Γ_R with radius R, both centered at 0, and with two edges from ϵ to R, one slightly above the x-axis, and the other slighly below, so that D avoids $[0, \infty)$. Then we can apply the residue theorem to $f(z)$ over ∂D.

The important part of this process is the expression of z once we trace over ∂D. When we start by tracing along the top edge from ϵ to R, we can choose $z = xe^{i0} = x$, where x goes from ϵ to R. Then as we trace over Γ_R, $z = Re^{i\theta}$ where θ changes from 0 to 2π. So as we trace along the lower edge from R to ϵ, $z = xe^{i2\pi}$, where x decreases from R to ϵ. Finally, over γ_ϵ, $z = \epsilon e^{i\theta}$, and θ goes from 2π back to 0. Note that along the lower edge $z = xe^{i2\pi}$, where x decreases from R to ϵ, if we don't have the "fractional "power z^{-a}, we can just remove $e^{i2\pi}$ as it's just 1. However, if we plug $z = xe^{i2\pi}$ into $f(z)$, we have $\frac{(xe^{i2\pi})^{-a}}{1+xe^{i2\pi}} = e^{-i2\pi a}\frac{x^{-a}}{1+x}$, where $e^{-2i\pi a} \neq 1$. This will contribute a term to the contour integral in the equation from the residue theorem.

Now about fractional residues. Certain contours have to avoid a singularity of $f(z)$ on its way, by moving along an arc of a circle with angle α around the singular point. When this point is a simple pole, then as the radius of the small circular arc goes to 0, the integral along this small circular arc is $\alpha i a_{-1}$ where a_{-1} is the residue of $f(z)$ at this singular point, i.e., the coefficient of the -1th power in the Laurent series expansion of $f(z)$ at this simple pole. This is called the fractional residue theorem. If the arc is the full circle, then we recover the usual residue theorem for simple pole.

The fractional residue theorem is quite useful, in its flexibility with the choice of the angle α and thus of the choice of the shape of D. However, it doesn't hold for poles of orders higher than 1 in general. There is always a balance.

Lesson 34 The argument principle

1. Let $f(z) = (z-1)^{2023}(z-2)^{2024}(z-3)^{2025}(z-4)^{2028}$. Prove that
$$\frac{1}{2\pi i}\int_{|z|=5}\frac{f'(z)}{f(z)}dz = 90^2.$$

2. Use the argument principle to show that
$$e^z = 1$$
has one and only one solution inside the horizontal strip $-\pi < \mathrm{Im} z < \pi$. [127]

3. Suppose $f(z)$ is analytic on D and γ is a simple closed curve in D such that the value of $f(z)$ on γ is in $\mathbb{C}\backslash(-\infty, 0]$. Prove that $f(z)$ does not have any zero inside the curve γ.

4. Suppose D is bounded with piecewise smooth boundary ∂D. Let f be meromorphic on D and extends to be analytic on $D \cup \partial D$ such that $f(z) \neq 0$ on ∂D. Furthermore, let $g(z)$ be analytic on $D \cup \partial D$. Show that
$$\frac{1}{2\pi i}\int_{\partial D} g(z)\frac{f'(z)}{f(z)}dz = \sum_{i=1}^{n} N_i g(z_i),$$
where $z_1, \cdots, z_n$ are the distinct zeros and poles of $f(z)$ with order N_i. This is called the *generalized argument principle.* Its proof should be almost the same as that for the argument principle.

5. Here continues Problem 5 of Lesson 31. Recall that a function is doubly periodic if there are complex numbers ω_1 and ω_2 which do not lie on the same line such that $f(z+\omega_1) = f(z)$ and $f(z+\omega_2) = f(z)$. Let D be the parallelogram $\{z \big| z = t\omega_1 + s\omega_2, 0 < t, s < 1\}$. Prove the following **Liouville's Third Theorem**. [128]

   Suppose f is meromorphic and doubly periodic on $\mathbb{C}$ with neither zeros nor poles on ∂D. Then the number of zeros of f equals the number of poles of f, both counted with multiplicity.

