Large Multi-Unit Auctions with a Large Bidder

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Abstract

We compare equilibrium bidding in uniform-price and discriminatory auctions when a single large bidder (i.e., with multi-unit demand) competes against many small bidders, each with single-unit demands. We show that the large bidder prefers the discriminatory auction over the uniform-price auction, and we provide general conditions under which small bidders have the reverse preference. We use examples to show that the efficiency and revenue rankings of the two auctions are ambiguous.

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1 Introduction

Multi-unit auctions are often used to sell many units of a homogeneous good in markets with many buyers. Prominent examples include the markets for treasury bills, Initial Public Offerings of stock, and carbon emissions permits. Most of the multi-unit auctions used in practice are variants of the uniform-price (UP) or the discriminatory-price (DP) auction. While there is no explicit characterization of the equilibrium of either auction in a general setting, Swinkels [1999, 2001] shows that it is possible to characterize each equilibrium when the auction is large (i.e., it involves many bidders and objects) and all bidders demand a negligible amount of the total issuance. Asymptotically no bidder in Swinkels' model influences the market's clearing price (i.e., the lowest price at which goods are awarded) with her actions and hence no bidder has market power. Swinkels uses this observation to show that all bidders, as well as the seller, are approximately indifferent between the two formats.

Yet the presence of market power is an important feature in many large auction settings, and furthermore the degree of market power is not uniformly distributed across bidders. For example, Hortaçsu and Puller [2008] analyze the difference in bid behavior between small and large bidders in the Texas electricity sport market. While many bidders compete for the right to sell electricity, the largest bidder controls 24% of distribution. There is evidence that bidders have market power in U.S. Treasury auctions as well. Hortaçsu et al. [2015] report that primary dealers in U.S. Treasury auctions are allocated 46% to 76% of the competitive demand.[1]

We take a first step in augmenting a large auctions model to allow for market power. In our model, a single large bidder demands a non-negligible measure of the total issuance and competes against a continuum of small bidders. Small bidders are heterogeneous and have negligible demand when considered as a fraction of the total issuance. All bidders have private values. If we think of Swinkels [1999, 2001] as modeling perfect competition in a multi-unit auction, then we model a market akin to a monoposony.[2] Our modeling approach of studying a single large player who competes against many small players has previously been employed to study competition between a monopolist and "fringe firms" in an auction [Krishna, 1993]. Outside of the auctions literature, economists have used similar models to study price setting behavior by a dominant firm that competes against a competitive fringe [Carlton and Perloff, 1990].[3]

Although the revenue and efficiency rankings of the UP and DP auctions are generally ambiguous in our model, we obtain clear predictions for the bidders' preferences between the two formats. A straightforward argument establishes that the large bidder prefers the DP auction in this environment. Similar to the bidders in Swinkels' model, the small bidders' bids do not influence the

[1]Large bidders have more influence over the clearing price than small bidders by virtue of the number of units they purchase, but they may also have an informational advantage, a hypothesis that is supported by recent empirical research on treasury auctions [Hortaçsu and Kastl, 2012].

[2]There is a continuum of bidders in our model, but we use the term monopsony because only one bidder has market power, who can be understood as purchasing units from a residual supply curve.

[3]The modeling strategy of studying a game with a single large player and a continuum of infinitesimal small players has also been use by Menzio and Trachter [2015] to model competition among firms, Ekmekci and Kos [2016] to model a corporate takeover, and Corsetti et al. [2004] to model a currency attack.

clearing price in either format but do affect the price they pay in the DP auction. They consequently shade their bids below their values in the DP auction but not the UP auction. On the other hand, any serious bid by the large bidder determines the clearing price in both formats. We use this to show that the large bidder's marginal cost of winning additional units is always lower in the DP auction than the UP auction when the large bidder's small rivals bid according to an undominated strategy. Hence, the large bidder favors the DP auction. At the same time, we also provide general conditions under which small bidders typically have the reverse preferences over auction formats.

Bidder preferences over pricing rules have important practical implications. An immediate implication is that the bidder preferences can influence the choice of auction format. Maskin and Riley [2000a] report that "Similarly, in the lumber tract auctions in the Pacific Northwest, the local 'insiders' with neighbouring tracts have forcefully (and successfully) lobbied for open auctions and the elimination of sealed high-bid auctions" (pg. 425). They suggest that this outcome was the result of a strong bidder's preference for a first-price auction over a second-price auction, where "strong" means that their distribution stochastically dominates their opponent's in the reversed hazard rate order.[4]

Our model suggests that when comparing the UP and DP formats in a multi-unit setting size, not strength, determines bidder preferences. Instead, we show that the relative strength of bidders' distributions is important for comparing bidder preferences between the DP auction and the Vickrey auction. More precisely, we show that determining equilibrium bid behavior in the DP auction can be reduced to characterizing bid behavior in an asymmetric first-price auction. We also show that a similar connection exists between the Vickrey auction and the second-price auction. These connections allows us to use the results of Maskin and Riley [2000a] to show that a large bidder prefers the Vickrey auction to the DP auction when she is sufficiently strong relative to her rivals, and that small bidders have the reverse preference.[5]

While our main focus is on the case where there is a single large bidder with market power, we also study how our results extend to a setting with multiple large bidders. In an augmented model with two large bidders, we again show that determining bid behavior in the DP auction can be reduced to characterizing bid behavior in an asymmetric first-price auction. Furthermore, we can again use the connection between the DP auction and asymmetric first-price auctions to show that the relative strength of the large bidder determines her preference between the DP auction and the Vickrey auction. However, characterizing equilibrium bid behavior in the UP auction is made more difficult by allowing for multiple large bidders. To illustrate this difficulty, we prove an impossibility result about the existence of a tractable equilibrium in the UP auction when there are multiple privately informed large bidders.[6] Although this result inhibits us from presenting

[4] We define this ordering in Definition 1.

[5] Kirkegaard [2009] extends the results of Maskin and Riley [2000a] using a nice approach.

[6] By tractable equilibrium we are referring to an equilibrium in monotone pure strategies in which each bidder bids weakly below their valuation and each bidder's choice of clearing price is optimal after the opponents' private information is revealed. Equilibria satisfying these criteria are the focus of many of the papers in the literature that consider uniform-price auctions with privately informed bidders (cf., Wang and Zender [2002]) or decentralized market games in which uniform-price rules are used to select the market price with privately informed firms (cf.,

a general preference ranking of the two auction formats, we present two cases in which we can establish that the large bidder prefers the DP auction to the UP auction. In the first case, we show that large bidders strictly prefer the DP auction to the UP auction when large bidder capacities are sufficiently small.[7] Second, we consider the case in which there is no asymmetric information between large bidders.[8] We characterize equilibrium bid behavior in both auctions, and we use this characterization to show that a large bidder prefers the DP auction to the UP auction.

The remainder of the paper is organized as follows. Section 2 describes the main model. Section 3 gives our main results, and Section 4 gives an overview of our results on the two large bidder model. Section 5 concludes. Proofs and a more formal discussion of two large bidder model are found in the online appendix.

