A Floatingly Falling Feline

Yongheng Zhang

2.8.2024

Abstract

This paper tells the story of a machine cat, a "kinetic sculpture" named in the title, made to artistically demonstrate some well-known solutions to the falling cat problem. Driven by servo motors, it drew inspiration from Alexander Calder's self-balancing but motor-free C(h)at Mobile.

Introduction

Once released by hand, an initially motionless cat with tummy pointing up would *quickly reorient itself while falling* so as to safely land on its feet. The whole procedure happens almost instantly, too fast for our brain to properly process the visual input and appreciate the cat's grace and ingenuity, and our fear of hurting the cat from the fall prevents us from even trying in the first place. Slow motion photography and zero gravity flights were effective technologies to aid our understanding, but I wondered: would there be another way?

Figure 1: *`Cat Mobile` @ MCA Chicago, 1966.*

Figure 2: *`AFFF` @ Amherst, 2024.*

Not sure if Alexander Calder could have thought about the same problem, I'm certain he is fond of cats. In his first published book, *Animal Sketching*, cats were the first about which Calder spent an entire chapter (two pages). He wrote: "*If it is alertly awake, get the attitude; all the muscles tense, ears erect, eyes observant, tail poised to give the best balance for a sudden spring.*" Calder's later kinetic sculpture Chat-Mobile, or `Cat Mobile` (Figure 1), currently archived at the Museum of Contemporary Art (MCA) Chicago, well captures a cat's potential energy while keeping a balance amid random movements generated by air currents. Calder's earlier mobiles, however, were driven by motors [1], so speed control to slow down becomes an option. If the mobiles were suspended by strings and their momenta were carefully regulated, then their centers of mass could be kept still. Thus, a floatingly falling feline (https://youtu.be/ncSC0xm6KEQ), called `AFFF` for short (Figure 2), was born. Its two cylindrical bodies simulate some well-known mathematical solutions to the falling cat problem, while all motions are generated by the "invisible" connecting middle body.

This paper is about `AFFF`. On the next three pages, its math, art, and their bridging roles will be discussed.

The Math

Mathematically, AFFF consists of two cylindrical bodies, the red front (+) cylinder and the white back (−) cylinder. Each cylinder can turn through an angle $\theta_\pm$ around its axis of symmetry, which in turn rotates vertically by an angle $\phi_\pm$ about a horizontal axis perpendicular to a stationary connecting middle body. Positive rotations of the front and back cylinders follow the right-hand rule and the left-hand rule, respectively. Figure 3 shows the angles in a simpler model afff of AFFF. An imaginary spine connects the two cylinders. It is generally assumed that the spine can bend, but it cannot twist. This translates to the relation $\theta_+ = \theta_-$. We also impose the further symmetry $\phi_+ = \phi_-$, which will be broken on the next page. Thus, AFFF has two degrees of freedom. Let the common value of $\theta_\pm$ be θ, and that of $\phi_\pm$ be ϕ.

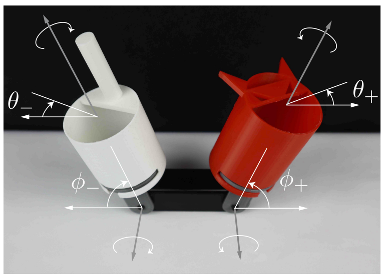


Figure 3: *Motor angles $\theta_\pm$, $\phi_\pm$.*

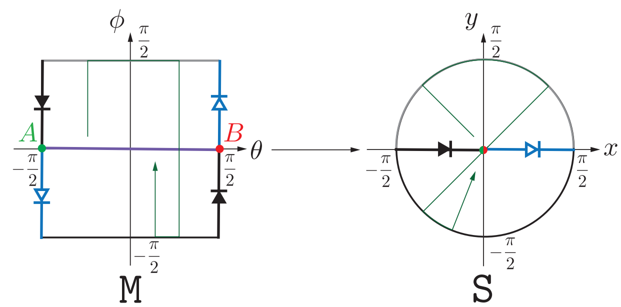
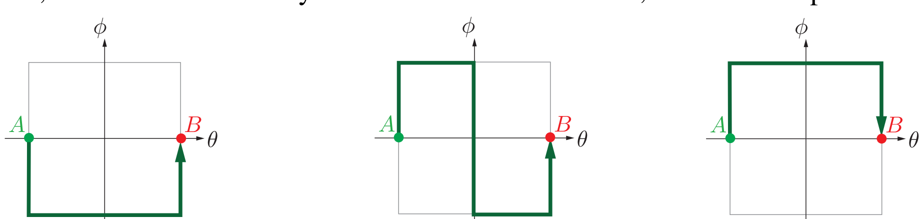