[127] Consider the logarithmic integral of $f(z) = e^z - 1$ over the rectangle with vertices $\pm R \pm i\pi$ and then let $R \to \infty$.

[128] Apply the argument principle to $f(z)$ and also use the proof of Liouville's Second Theorem.

Lesson 34 Summary

The argument principle relates the number of zeros and poles of a meromorphic function (or analytic function if no poles) inside a region, which is number-theoretic, to the number of times a curve wraps around the origin, which is topological. These two seemingly unrelated notations are connected by analysis, using an integral appearing in the residue theorem.

In fact, the argument principle is a special application of the residue theorem, where the function integrated is $\frac{f'(z)}{f(z)}$. You may have recognized that $\frac{f'(z)}{f(z)}$ looks like the derivative of a logarithmic function: it is $(\mathrm{Log} f(z))'$ by the chain rule, and $\mathrm{Log} f(z)$ by definition is the sum $\ln|f(z)|+i\mathrm{Arg} f(z)$. That's where angle comes into play. There is a caveat there: $f(z)$ may not be in the domain of $\mathrm{Log}(\ )$, which is a slit plane, not the whole plane. We will come back to this later. On the other hand, the residue of $\frac{f'(z)}{f(z)}$ at a pole turns out to be the multiplicity of a zero or a pole of $f(z)$.

Here is a precise statement of the argument principle.

The argument principle. Let D be a bounded domain with piecewise smooth ∂D, and f a meromorphic function on D which extends analytically across ∂D such that $f(z) \neq 0$ on ∂D. Then

$$N_0 - N_\infty = \frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz = \frac{\Delta \arg f(\partial D)}{2\pi},$$

where N_0 is the total number of zeros of f inside D counted with multiplicity, N_∞ is the total number of poles inside D counted with multiplicity, and $\Delta \arg f(\partial D)$ is the total angle change if we trace along the curve $f(\partial D)$ once, and thus $\frac{\Delta \arg f(\partial D)}{2\pi}$ is the number of times $f(\partial D)$ wraps around the origin, called the winding number of $f(\partial D)$.

As $f(z) \neq 0$ over ∂D, $f(z)$ is always a nonzero vector, and thus $\arg f(z)$ always makes sense.

Let's prove both equalities. For the first, residue theorems says $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz$ is the sum of residues of $\frac{f'(z)}{f(z)}$ inside D. To see what a residue is, let z_0 be a zero or a pole of $f(z)$, so $f(z) = (z-z_0)^N g(z)$ for some analytic $g(z)$ with $g(z_0) \neq 0$ and some nonzero integer N. When N is positive, z_0 is a zero of order N. When N is negative, z_0 is a pole of order $-N$ (or we can just say it's a pole of order N depending on the context). Thus, $\frac{f'(z)}{f(z)} = \frac{N(z-z_0)^{N-1}g(z)+(z-z_0)^N g'(z)}{(z-z_0)^N g(z)} = \frac{N}{z-z_0} + \frac{g'(z)}{g(z)}$, where $\frac{g'(z)}{g(z)}$ is analytic near z_0 as $g(z_0) \neq 0$. Therefore, the residue of $\frac{f'(z)}{f(z)}$ at z_0 is N, the order of the singularity at z_0! Thus, summing over all the zeros and poles, we have the first equality.

For the second, as the antiderivative of $\frac{f'(z)}{f(z)}$ locally is $\log f(z)$, we have $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\log f(z)\big|_{z=z_0}^{z=z_f}$ where z_0 is any initial point on ∂D and z_f is the final point on ∂D as we traverse it once, which presumably is the same as z_0. However, what we really mean here is that we are keeping track of how $\log f(z)$ changes as we move along ∂D. $f(z_0)$ and $f(z_f)$ should be interpreted as living on the Riemann surface of $\log(\ )$. Furthermore, as $\log f(z) = \ln|f(z)| + i\arg f(z)$, and $\ln|f(z)|\big|_{z=z_0}^{z=z_f} = 0$, we have $\frac{1}{2\pi i}\int_{\partial D} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi}\Delta \arg f(\partial D)$, the total angle change along $f(\partial D)$ divided by 2π. Practically, to see how the angle accumulates, we can cut $f(\partial D)$ into several pieces such that each piece lies in a slit plane, then we can use an analytic branch of $\log z$ on this slit plane to calculate the angle change. Finally, we add them up, which is the total angle change.