2 Model

A continuum of infinitesimal bidders of measure μ_s, referred to as the small bidders, compete in an auction for one unit of a divisible good against a large bidder with demand for a non-zero measure, μ_L, of the good. The small bidders are "single-unit bidders" each with value for an increment dq denoted by v_S and distributed according to the commonly known absolutely continuous distribution function $F_S(v_S) : [0, 1] \to [0, 1]$. We assume that the density, $f_S(v_S)$, is strictly positive on its support.

The large bidder has multi-unit demand and a constant marginal value for additional units. Her willingness to pay for each marginal unit is given by her type, v_L.[9] The quantity demanded by the large bidder is bounded above by her "capacity" μ_L. Thus, the large bidder gets zero marginal value from winning any quantity of units beyond μ_L. We assume throughout that $\mu_S + \mu_L > 1$ (i.e., that there is excess demand for all units) and $\mu_L \leq 1$. The large bidder's type is distributed according to the commonly known absolutely continuous distribution function $F_L(v_L) : [0, \overline{v}_L] \to [0, 1]$ with density $f_L(v_L)$, which we also assume to be strictly positive on its support.

We evaluate three pricing rules, the discriminatory-price rule (DP), the uniform-price rule (UP), and the Vickrey rule.[10] In all three auctions, a type-v_S small bidder submits a bid $b(v_S)$ and a type-v_L large bidder submits a nonincreasing function of q, $B(q, v_L) : [0, \mu_L] \times [0, \overline{v}_L] :\to \mathbb{R}_+$. Given the

Vives [2011]). See Section 4 for more discussion of these results.

[7] This is also a robustness check on Swinkels [2001]. We show that bidders with a small amount of market power strictly prefer the DP to the UP auction, and this finding is consistent with our main result.

[8] Examples of papers taking this approach towards characterizing equilibrium behavior in UP auctions include Wang and Zender [2002] and Ausubel et al. [2014].

[9] This flat demand assumption allows us to use results from the literature on asymmetric first-price auctions directly in our analysis of the DP auction. A couple of our results generalize easily to cases where the large bidder's demand is downward sloping, and we note these cases. Generalizing all of the results to the downward-sloping demand case would require an extensive analysis of a game similar to but distinct from an asymmetric first-price auction.

[10] Our analysis focuses on the UP and DP auctions, but we include the Vickrey auction as an efficient benchmark that is useful for comparison.

large bidder's type and the functions b and B, the total quantity demanded at price p is

$$Q(p; v_L, b, B) = \int_{b(v_s) \geq p} f_S(v_S)\, dv_S + \inf\{q | B(q, v_L) \geq p\}.$$

The clearing price, p^*, is determined by

$$p^* \equiv \sup\{p' | Q(p'; v_L, b, B) \geq 1\}.$$

In the UP auction, goods are awarded to all bidders with bids above the clearing price. The payment for each increment is $p^* dq$. In the DP auction payments are determined by bids directly, so a winning small bidder of type v_S gets payoff $v_S - b(v_S)$. The large bidder gets payoff,

$$\int_{B(q, v_L) \geq p^*} (v_L - B(q, v_L))\, dq.$$

In the Vickrey auction, the large bidder pays the integral sum of the defeated small bidders' values. The winning small bidders pay the clearing price.

3 Main Results

3.1 Bidder Preferences

Swinkels [2001] shows that bidders are indifferent between participating in a DP and a UP auction when no bidder has market power in a large auction. The Swinkels result holds because in each auction all bidders face a decision problem that is asymptotically similar when they have negligible demand and they face many rivals, even if bidders are ex ante asymmetric. In this section we show that bidders generally have clear preference rankings over the DP and UP auctions in our model where there is a large bidder with market power. As in Maskin and Riley's model, the bidder's preference is driven by ex ante asymmetries. Yet, like Swinkels [2001] we show that ex ante differences in bidder strength do not influence bidder rankings of the two auctions. Instead we find that a bidder's size — whether it be large or small — determines a bidder's preferences over the two auctions.

We first characterize equilibrium bid behavior in the UP auction. In the UP auction bid behavior is determined using two rounds of iterative elimination of dominated strategies. First, note that it is always a best response for a small bidder to bid truthfully. This is because the small bidder does not have any impact on the market clearing price. By bidding truthfully, the small bidder wins if and only if the market clearing price is below her value.

Remark 1. *In the UP auction, bidding $b_S(v) = v$ is a weakly dominant strategy for a small bidder.*

Remark 1 converts the large bidder's decision problem into a standard monopsony pricing problem. If the large bidder's highest losing bid is B,[11] the large bidder wins the quantity determined by

[11] We ignore complications in the choice of price that may arise from a discontinuous bid curve here because the

the residual supply curve at price B, $S(B) := \max\{0, 1 - (1 - F_S(B))\mu_S\}$. Thus, the large bidder's payoff for a given highest losing bid B is

$$\Pi_U(B, v_L) = S(B)(v_L - B).$$

The optimal bid maximizes the above expression. It is without loss of generality to assume that the large bidder submits a flat bid curve in this case, as only the highest losing bid affects the outcome of the auction. Note that the large bidder's bid is nondecreasing in her type.

The UP auction is inefficient because the large bidder buys too few units relative to the efficient benchmark. This is because the large bidder has a demand reduction incentive that is equivalent to a monopsonist's incentive to lower the market price below the perfectly competitive benchmark. Small bidders do not affect the price they pay when they win, and hence have no incentive to shade their bids. This is not true in the DP auction, which we analyze next.

In the DP auction, small bidders do not have a dominant strategy and they shade their bids. The large bidder again determines her bid by solving an optimization problem akin to that of a price-setting monopsonist. However, the residual supply available to the large bidder depends on the strategies of small bidders. In equilibrium the small bidders who do not win with probability one bid according to an increasing bid strategy $b_S(v)$ with inverse $\phi_S(b)$. Small bidders who win with probability one in equilibrium will all place the same bid. Thus, for these bidders, there is no well-defined inverse. In equilibrium these bidders always bid weakly higher than the large bidder and thus they can be ignored when evaluating the large bidder's objective. For the sake of exposition, we will talk about $\phi_S(b)$ as "the inverse bid function". The large bidder optimizes by considering the residual supply curve

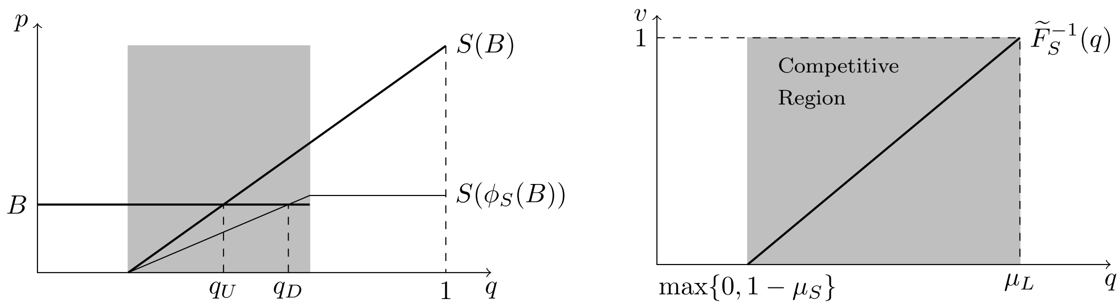
$$S(\phi_S(B)) := \max\{0, 1 - (1 - F_S(\phi_S(B)))\mu_S\}.$$

In the equilibrium of the DP auction, the large bidder submits a flat demand curve. A bid function that is strictly decreasing over some interval is never a best reply to any bid strategy of the small bidders. The large bidder receives a strictly higher payoff by submitting a flat demand curve that equals the clearing price. By submitting a flat demand curve that equals the clearing price, the large bidder wins the same number of units that she wins when she submits the strictly decreasing bid function, but pays a lower price per unit. This observation is important, because it allows us to parameterize the large bidder's bid function by a single dimensional variable — the value of her flat bid. Thus, given the small bidders' strategy ϕ_S and the large bidder's type t, the large bidder selects B to maximize Π_D where

$$\Pi_D(B, v_L; \phi_S) := S(\phi_S(B))(v_L - B).$$

The large bidder prefers the DP auction to the UP auction in a strong sense (i.e., for all un-

equilibrium bid curve will be a constant function.