Figure 4: *The map $\pi : \mathtt{M} \to \mathtt{S}$, a path in M and its image in S.*

The values of $\theta_\pm$ and $\phi_\pm$ are controlled by four hobby servo motors. Each can rotate both forward and backward up to 90°. Thus, the configuration space, with both shape and orientation of AFFF taken into account, can be modeled by the motor space $\mathtt{M} := \{(\theta, \phi) \big| -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, -\frac{\pi}{2} \le \phi \le \frac{\pi}{2}\}$. The shape space $\mathtt{S} := \mathtt{M}/\mathtt{SO(3)}$, where SO(3) acts on M by rotating the entire AFFF, forms the base space of the projection $\pi : \mathtt{M} \to \mathtt{S}$. It is interesting to note that while M is a square, S can be modeled as a disk $\{(x, y) \big| x^2 + y^2 \le \frac{\pi^2}{4}\}$. In order to project M down to S (Figure 4), the left and right edges of M are glued in opposite directions to form a Möbius strip, and then the equator $\phi = 0$ (configurations of the straightened cat) is shrunk to a point. Alternatively, we can contract $\phi = 0$ to a point first to get two sectors, and then glue the two pairs of half edges. The use of the semiconductor diode symbols for the half edges is intentional: diode carries current only in one direction, and the “filled-in” symbols on the upper left and lower right half-edges of M fit the 1975 standard of IEEE, after which the acronym AFFF was modeled. Thus, it reads “A-triple-F”.



(a) Rademaker-der Braak (b) Kane-Scher (c) Montgomery

Figure 5: *Three solutions of the falling cat problem: https://youtu.be/YQEpPRinWtI*

A cat with tummy pointing upward (downward) corresponds to the point A (B) in M. A solution to the falling cat problem (for AFFF) is a path from A to B in M (and thus a closed loop based at the origin in S with a 180 degrees holonomy), satisfying the physical constraint that the total angular momentum is kept zero and the connecting middle body is kept skill. Figure 5 shows three solutions, due to Rademaker and der Braak, Kane and Scher [3], and Montgomery [4], respectively. The solution by Kane and Scher avoids backbend of the spine. It was used in the training of astronauts and athletes. The other two are simpler and symmetric.

Before leaving, it’s a delight to realize that there is a formula for π: $\pi(\theta, \phi) = (\phi \sin\theta, \phi\cos\theta)$.

The Art

"It was early one morning on a calm sea, off Guatemala, when over my couch—a coil of rope—I saw the beginning of a fiery sunrise on one side and the moon looking like a silver coin on the other. Of the whole trip this impressed me most of all; it left me with a lasting sensation of the solar system," recalled Calder in his 1966 *Autobiography with Pictures*. This precious experience in 1922 could be the genesis of Calder's art, an art not of isolated singular objects but of a dialogue between them [1], a unity among disparity.

Figure 6: *Adapted from Plate VIII-1 of Josef Albers, Interaction of Color https://interactionofcolor.com/*

Within walking distance from the MCA, down the avenue, and then along the river, Calder's 1974 monumental composition *The Universe* (https://calder.org/works/monumental-sculpture/universe-1974/) used to be installed in the lobby of the Sears Tower (now Willis Tower). It revived motorized elements and recapitulated earlier images [2], including the 1934 mobile *A Universe* (https://www.moma.org/collection/works/81054), whose association with Figure 6 is clear: coordinated motion of the red and white in darkness. It is said that white (bright), black (dark), and then red were the first three words for color in many languages. We just saw they were also the first for Calder. In `AFFF`, the wires from the battery and the two larger motors are also of the color white (signal[1]), black (ground), and red (power).

Figure 7: *"Nuts and bolts" of* `AFFF`*.*

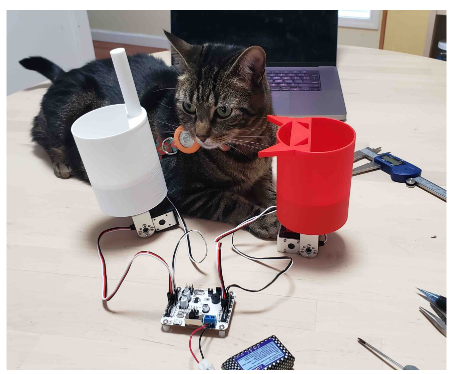


Figure 8: `AFFF` *in progress and muon.*

Figure 9: *Luminescence.*

The dark background is the central component. In `AFFF`, all motions were generated by the "invisible" connecting middle body, consisting of all pieces except the two 3D printed cylinders in Figure 7, playing the role of `Kuroko` (https://en.wikipedia.org/wiki/Kuroko). It's funny that the cat muon once appeared to connect the cylinders in Figure 8, reminding us that the falling cat problem is a translation of gauge theory used in particle physics to the macroscopic world [4]. Calder's cat further brought astronomy down to earth.