Lesson 35 Rouché's Theorem

1. Prove that $2z^5 + 6z - 1$ has four roots in the annulus $1 < |z| < 2$.

2. Let D be the rectangle with vertices iR, $-R+iR$, $-R-iR$ and $-iR$. For any $R > 4$, prove that $e^z = z + 2$ has exactly one solution in D. [129]

3. Let's prove the **Fundamental Theorem of Algebra** one last time. Suppose
$$p(z) = a_n z^n + \cdots + a_0$$
is a polynomial of degree $n \geq 1$. Show that $p(z) = 0$ has a solution. [130] [131]

4. In Rouché's Theorem, f and h are assumed to be analytic on $D \cup \partial D$, so only zeros were considered. In this problem, poles are also considered. So let domain D be bounded with piecewise smooth ∂D, and suppose $f(z)$ and $h(z)$ are meromorphic on $D \cup \partial D$ with poles only in D. Furthermore, suppose
$$|h(z)| < |f(z)|$$
on ∂D. Show that the number of zeros minus the number of poles inside D are the same for both $f(z)$ and $f(z) + h(z)$. [132]

5. Prove **the symmetric version of Rouché's Theorem**: Let domain D be bounded with piecewise smooth ∂D. Suppose $f(z)$ and $g(z)$ are analytic on $D \cup \partial D$ and
$$|f(z) + g(z)| < |f(z)| + |g(z)|$$
over ∂D. Show that $f(z)$ and $g(z)$ have the same number of zeros in D. [133]

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[129] $f(z) = z + 2$, $h(z) = -e^z$.

[130] The proof is on page 230 of the textbook. To fill in more details, let $f(z) = a_n z^n$ and $h(z) = a_{n-1}z^{n-1} + \cdots + a_0$. Then $\lim_{z\to\infty} \frac{h(z)}{z^n} = 0$. Thus, responding to $\epsilon = |a_n|$, there is R large enough such that if $|z| \geq R$, $|\frac{h(z)}{z^n}| < |a_n|$, which means $|h(z)| < |a_n z^n| = |f(z)|$.

[131] In fact, the proof shows directly that $p(z) = 0$ has n solutions without using the iterative Euclidean algorithm.

[132] The proof is the same as that of Rouché's Theorem, except that the number of poles is added to the equation.

[133] Similar to the proof of the ordinary version of Rouché's Theorem, we write $f(z) = g(z)\frac{f(z)}{g(z)}$ and thus $\arg f(z) = \arg g(z) + \arg \frac{f(z)}{g(z)}$. On the other hand, divide $|g(z)|$ from both sides of $|f(z) + g(z)| < |f(z)| + |g(z)|$, then we have $|\frac{f(z)}{g(z)} + 1| < |\frac{f(z)}{g(z)}| + 1$. This shows $\frac{f(z)}{g(z)}$ has to fall in $\mathbb{C}\backslash[0, \infty)$, which can be shown by noticing that $|\frac{f(z)}{g(z)} + 1| = |\frac{f(z)}{g(z)}| + 1$ if $\frac{f(z)}{g(z)}$ is a nonnegative real number.

Lesson 35 Summary

The argument principle we discussed in Lesson 35 applies to meromorphic functions whose zeros and poles are not on the boundary ∂D of the bounded domain D. Rouché's Theorem is simply an application of this principle to the simpler analytic functions which can be decomposed into a big function and a small function. The theorem says the small function can be neglected if you want to count the number of zeros of the original function inside D. To be precise:

Rouché's Theorem. Let D be a bounded domain with piecewise smooth boundary ∂D. We also assume $f(z)$ and $h(z)$ are analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ on ∂D, then the number of zeros of $f(z) + h(z)$ and $f(z)$ in D are the same.