(a) Greater Supply in the DP Auction

(b) Construction of $\widetilde{F}_S(v)$

Figure 1: Large Bidder's Residual Supply Curves

dominated strategies that may be used by the small bidders) because it is a dominated strategy for small bidders to bid at or above their value in the DP auction, but not in the UP auction. This implies that $S(B) \geq S(\phi_S(B))$. Thus, the large bidder is better off in the DP auction because she wins weakly more units for any flat bid B. We illustrate the large bidder's respective residual supply curves in Figure 1a. In the figure, we see that if a large bidder bids places the flat bid B for a μ_L quantity of units, then the large bidder wins q_D units in the DP auction and q_U units in the UP auction, where $q_D > q_U$. In each auction the large bidder pays a price of B per unit that she wins. An identical argument also shows that the large bidder prefers the DP auction over the UP auction in a setting where the large bidder has declining marginal values.

Proposition 1. *The large bidder gets a higher (interim) payoff from the DP auction than from the UP auction for all $v_L \in [0, \overline{v}_L]$.*

Proof. Suppose $\phi_S(b) \geq b \ \forall b$. Then, $S(\phi_S(B)) \geq S(B)$ because S is weakly increasing in B by construction. If $B^* \in \arg\max_B \Pi_U(B, v_L)$, clearly $B^* \leq v_L$, implying

$$\Pi_U(B^*, v_L) \leq S(\phi_S(B^*))(v_L - B^*) = \Pi_D(B^*, v_L; \phi_S) \implies \max_B \Pi_U(B, v_L) \leq \max_B \Pi_D(B, v_L; \phi_S), \ \forall v_L.$$

□

Next, we show that the DP auction is strategically similar to an asymmetric first-price auction between two bidders, and we use this for further insight into bidders' preferences over auction formats. To see the connection between our DP auction setting and the asymmetric first-price auction, consider an equilibrium of the DP auction where the large bidder bids a flat bid $B(v)$ and small bidders bid $b(v)$. In equilibrium, there is some interval (b_ℓ, b_h) on which bids are "competitive", meaning they win with probability strictly between zero and one. If a small bidder's value is such that her bid is in this open interval, then her bid wins if and only if her bid exceeds the flat bid of the large bidder.

Similarly, the large bidder wins a quantity $q \in (\max\{0, 1 - \mu_S\}, \mu_L)$ if and only if her (flat) bid is in the competitive interval (b_ℓ, b_h). Thus, the interaction between a large bidder, and a small bidder with value v such that $b(v) \in (b_\ell, b_h)$ is analogous to the interaction between two bidders in the corresponding asymmetric first-price auction. The large bidder's interim payoff is written identically to the interim expected payoff of a bidder in an asymmetric first-price auction who bids against a rival that employs bid strategy $b(v)$ and has value that is distributed over $[\phi_S(b_\ell), \phi_S(b_h)]$, where ϕ_S is inverse bid function mentioned earlier. We let $\widetilde{F}_S(v)$ be the conditional distribution of small bidder types in the interval $[\phi_S(b_\ell), \phi_S(b_h)]$, where

$$\widetilde{F}_S(v) = \frac{1}{\mu_L} - \frac{\mu_S}{\mu_L}(1 - F_S(v)),$$

. Note that the distribution $\widetilde{F}_S$ is determined by the exogenous quantities μ_S and μ_L (see Figure 1b). We conclude that the equilibrium bid functions in the discriminatory auction are equivalent to the equilibrium bids in an asymmetric auction where one bidder has distribution F_L and the other has distribution $\widetilde{F}_S$.

Lemma 1. *An equilibrium in the DP auction can be constructed from the equilibrium of a first-price auction between two bidders with values distributed according to* $\widetilde{F}_S(v)$ *and* $F_L(v)$*. Small bidders who are sure winners (i.e., those with value* $v > v_h$ *s.t.* $\widetilde{F}_S(v_h) = 1$*) bid the largest equilibrium bid from the constructed first-price auction, while any sure losers (i.e., those with value* $v < v_\ell$ *s.t.* $\widetilde{F}_S(v_\ell) = 0$*) bid their values.*

A similar construction connects the Vickrey auction to the equilibrium of a single-unit second-price auction. The logic is straightforward to see because it is an equilibrium to bid one's value on every marginal unit. Small bidders in the competitive region win if and only if their value exceeds the large bidder's. The large bidder wins a quantity that equals the fraction of small bidders in the competitive region that have values below the large bidder's value. This observation is stated in Lemma 2 below.

Lemma 2. *An equilibrium in the Vickrey auction can be constructed from a second-price auction between two bidders with values distributed according to* $\widetilde{F}_S(v)$ *and* $F_L(v)$*.*

Several corollaries follow from the connections between the DP and Vickrey auctions and their single unit auction counterparts. The first corollary establishes bidder preference between the Vickrey and UP auctions. Recall, that in the UP auction, the large bidder wins relatively fewer units than she does in the efficient outcome, and small bidders win relatively more frequently than they do in the efficient outcome.

Corollary 1. *The large bidder prefers the Vickrey auction to the UP auction. The small bidder prefers the UP auction to the Vickrey auction.*

In the literature on asymmetric first-price auctions, bidders are labeled "strong" or "weak" according to how their respective distributions are stochastically ordered. It is typical to use the reversed hazard rate order. This is defined below.

Definition 1. $F \succeq_{rh} G \iff \frac{F(x)}{G(x)}$ *is nondecreasing for all* $x \in [0, \max\{1, \overline{v}_L\}]$.[12]

[12]When $f(x)$ and $g(x)$ both exist, this implies $f(x)/F(x) \geq g(x)/G(x)$.

Maskin and Riley [2000a] show that in an asymmetric single-unit auction the weaker bidder prefers a first-price auction over a second-price auction, while the strong bidder has the reverse preference. Thus, Lemmas 1 and 2 imply that if the large bidder's distribution is strong relative to $\widetilde{F}_S(v)$ (i.e., $F_L \succeq_{rh} \widetilde{F}_S$), the large bidder prefers the Vickrey auction to the DP auction; and conversely if the large bidder's distribution is weak relative to $\widetilde{F}_S(v)$, the large bidder prefers the DP auction to the Vickrey auction.