I also experimented with fluorescent materials for the two cylinders (Figure 9), but the glowing light diminishes much faster than the onboard battery drains its power. To break the boundary of its transient life, YouTube Short (https://youtube.com/shorts/EjbA8oN_lLc) was used to generate the effect of perpetual motion for `AFFF` to wrap the time interval $[0, \infty)$ around $\partial \mathtt{M}$ counterclockwise. Once the symmetry $\phi_+ = \phi_-$ was broken, `AFFF` was also able to perform acrobatics (https://youtu.be/xOIC7xJ8TMQ), as if experiencing Calder's Circus.

[1]Does the white cylinder remind you of *mobile* phones in the 1980s? An actual *Chat-Mobile* it is, when `ChatGPT` emerges.

And the Bridges

On December 26, 2023, I flew from Boston to Chicago just to see Cat Mobile at MCA. At admission, I learned that it was not on display, when the museum staff kindly offered a discount to make me not feel at a total loss. Back to the hotel, I immediately started to sketch the final version of AFFF in every detail, before returning Amherst to make measurements and build it up, while being content with studying the online pictures of Cat Mobile and continuing reading about Calder.

During the Summer of 2023, 7-year old Benjamin made the following art (Figure 10) using recycled materials in teacher Maya Lapping Rivera's art studio at Amherst. I'm very fond of these two cats, and was curious if they were a mom cat and a baby cat. He clarified that those are just a big cat and a small cat, and the smaller is not necessarily younger. Electrified, I realized that we should not limit ourselves by physical space and scale. When space is bridged with time, we could push the limit to gain new perspectives: the smaller could be a mom while the bigger a baby, and the smaller and the bigger could also be two versions of the same one. In the case of AFFF, the prototype afff I made in July 2023 is the mom of AFFF, and afff is a future simpler mathematical self of AFFF, all projected from spacetime to space alone (Figure 11).

Figure 10: *Small cat and big cat.*

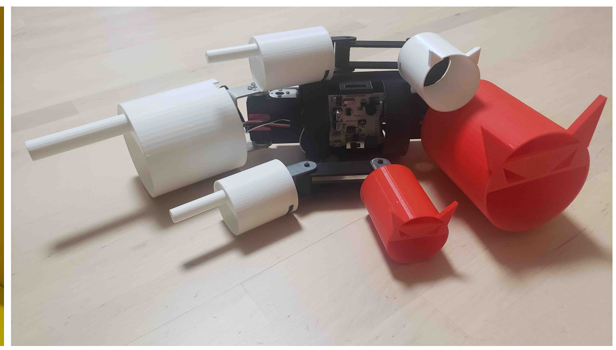


Figure 11: *AFFF, afff, and afff.*

AFFF was made during a span of eight months, from July 2023 to February 2024, but it started seventeen years ago, since January 2007, when I was a teaching assistant in a robotics and manufacturing lab during my last undergraduate semester. In the many years afterwards, I was (still) uncertain about the future. Sometimes, I drove to a beach in the industrial northwest corner of Indiana, building sandcastles, drawing formulas, watching the Chicago skyline across Lake Michigan, not aware of the modern and humble building hidden by the towering skyscrapers, nor expecting to be mathematically and artistically connected with a cat in it. The original idea of a fully automatic cat in 2007 was (much) more complicated. I do not know which would be better, but I adore its growth to be the simple AFFF as it is now, supported by everything, and everyone.

Acknowledgements

I would like to thank Benjamin, Donna, muon, and Terrence, for inspiration, criticism, and suggestion.

References

[1] J. Perl. *Calder: The Conquest of Time: The Early Years: 1898-1940*. Knopf, 2017.

[2] J. Perl. *Calder: The Conquest of Space: The Later Years: 1940-1976*. Knopf, 2020.

[3] G. J. Gbur. *Falling Felines and Fundamental Physics*. Yale University Press, New Haven, 2019.

[4] R. Montgomery. "Gauge Theory of the Falling Cat." *Fields Institute Communications*, vol. 1, pp. 193-218, 1993. https://montgomery.math.ucsc.edu/papers/cat.PDF