The proof contains an interesting topological idea. First of all, the inequality $|h(z)| < |f(z)|$ guarantees that no zeros of $f(z)$ and $f(z) + h(z)$ are on ∂D, which is a condition to quote the argument principle. If $f(z)$ is ever 0 on ∂D, then $|h(z)| < |0|$, which cannot be true. If $f(z) + h(z)$ can ever be 0, then $|h(z)| = |-f(z)| = |f(z)|$, contradicting $|h(z)| < |f(z)|$. Then, by the argument principle, the number of zeros inside D of $f(z)$ and $f(z)+h(z)$ are $\frac{\Delta \arg(f(\partial D)}{2\pi}$ and $\frac{\Delta \arg(f(\partial D)+h(\partial D))}{2\pi}$, respectively. However, as $f(z) + h(z) = f(z)(1 + \frac{h(z)}{f(z)})$, $\arg(f(z) + h(z)) = \arg f(z) + \arg(1 + \frac{h(z)}{f(z)})$. Since $|h(z)| < |f(z)|$, we have $|\frac{h(z)}{f(z)}| = \frac{|h(z)|}{|f(z)|} < 1$, and thus $\frac{h(z)}{f(z)}$ lies inside the unit disk. Hence, $1 + \frac{h(z)}{f(z)}$ lies in the unit disk centered at 1. In particular, it misses the slit $(-\infty, 0]$. Therefore, once z travels over ∂D once, $1 + \frac{h(z)}{f(z)}$ wraps around 0 zero times. So $\frac{\Delta \arg(f(\partial D)}{2\pi} = \frac{\Delta \arg(f(\partial D)+h(\partial D))}{2\pi}$, which implies that $f(z) + h(z)$ and $f(z)$ have the same number of zeros minus poles in D.

That $\frac{\Delta \arg(f(\partial D)}{2\pi} = \frac{\Delta \arg(f(\partial D)+h(\partial D))}{2\pi}$ above was proved partially by algebra and partially by analysis. However, if you model it using real-world examples, it's quite clear. Imagine $f(z)$ describes the motion of you on the plane as z goes over ∂D, and $h(z)$ the motion of a puppy relative to you. Since $|h(z)| < |f(z)|$ is always true, as z goes over ∂D once, the puppy $f(z) + h(z)$ and you $f(z)$ wrap around 0 the same number of times, no matter how wildly the puppy runs around you. Draw a picture, or just imagine it![134] On the other hand, we can also model $f(z)$ as the motion of the earth around the sun, and $h(z)$ as the motion of the moon relative to the earth. Then the winding number of the moon around the sun and that of the earth around the sun have to be the same for the same reason. [135]

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[134] I learned this in a complex analysis class taught by Steven R. Bell in graduate school.

[135] I learned this from an article by Donald G. Saari. The article quoted Wintner's 1941 book, *The Analytical Foundations of Celestial Mechanics*, which mentioned the astronomy origins of Cauchy and Rouché's discoveries in complex analysis.

Lesson 36 The Schwarz lemma and automorphisms of the unit disk

1. Prove[136] the following version of the Schwarz Lemma for disks not necessarily centered at the origin: Suppose f is analytic on $|z-z_0|<R$ and $|f(z)|\leq M$. If $f(z_0)=0$, then

   (a) $|f(z)| \leq \frac{M}{R}|z-z_0|$ if $|z-z_0|<R$.

   (b) If $|f(z')| = \frac{M}{R}|z'-z_0|$ for some z' in $0<|z-z_0|<R$, then $f(z)=\lambda(z-z_0)$ for some λ with $|\lambda|=\frac{M}{R}$.