Corollary 2. *If $\widetilde{F}_S(v) \succeq_{rh} F_L(v)$, then the large bidder prefers the DP auction to the Vickrey auction. If $F_L(v) \succeq_{rh} \widetilde{F}_S(v)$, then the large bidder prefers the Vickrey auction to the DP auction.*

In each case, small bidders have the reverse preference between the two auctions. If the small bidders prefer the Vickrey auction to the DP auction, which is the case when they are relatively strong, then a ranking between the UP auction and the DP auction follows immediately.

Corollary 3. *If $\widetilde{F}_S(v) \succeq_{rh} F_L(v)$, then the small bidders prefer the UP auction to the DP auction.*

Thus, we have provided conditions under which we can obtain a ranking of the DP, UP, and Vickrey auctions from the perspective of the large bidder. For the small bidder, we have shown that a UP auction is preferred to a Vickrey auction, and we have given conditions that determine whether the small bidders prefer the DP auction to the Vickrey auction.

Example 1 gives an illustration of the large bidder's ranking of the three auctions.

Example 1. Suppose that $\mu_L = \mu_S = 1$ and let

$$F_L(v_L) = \left(\frac{v_L}{2\alpha}\right)^{\frac{\alpha}{1-\alpha}}$$

where $0 < \alpha < 1$ and $0 \leq v_L \leq 2\alpha$. Also let the small bidder's values be uniformly distributed on $[0, 1]$. Let B_A, Π^*_A with $A \in \{U, D, V\}$ be respectively the equilibrium bid function and interim expected payoffs of the large bidder. Similarly, we let $S_A(v_L)$ be the total surplus generated by each format conditional on the large bidder's type. In equilibrium, we find that

$$\begin{array}{lll} B_U(v,q) = \dfrac{v}{2} & \Pi^*_U(v) = \dfrac{v^2}{4} & S_U(v) = \dfrac{1}{2}\left(1 + \dfrac{3v^2}{4}\right) \\ B_D(v,q) = \dfrac{v}{2} & \Pi^*_D(v) = \dfrac{v^2}{4\alpha} & S_D(v) = \dfrac{1}{2}\left(1 + \dfrac{(4\alpha-1)v^2}{4\alpha^2}\right) \\ B_V(v,q) = v_L & \Pi^*_V(v) = \dfrac{v^2}{2} & S_V(v) = \dfrac{1}{2}\left(1 + v^2\right) \end{array}$$

and that the small bidders bid according to $b_D(v) = \alpha v$ in the DP auction. For any v_L the revenue in the UP auction is $B_U(v_L, q)$, making the expected revenue $\alpha/2$. The expected revenue in the DP auction is greater at $2\alpha/3$.

A difficulty arises in determining the small bidders' ranking between the DP and UP auctions. In Example 1 all small bidders receive a higher interim expected payoff in the UP auction than they do in the DP auction. This is because the large bidder bids according to the same bid function in both auctions, but small bidders shade their bid in the latter, but not in the former. Thus, a small bidder with type v wins with a higher probability in the UP auction, $q_U(v)$, than the DP auction, $q_D(v)$. If we let π_U and π_D be the interim expected payoff of the small bidder in the two auctions, then a standard envelope theorem argument shows that $q_U(v) \geq q_D(v), \forall v$ implies $\pi_U(v) \geq \pi_D(v), \forall v$.[13]

While in Example 1 all types of small bidders have higher interim win probabilities in the UP auction than the DP auction, this is not true generally, preventing us from using an envelope theorem argument to rank the auctions for the small bidders. In Example 4 in the Appendix, we show that when the distribution of small bidders who have values in the competitive region ($\widetilde{F}_S$) is sufficiently weak relative to the large bidder's, then some types of small bidders can have greater interim win probabilities in the DP auction.[14] However, in Example 4 we still see that all types of small bidders prefer the UP auction to the DP auction, even though higher types of small bidders win with greater probability in the latter.

Although we are able to construct examples where a small bidder prefers the DP auction to the UP auction (see Example 5 in the Appendix), we find that this is not typically the case and we provide a sufficient condition under which all small bidders prefer the UP auction to the DP auction. To introduce the sufficient condition for comparing the small bidders' preferences between the two auctions, let $\phi_L(b)$ be large bidder's inverse bid function in the DP auction. Let $b(v)$ be the bid small bidders' equilibrium bid function in the DP auction. Thus, a type-v small bidder wins in the equilibrium of the DP auction if and only if the large bidder's value is below $\phi_L(b(v))$.

While the value of $\phi_L(b(v))$ cannot generally be given in closed form, the analogous quantity in the UP can be. A type-v small bidder wins a unit in the UP auction if and only if the large rival has a value below $\tau(v)$ where τ is derived by studying the large bidder's first-order condition.

Using the large bidder's first-order condition for the UP auction, the type of large bidder that a type-v small bidder ties with in the UP auction is defined in terms of primitives as[15]

$$\tau(v) \equiv \begin{cases} 0 & v < v_\ell \\ \min\left\{ v + \frac{1-\mu_S(1-F_S(v))}{\mu_S f_S(v)}, \overline{v}_L \right\} & v \geq v_\ell. \end{cases}$$

[13] The envelope theorem implies that $\pi_A(v) = \int_0^v q_A(v)\, dv$ for auction A.

[14] This phenomenon occurs generally whenever the large bidder's capacity μ_L is small. When μ_L is small in the UP auction and the large bidder is a high type (i.e., a type near $\overline{v}_L$), then the large bidder typically exhausts her capacity and reports a bid that ensures she wins μ_L units. However, similar to a first-price auction, it is only the highest type of large bidder that purchases the full amount μ_L in the DP auction. Any other type purchases less than the full amount. This implies that the higher types of small bidders must be outbidding more large bidder types in the DP auction when μ_L is small. Hence, they can have a higher interim win probability in the DP auction.

[15] The small bidder value v_ℓ and v_h are the cutoff values for sure winners and sure losers defined in Lemma 1. Note that the set of sure winners and losers is the same in either auction. In this definition we implicitly require that $\tau(v)$ is nondecreasing, which we assume in Condition 1. This is analogous to the requirement that the marginal cost associated with purchasing from the large bidder's residual supply curve in the UP auction is nondecreasing. Finally, note that it is possible that $\tau(v)$ jumps at v_ℓ and/or v_h. With a jump at v_ℓ, a large bidder with type $0 < v_L < \tau(v_\ell-)$ bids zero. With a jump at v_h, a large bidder with type $\tau(v_h+) < v_L < \overline{v}_L$ places the same bid as the $\tau(v_h+)$ type.

The condition we provide guarantees that the following inequality holds, which implies that a small bidder with type v is weakly better off in the UP auction compared to the DP auction.

$$\int_{v_\ell}^{v} F_L(\tau(x))\, dx \geq \int_{v_\ell}^{v} F_L(\phi_L(b(x)))\, dx.$$

If this holds for all $v \in [v_\ell, v_h]$, then this is equivalent to saying that the distribution $F_L(\phi_L(b(x)))$ second-order stochastically dominates the distribution $F_L(\tau(x))$. We show that second-order stochastic dominance holds in this environment under the following sufficient condition.

Condition 1. *For $v \in [v_\ell, v_h]$, either*

(i) $\widetilde{F}_S(v) \succeq_{rh} F_L(v)$; or

(ii) $F_L(v) \succeq_{rh} \widetilde{F}_S(v)$, $\tau(v)$ is nondecreasing and $\widetilde{F}_S(v) \succeq_{rh} F_L(\tau(v))$.

Corollary 3 implies that the small bidders prefer the UP auction to the DP auction when (i) holds in Condition 1. This is because when the small bidders in the competitive region are strong (in the reverse hazard rate sense) relative to the large bidder, they bid less aggressively than the large bidder in the DP auction, winning less than the efficient quantity. In contrast, they win more than the efficient quantity in the UP auction, because the large bidder shades her bid. Thus they have higher interim win probabilities in the UP auction and favor that format.

We can also establish that the small bidders prefer the UP auction to the DP auction when the second part of Condition 1 holds. The second part of Condition 1 requires that the small bidders are strong compared to the hypothetical large bidder with type distribution $F_L(\tau(v))$. Given the definition of $\tau(v)$, the small bidders are indifferent between facing a large bidder in the UP auction with type distribution $F_L(\tau(v))$ and a large bidder in the Vickrey auction with type distribution $F_L(v)$. It follows from Corollary 2 that if $\widetilde{F}_S(v) \succeq_{rh} F_L(\tau(v))$ the small bidders prefer competing against a large bidder with type distribution $F_L(v)$ in the UP auction to competing against a large bidder with type distribution $F_L(\tau(v))$ in the DP auction.

In the proof of Proposition 2 we strengthen this statement by showing that under Condition 1 the small bidders prefer competing against the large bidder with type distribution $F_L(v)$ in the UP auction to competing against the large bidder with the *same* type distribution in the DP auction. The argument proceeds in two steps. First, we argue that under the condition $\tau(v)$ may only cross $\phi_L(b(v))$ once from above, and second that the mean of the random variable with distribution $F_L(\phi_L(b(x)))$ (or equivalently the highest bid made in the DP auction) is weakly larger than the mean of the random variable with distribution $F_L(\tau(x))$ (or the expected payment of the highest small type in the UP auction). Together these facts imply that the random variable with distribution $F_L(\tau(v))$ second-order stochastically dominates the random variable with distribution $F_L(\phi_S(b(v)))$, and this is sufficient to determine the small bidders' preferences.

Proposition 2. *Under Condition 1, the small bidders prefer the UP to the DP auction.*

The following example illustrates a case in which the second part of Condition 1 applies.

Example 2. Suppose that there is a unit measure of small bidders, and the large bidder has capacity for all availble units, $\mu_L = \mu_S = 1$. In addition, suppose that small bidder values have distribution $U[0, 2]$ and the large bidder's values have distribution $U[0, 3]$. In this case, the large bidder is stronger than small bidder and hence part (i) of Condition 1 does not hold. However, part (ii) of Condition 1 holds because $\tau(v) = 2v$ and hence $F_S(v) = \widetilde{F_S}(v) = .5v$ and $F_L(\tau(v)) = \frac{2}{3}v$. Thus, Condition 1 holds and Proposition 2 implies that small bidders prefer the UP auction to the DP auction, even when their distribution is weak relative to their large rival.

3.2 Revenue and Efficiency Ambiguity of UP and DP Auctions

Ausubel et al. [2014] uses examples to show that the revenue and efficiency rankings of the UP and DP auctions are ambiguous. We obtain similar results in our large auction setting.[16]

In Example 1, the DP auction gives greater revenue than the UP auction. This is immediate to see as the clearing price is the same in both auctions (the large bidder bids the same flat bid in both), and all small bidders who win units pay an amount above the clearing price in the DP auction. In addition, if we assume that $\alpha = \frac{1}{2}$ in Example 1, then the DP auction is efficient. This is because equilibrium bid behavior is equivalent to bid behavior in a symmetric first price auction. At the same time, the UP auction is inefficient because the large bidder engages in bid shading.

In Example 3 below, we illustrate that the two auctions have ambiguous revenue and efficiency rankings by constructing a case where the UP auction is more efficient (yields greater expected surplus) and has greater expected revenue when compared with the DP auction.[17]

Example 3. Suppose that $\mu_S = \mu_L = 1$ and the large bidder's value, v_L, takes values in $\{0, 2\}$ with probability $\frac{1}{2}$ each.[18] Small bidders values are uniformly distributed over $[0, 1]$. In the UP auction, the large bidder bids 0 if $v_L = 0$, and any bid greater than or equal to 1 is a best response if $v_L = 2$. Therefore, expected revenue is $\frac{1}{2}$. In the DP auction, the large bidder bids 0 if $v_L = 0$. Thus, all small bidders know that they win with probability of at least $\frac{1}{2}$ if they bid any amount $\epsilon > 0$. Moreover, no small bidder submits a bid above $\frac{1}{2}$ because

$$\lim_{\epsilon \to^+ 0} \frac{1}{2}(v - \epsilon) = \frac{v}{2} \geq v - \frac{1}{2} \geq p(v - \frac{1}{2}) \; \forall p \in [0, 1], \; v \in [0, 1].$$

[16] While we cannot generally rank the revenues of the DP auction and the Vickrey auction, the connection to the first- and second-price auctions suggests that existing work on the revenue raised by asymmetric first-price auctions [Maskin and Riley, 2000a, Kirkegaard, 2012] would apply here. When all small bidders are in the competitive region and all of the large bidder's bids compete with small bidders (i.e., $\mu_L = \mu_S = 1$), the revenues generated by the DP and Vickrey auctions are the same as the revenues from the corresponding single-unit auctions. Note that in this case $\widetilde{F_S}(v) = F_S(v)$. However, with small bidders that are sure winners and/or sure losers, this is no longer the case, because the strategies are determined using the $(F_L, \widetilde{F_S})$ single-unit auctions for which the corresponding revenues will differ from the DP and Vickrey auctions. Consider the fact that the revenue from the $(F_L, \widetilde{F_S})$ single-unit auction does not incorporate payments from sure winners. We therefore do not get immediate revenue rankings in these cases.

[17] Example 1 already illustrates the ambiguous efficiency rankings of the two auctions. While the DP auction generates greater ex post surplus when $\alpha = \frac{1}{2}$, the UP auction generates greater ex post surplus when $\alpha < \frac{1}{3}$.

[18] This violates our assumption that the large bidder has values that are draws of a random variable that has density f_L with full support over an interval. However, we could construct an almost equivalent example where the large bidder has type v_L that is the draw of a random variable that has an associated density f_L that has full support over $[0, 2]$, yet arbitrarily large density near 0 and 2 and arbitrarily small density over the interval $(\epsilon, 2 - \epsilon)$.

The large bidder never submits a bid above $\frac{1}{2}$ because small bidders never bid above $\frac{1}{2}$. Since small bidders never bid above their value, then an upper bound on the small bidders' bid function is $\overline{b}(v) = \min\{v, \frac{1}{2}\}$. Thus, if the large bidder has value $v_L = 0$, the upper bound on revenue is

$$\int_0^1 \overline{b}(v)dv = \frac{3}{8}.$$

If the large bidder has value $v_L = 2$, the upper bounds on revenue is $\frac{1}{2}$. Therefore, expected revenue of the DP auction bounded above by $\frac{7}{16}$.