2. The above version of the Schwarz lemma just assumes that z_0 is a zero f without specifying its order. If we do, then we get a sharper version. Prove the following: Suppose f is analytic on $|z-z_0|<R$ and $|f(z)|\leq M$. If z_0 is a zero of order m of $f(z)$, then

   (a) $|f(z)| \leq \frac{M}{R^m}|z-z_0|^m$ if $|z-z_0|<R$.

   (b) If $|f(z')| = \frac{M}{R^m}|z'-z_0|$ for some z' in $0<|z-z_0|<R$, then $f(z)=\lambda(z-z_0)^m$ for some λ with $|\lambda|=\frac{M}{R^m}$.

3. Prove[137] the following infinitesimal version of the Schwarz Lemma: Let f be analytic on $\mathbb{D}$. Suppose $|f(z)|\leq 1$ for each $z\in\mathbb{D}$ and $f(0)=0$, then

   (a) $|f'(0)|\leq 1$.

   (b) If $|f'(0)|=1$, then $f(z)=\lambda z$ for some λ with $|\lambda|=1$.

4. In the above infinitesimal version of the Schwarz Lemma, the condition $f(0)=0$ can be removed. Show that if f is analytic on $\mathbb{D}$ and $|f(z)|\leq 1$ for each $z\in\mathbb{D}$, then $|f'(0)|\leq 1$. [138]

5. Let's prove the following fact used in class: If $f: D\to\mathbb{C}$ is analytic and one-to-one on D, then $f'(z)\neq 0$ for all $z\in D$. (Note that this is false in calculus, as $f(x)=x^3$ is analytic and one-to-one on $\mathbb{R}$ but $f'(0)=0$.) [139]

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[136] It's on page 260 and 261 of our textbook. I would say adapting the proof of the Schwarz Lemma directly is easier.

[137] It's on page 261 of our textbook.

[138] Say $f(0)=a$. Then apply the above infinitesimal version of the Schwarz Lemma to $\varphi_a\circ f(z)$, where $\varphi_a(z)=\frac{z-a}{1-\bar{a}z}$.

[139] We can prove, for example, of the contrapositive. Suppose f is analytic on D and there is $z_0\in D$ such that $f'(z_0)=0$. Then it will turn out that f cannot be one-to-one. This can done as follows: Consider the power series expansion $\sum_{k=0}^{\infty}a_k(z-z_0)^k$ of f. As $a_1=f'(z_0)=0$, this power series becomes $f(z)=a_0+a_N(z-z_0)^N+\cdots$ for some $N\geq 2$. Thus, $f(z)=a_0+(z-z_0)^N h(z)$ where $h(z_0)\neq 0$. Similarly to what we did for Problem 3 of Lesson 27, we have $f(z)=a_0+g(z)^N$ with $g(z_0)=0$ and $g'(z_0)\neq 0$. Thus, by the inverse function theorem (Lesson 8), there are open sets U of z_0 and V of 0 such that $g: U\to V$ is one-to-one. So, for r and $re^{i2\pi/N}$ in V, there are $z_1\neq z_2$ in U such that $g(z_1)=r$ and $g(z_2)=re^{i2\pi/N}$. You see, the proof will be done after one more sentence.

Lesson 36 Summary

The automorphism group of the unit disk $\mathbb{D}$, denoted by $\mathrm{Aut}(\mathbb{D})$, consists of all bijective analytic maps from $\mathbb{D}$ to itself. Composition of functions, which is associative, serves as the group operation. The identity element is the identity map $z \mapsto z$. For any $f \in \mathrm{Aut}(\mathbb{D})$, $f^{-1} : \mathbb{D} \to \mathbb{D}$ is also analytic, and thus $f^{-1} \in \mathrm{Aut}(\mathbb{D})$. To see why f^{-1} is analytic, note that f being one-to-one implies that $f'(z) \neq 0$ for all $z \in \mathbb{D}$ (This is Problem 5). Thus, by the complex inverse function theorem, f^{-1} is locally and thus globally analytic. Because $f'(z) \neq 0$, we see that f is conformal. Indeed, $\mathrm{Aut}(\mathbb{D})$ is called the group of conformal self-maps of the unit disk.