Thus, we see that there are cases where the UP auction has higher revenue than the DP auction. Notice also that the outcome of the UP auction is Pareto efficient for any realization of the large bidder's value. If $v_L = 0$, the large bidder's value for additional units is below the value of all small bidders, and the large bidder does not win any units. If $v_L = 2$, the large bidder's value for additional units exceeds any small bidder's value for units, and the large bidder wins all units. At the same time the DP auction is not Pareto efficient, because when the large bidder has value $v_L = 2$ she does not win all units with probability one. Thus, there is a positive probability there are Pareto improving trades where the large bidder buys units from small bidder. This differs from the revenue and efficiency rankings presented in Example 1. Therefore Examples 1 and 3 show that the two auctions have ambiguous revenue and efficiency rankings.

4 Extensions with Multiple Large Bidders

We study three extensions of our benchmark model to show how our results extend to settings with multiple large bidders. In each case, we assume that there are two ex ante identical large bidders that each have capacity $\mu_L > 0$ and a privately known constant value for all marginal units up to the capacity. Large bidder i's private value v_{Li} is drawn independently from the distribution F_L. We continue to assume that there is a continuum of small bidders distributed according to distribution F_S, and each small bidder has infinitesimal demand. We give an overview of our results on the multiple large bidder case below, and we provide a complete discussion of these results in the online appendix.

We first study a model where each large bidder has capacity for all available units $\mu_L = 1$. Similar to the case with one large bidder, we show that an equilibrium in the DP auction can be constructed by studying a corresponding asymmetric first-price auction involving three bidders, two of whom have the same distribution (Proposition 3).[19] Similarly, an equilibrium in the Vickrey auction can be constructed from a corresponding single-unit second-price auction. From these results we find that the large bidders prefer the DP auction to the Vickrey auction if the large bidders have a weaker type distribution than small bidders, and that in this case the small bidders have the reverse ranking. The preferences of each type of bidder reverse if the large bidders have stronger type distributions.

[19] The arguments in Proposition 3 extend to a setting with an arbitrary number of large bidders.

While our comparison of the large bidder's preference over the DP auction and the Vickrey auction can be extended to a setting with multiple large bidders, we show that finding an equilibrium of the UP auction is more difficult. Toward this end, we provide an impossibility result. To be more preicse, we search for a pure-strategy equilibrium satisfying properties guaranteeing that the clearing price is optimal for each large bidder ex post, meaning a large bidder would not choose a different clearing price after the private information is revealed. We prove that no such equilibrium exists in our model with multiple privately informed large bidders (Proposition 4). Equilibria with these properties are the focus of much of the literature on UP auctions and related games, due to their tractability.[20] If the UP auction has an ex post equilibrium, then it must be that the large bidder's bid curve is always an ex post best reply to her rival. In other words, the large bidder reports a bid curve that selects her preferred price-quantity pair for every realization of her rival's type. We show that any candidate solution to this pointwise maximization problem would violate the monotonicity restriction on bid curves in the UP auction. Hence there is no ex post equilibrium of the UP auction.[21]

Although Proposition 4 impedes a general characterization of bid behavior in the UP auction, we are able to determine the large bidders' preferences over the DP and UP auctions in two cases. In the first, large bidders do not demand all available units. We show that if large bidders have sufficiently small capacity, then each large bidder gets a greater expected payoff in the DP auction than the UP auction (Proposition 5). We obtain a ranking of large bidder preferences over the two auctions by placing bounds on bid behavior, instead of directly characterizing equilibrium. In the UP auction we assume that the large bidders play a strategy that is undominated given that her small rivals bid truthfully. We are able place a bound on the set of all bid strategies that are undominated given that the large bidder's small rivals bid truthfully. We use this bound on bid behavior to obtain an upper bound on a large bidder's expected payoff in the UP auction. We then show that the upper bound on the large bidder's expected payoff in the UP auction is below a lower bound on her expected payoff in the DP auction. Thus, the large bidder prefers the DP auction to the UP auction when the large bidder has sufficiently small capacity. This result is also a robustness

[20] Wang and Zender [2002] is an example of such a paper studying uniform-price auctions. When they consider privately informed bidders they focus on equilibria that are ex post optimal. The route they take is slightly different from ours. They search for Bayesian Nash equilibria in which there exists a sufficient statistic of opponent's information uniquely determining the clearing price and show that these equilibria must be ex post optimal (see pg. 686). Vives [2011] studies a market in which privately informed firms submit supply curves against a known demand curve. The rules determining the market price and the payoffs are analogous to a uniform-price auction. Vives [2011] shows that the equilibrium price reveals the relevant information contained in the signals of the opposing firms. An implication is that each bidder's choice of clearing price is optimal given other bidders' information.

Back and Zender [1993] describe another type of equilibrium in the UP auction. They show that it can be an equilibrium for private informed bidders in a divisible good UP auction to bid in such a way to split the quantity auctioned at a very low price. Bidders in the Back and Zender [1993] construction bid very high for their split of the good and drop their demand to the low price after (see their Theorem 1). This construction works by presenting any opposing bidder with a residual supply curve that has a vertical marginal cost at the quantity they are intended to purchase. Such construction does not work in our model, because the presence of the small bidders puts a lower bound on the slope of the residual supply curve in the UP auction.

[21] A Bayesian Nash equilibrium that is not an ex post equilibrium necessarily involves endogenously binding monotonicity constraints. See the Appendix for more discussion of this point.

check of Swinkels [2001], since it establishes that the large bidder strictly prefers the DP auction over the UP auction when both large bidders have small, but non-negligible, capacities.

Finally, we establish that large bidders have higher equilibrium payoffs in the DP auction than they do in the UP auction if there is no asymmetric information among large bidders. This information structure is analogous to others used in the literature on multi-unit auctions.[22] In this setting, we can explicitly characterize equilibrium behavior in both auctions (although there are multiple equilibria in the UP auction), and we show that the large bidders' payoffs in the equilibrium of the DP auction may exceed the large bidders' payoffs in the UP auction (Proposition 6).

5 Conclusion

We introduce a tractable model of large multi-unit auctions with a single large bidder. We show that the large bidder has a clear preference for a discriminatory pricing rule compared to a uniform pricing rule. We give sufficient conditions under which small bidders have the have the reverse preference and prefer the uniform-price auction. We also give examples that show that revenue and efficiency comparison between the uniform-price and discriminatory-price auctions are generally ambiguous.

Our model is tractable in comparison to more general models of multi-unit auctions. This is because we are able to reduce the discriminatory-price auction to a asymmetric first-price auctions for a single-unit, even when there is more than one large bidder. Therefore, this model may prove fruitful in future research.

A Online Appendix

A.1 Proofs and Examples Omitted from Paper

Proof of Lemma 1 Let $\widetilde{F}_S(x) = \frac{1}{\mu_L} - \frac{\mu_S}{\mu_L}(1 - F_S(x))$. This is a distribution function for an appropriately defined support, $[v_\ell, v_h]$. If $\mu_S > 1$ choose v_ℓ to solve $\widetilde{F}_S(v_\ell) = 0$, setting $v_\ell = 0$ otherwise. Note $\widetilde{F}_S(x)$ has a mass point at $x = 0$ if $\mu_S < 1$. Choose v_h to solve $\widetilde{F}_S(v_h) = 1$.