Mathematicians are greedy, as they often want to find all things with a certain property. In the case of $\mathrm{Aut}(\mathbb{D})$, they are quite successful:

Theorem. $\mathrm{Aut}(\mathbb{D}) = \{\lambda \dfrac{z-a}{1-\bar{a}z} : |a| < 1, |\lambda| = 1\}$. So each element in this group is a special fractional linear transformation $\varphi_a(z) := \frac{z-a}{1-\bar{a}z}$ we studied in Lesson 1 followed by a rotation, and as a topological space, $\mathrm{Aut}(\mathbb{D})$ is a solid donut without glaze.

The proof of this theorem depends on a lemma with far-reaching consequences beyond Lesson 36, though its proof uses nothing but power series and the maximum principle.

The Schwarz Lemma. Let f be analytic on the open unit disk $\mathbb{D}$. Suppose $|f(z)| \leq 1$ for each $z \in \mathbb{D}$ and $f(0) = 0$. Then

(1) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

(2) If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $f(z) = \lambda z$ for some λ with $\lambda = 1$.

The power series part comes in the condition $f(0) = 0$, from which we get $f(z) = zg(z)$ for some analytic function $g(z)$ after factoring z out of the terms of the power series expansion of $f(z)$ at 0. So $|f(z)| = |z||g(z)|$. If we can show $|g(z)| \leq 1$, then we done. To see why this is the case, let $0 < r < 1$. And consider the circle $|z| = r$, on which we have $|g(z)| = \frac{|f(z)|}{|z|} = \frac{|f(z)|}{r} \leq \frac{1}{r}$. Thus, by the maximum principle, $|g(z)| \leq \frac{1}{r}$ over $|z| \leq r$. Therefore, for any $z \in \mathbb{D}$, $|g(z)| = \lim_{r\to 1} |g(z)| \leq \lim_{r\to 1} \frac{1}{r} = 1$, and hence $|g(z)| \leq 1$. Thus, (1) $|f(z)| \leq |z|$. To prove (2), note that if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $|g(z_0)| = 1$, which means the maximum value of $|g(z)|$ is attained in the interior of $\mathbb{D}$. Thus, $g(z)$ is a constant with $|g(z)| = 1$. Letting this constant be λ concludes the proof. A few generalizations of the Schwarz Lemma are proved in similar ways.

One application of the Schwarz Lemma is the following **Corollary**: If $f \in \mathrm{Aut}(\mathbb{D})$, and $f(0) = 0$, then $f(z) = \lambda z$ for some λ with $|\lambda| = 1$. The proof shows a typical way to deal with a function which can go both ways. If we apply the Schwarz Lemma to f, then $|f(z)| \leq |z|$. If we apply the Schwarz Lemma to f^{-1}, then we get $|f^{-1}(w)| \leq |w|$. Writing $w = f(z)$, we have $|z| \leq |f(z)|$. Thus $|f(z)| = |z|$, which by the second part of the Schwarz Lemma, says $f(z) = \lambda z$ for some λ with $|\lambda| = 1$.

Now the proof of the **Theorem** follows by using $\varphi_a(z)$, which is itself an element in $\mathrm{Aut}(\mathbb{D})$ as φ_a maps the unit circle to the unit circle as shown in Problem 4 of Lesson 1 and f maps a to 0 and thus the inside the circle to the inside the circle. For any $h \in \mathrm{Aut}(\mathbb{D})$, let $a = h^{-1}(0)$ and consider $h \circ \varphi_a^{-1}$, which is in $\mathrm{Aut}(\mathbb{D})$. Note that this composite map sends 0 to 0, so we can apply the above corollary to it, from which we get $h \circ \varphi_a^{-1}(w) = \lambda w$ and thus $h(z) = \lambda \varphi_a(z)$.