Temporarily ignore the small bidders with valuations outside of the interval $[v_\ell, v_h]$. Observe that it is a weakly dominant strategy to submit a flat bid curve for the large bidder,[23] and consider the first-price single-unit auction with two bidders in which the type distributions are F_L and $\widetilde{F}_S$. An equilibrium of this auction exists in which there is a common maximum bid $\bar{b}$ [Maskin and Riley, 2000b, 2003, Lebrun, 1999]. Let $(B, \widetilde{b})$ be the equilibrium bid functions. Define b on $[0, 1]$ by setting $b(v) = \widetilde{b}(v)$ for $v \in [v_\ell, v_h]$, $b(v) = v$ for any $v \leq v_\ell$ and $b(v) = \bar{b}$ for any $v \geq v_h$. We claim that this

[22]For example, both Ausubel et al. [2014] and Wang and Zender [2002] emphasize analyses of cases in which bidders are not asymmetrically informed. In the Industrial Organization literature, Klemperer and Meyer [1989] is a prominent example of a market game analogous to a UP auction in which firms submit supply curves and a uniform clearing price is determined. In their model, demand is allowed to be stochastic but there is no asymmetric information among firms.

[23]Any decreasing bid curve that crosses the residual supply curve at the same point as a flat bid curve must lead to a lower payoff for the When the same fraction of units are won at a higher cost.

is an equilibrium of the original game. For small bidders with $v < v_\ell$, any bid above their value is weakly dominated. For bidders with $v \in [v_\ell, v_h]$ the conditions for $(B, \widetilde{b})$ to be an equilibrium in the first-price auction game ensure that no deviation in $[v_\ell, \overline{b}]$ is profitable. In particular, bids above $\overline{b}$ are weakly dominated by $\overline{b}$ which wins with probability one. Bids below v_ℓ lose with probability one. For small bidders with $v > v_h$, the single crossing condition on their payoff ensures that they earn more from bidding $\overline{b}$ and winning for sure than by bidding at any lower level. Finally, the mass of small bidders that may arise at the upper end of the bid distribution does not cause any difficulties for the large bidder's proposed strategies because with a bid of $\overline{b}$ she wins all of the units that she has value for with probability one, and so cannot gain by increasing her bid.

Example 4 (Small bidders may win with higher interim probability in the DP auction)**.** Assume that the large bidder has capacity $\mu_L < 1/2$, and $v_L \sim U[0, 1]$. Assume that measure of small bidders is such that $\mu_S = 1$ and $v_S \sim U[0, 1]$. To construct an equilibrium, let $\widetilde{F}_S(v) = v/\mu_L$ for $v \in [0, \mu_L]$. The first-price auction with distributions F_L and $\widetilde{F}_S$ has equilibrium bid functions given by

$$B_D(v) = \frac{\sqrt{1 + kv^2} - 1}{kv} \qquad b_D(v) = \frac{1 - \sqrt{1 - kv^2}}{kv}$$

where $k = \mu_L^{-2} - 1$ and $\overline{b} = \mu_L/(1 + \mu_L)$ is the maximum bid.[24] For $v > \mu_L$ set $b_D(v) = \overline{b}$ to complete the description of equilibrium. The small bidders' win probabilities and payoffs in the DP auction are

$$q_D(v; \mu_L) = \begin{cases} \frac{v}{\sqrt{1-kv^2}} & \text{if } v \le \mu_L \\ 1 & \text{if } v > \mu_L \end{cases} \qquad \pi_D(v; \mu_L) = \begin{cases} \frac{v}{\sqrt{1-kv^2}}(v - b_D(v)) & \text{if } v \le \mu_L \\ v - \frac{\mu_L}{1+\mu_L} & \text{if } v > \mu_L. \end{cases}$$

In contrast, the small bidders' win probabilities and payoffs in the UP auction are given by

$$q_D(v; \mu_L) = \begin{cases} 2v & \text{if } v \le \mu_L \\ 1 & \text{if } v > \mu_L \end{cases} \qquad \pi_U(v; \mu_L) = \begin{cases} v^2 & \text{if } v \le \mu_L \\ v - \mu_L + \mu_L^2 & \text{if } v > \mu_L. \end{cases}$$

The small bidders prefer the UP to the DP auction. The large bidder has the reverse preference.

Example 5 (Small bidders may prefer the DP auction)**.** Suppose that $\mu_L = \mu_S = 1$ and let $F_L(v_L)$ be such that there is a $\frac{1}{2}$ probability that the large bidder is type $v_L = 0$ and there is a $\frac{1}{2}$ probability that the large bidder is type $v_L = 10$. Small bidders have type $F_S(v_L) = (1 - \epsilon)v_L$ if $v_L < 1$. There is also an $\epsilon > 0$ measure of small bidders with type $t = 100$.

In the UP auction, small bidders truthfully report their type and the large bidder best responds by bidding 0 if $v_L = 0$ and 1 if $v_L = 10$. Thus, a small bidder with type $v_L = 100$ gets an expected payoff of 99.5.

In the DP auction no bidder bids above their value. Thus, for a small bidder, and bid $b > 0$ wins with probability of at least $\frac{1}{2}$. This implies that any small bidder with type $v \le 1$ bids $b(v) \le \frac{1}{2}$

[24]This example is from Plum [1992] who allows for the distributions to take the form $F(x) = x^a$ for $a > 0$, but it is sufficient for our purposes to work with the $a = 1$ case.

because

$$\lim_{b\to 0}\frac{1}{2}(v-b)\geq v-\frac{1}{2},$$

where the left hand side is a lower bound a bidder's payoff from submitting an arbitrarily small bid and the right hand side is an upper bound on a bidder's payoff from bidding $\frac{1}{2}$.

Moreover, when the large bidder has type $v_L = 10$, she bids by submitting a flat bids that mixes over $(0, \bar{b})$ with no atoms, where $\bar{b}$. The large bidder mixes over a support that has zero as a lower bound, because if the large bidder mixed over a support where $\underline{b} > 0$ was the lower bound on her bid, then no small bidder would bid in the interval $(\epsilon, \underline{b})$. In addition, $\bar{b} \leq \frac{1}{2}$ because almost all small bidders bid below $\frac{1}{2}$.

Since the large bidder mixes over $(0, \bar{b})$ with no atoms, then it follows that small bidders bid according to strategy $b(v)$, that is continuous and such that $b(0) = 0$ and $b(1) = \bar{b}$. Thus, a small bidder's interim probability of winning given that she is type v is $q(v)$ where q is continuous, and such that $q(v) \geq \frac{1}{2}$ if $v > 0$ and $q(1) \approx 1$. A standard envelope theorem condition then implies that a small bidder with type $v = 1$ gets payoff

$$1-\bar{b}\approx q(1)(1-\bar{b})=\int_0^1 q(s)ds>\frac{1}{2}\implies \bar{b}<\frac{1}{2}.$$

Thus, a small bidder with type $v = 100$ knows that the clearing price is at most $\bar{b}$. Thus, the bidder with type $v = 100$ gets a payoff of at least $100 - \bar{b} > 99.5$ because $\bar{b} < \frac{1}{2}$.

Proof of Proposition 2 If the first part of Condition 1 holds, the result follows from Corollary 3. Therefore, assume the second part of the condition holds. We first show that if $\tau(v)$ and $\phi_L(b(v))$ cross, the latter is steeper. For a contradiction suppose that $\tau(v)$ and $\phi_L(b(v))$ cross at $\hat{v}$ and the former is steeper than the latter. Then at $\hat{b} = b(\hat{v})$, $\phi_L(\hat{b}) = \tau(\phi_S(\hat{b}))$ and $\phi'_L(\hat{b}) < \tau'(\phi_S(\hat{b}))\phi'_S(\hat{b})$. From the FOCs of the DP auction and Condition 1,

$$\frac{F_L(\phi_L(\hat{b}))}{f_L(\phi_L(\hat{b}))}\frac{1}{\phi_S(\hat{b})-\hat{b}}<\frac{\widetilde{F}_S(\phi_S(\hat{b}))}{\widetilde{f}_S(\phi_S(\hat{b}))}\frac{1}{\phi_L(\hat{b})-\hat{b}}\tau'(\phi_S(\hat{b}))\leq\frac{F_L(\phi_L(\hat{b}))}{f_L(\phi_L(\hat{b}))}\frac{1}{\phi_L(\hat{b})-\hat{b}},$$

which is a contradiction because the small bidders being weaker implies $\phi_S(b) < \phi_L(b)$ for all equilibrium bids [Maskin and Riley, 2000a, Proposition 3.5].

Next we show that the highest bid made in the DP auction is at least as big as the expected payment of v_h in the UP auction or that

$$\bar{b}=\int_{v_\ell}^{v_h} x\, dF_L(\phi_L(b(x)))\geq\int_{v_\ell}^{v_h} x\, dF_L(\tau(x)),$$

where $\bar{b}$ is the largest equilibrium bid in the DP auction. From the large bidder's preference for the

Vickrey auction (due to her strength) and Condition 1,

$$\bar{b} = \int_0^{\overline{v}_L} x \, d\widetilde{F}_S(\phi_S(B(x))) > \int_{v_\ell}^{v_h} x \, d\widetilde{F}_S(x) \geq \int_{v_\ell}^{v_h} x \, dF_L(\tau(x)).$$

Combined, the facts that $F_L(\tau(v))$ may only cross $F_L(\phi_L(b(v)))$ from below and that the payoff of the highest type of small bidder is higher in the UP auction prove the proposition.

A.2 Model Extension 1: Privately Informed Large Bidders

We first consider a straightforward extension of the original model. We now study a model with two large bidders. Again, each large bidder has a constant marginal value for additional units that equals her type v_L which is an i.i.d. draw of a random variable with distribution $F_L(v_L)$. We study the case where both large bidders value all units, $\mu_L = 1$, and the measure of small bidders is one, $\mu_S = 1$.

A.2.1 Equilibrium in the DP and Vickrey Auctions

We again show that we can characterize equilibrium in the DP auction by studying a corresponding asymmetric first price auction. Or in other words, we are able to extend the implication of Lemma 1 to the multiple large bidder setting. To see this, consider an asymmetric first-price auction with three bidders. Two of the bidders are type "L" bidders and have values that are independent draws of a random variable that has distribution F_L. The remaining bidder is a type "S" bidder that has a value that is an independent draw of a random variable with distribution F_S.[25] Lebrun [1999] characterizes equilibrium bid behavior in this case. He shows that an equilibrium of this asymmetric first-price auction exists and equilibrium bid behavior is described by two inverse bid functions, $\phi_L(b)$ and $\phi_S(b)$. Both inverse bid functions are defined on $[0, \bar{b}]$ and solve the system of equations below.

$$\phi'_L(b)\frac{f_L(\phi_L(b))}{F_L(\phi_L(b))} + \phi'_S(b)\frac{f_S(\phi_S(b))}{F_S(\phi_S(b))} = \frac{1}{\phi_L(b) - b} \tag{1}$$

$$2\phi'_L(b)\frac{f_L(\phi_L(b))}{F_L(\phi_L(b))} = \frac{1}{\phi_S(b) - b}. \tag{2}$$

Equations (1) and (2) correspond to the first-order conditions of the type L and S bidders respectively.

Proposition 3 shows that this first-price equilibrium also describes equilibrium bid behavior in our DP auction setting. The proposition would be immediate if the large bidders were restricted to submitting flat bid curves. The primary challenge is to show that any one large bidder does not want to deviate and submit a bid curve that decreases on some quantity interval. The argument proceeds in two steps. First we choose an arbitrary point on the small bidders' residual supply curve.

[25] Our assumptions in this section imply that $\widetilde{F}_S = F_S$.

The associated quantity, q_0, represents the quantity won by a large bidder when the opposing large bidder has a sufficiently low type. Given that a large bidder's bid curve intersects the small bidders' residual supply at this point, we prove that the bid curve must be flat to left of that point (for $q \leq q_0$).[26] This rules out that the bid curve is downward sloping on some interval $[q_a, q_b]$ with $q_b < q_0$. Having optimized for an arbitrary q_0 we then consider the optimal choice of q_0 and show that the resulting bid curve is flat at the first-price auction bid $b_L(v)$.

Proposition 3. *If $b_L(v)$ and $b_S(v)$ are the equilibrium bid strategies of the type L and type S bidders in an asymmetric first price auction with two type L bidders and one type S bidder, then in our DP auction it is an equilibrium for the large bidders to bid $b_L(v)$ for all units and the small bidders to bid according to $b_S(v)$.*

Proof. Consider the problem of a type-v_L large bidder when all other bidders follow the proposed strategy. We show bidding $b(q, v_L) = b_L(v_L), \forall q \in [0, 1]$ is a best reply for the large bidder among all nonincreasing bid schedules. Given a quantity q_0 to be purchased when the opposing large bidder has type zero (i.e., q_0 solves $F_S(\phi_S(b(q_0, v_L))) = q_0$), consider the choice of $b(q, v_L)$ for $q \leq q_0$ subject to the constraint $b(q, v_L) \geq b(q_0, v_L)$. Ignoring the monotonicity constraint, the large bidder's objective function for determining her bid for her q^{th} quantity increment is

$$\max_{b(q,v_L)} F_L(\phi_L(b(q, v_L)))(v_L - b(q, v_L)) \text{ s.t. } b(q, v_L) \geq b(q_0, v_L), \tag{3}$$

because conditional on $b(q, v_L) \geq b(q_0, v_L)$ the large bidder wins a unit $q < q_0$ if and only if $b(q, v_L) \geq b_L(V_L)$ where V_L is the type of the opposing large bidder. Let $\hat{b}(v_L) = \arg\max_b F_L(\phi_L(b))(v_L - b)$. There are two cases. If $b(q_0, v_L) \leq \hat{b}(v_L)$, then clearly $b(q, v_L) = \hat{b}(v_L)$ solves (3). Otherwise, if $b(q_0, v_L) > \hat{b}(v_L)$, then $b(q, v_L) = b(q_0, v_L)$ solves (3) because the objective is quasi-concave. Note that the monotonicity constraint is not violated in either case.

The second step is to choose q_0 optimally. If $b(q_0, v_L) < \hat{b}(v_L)$, the payoff is

$$q_0 F_L(\phi_L(\hat{b}(v_L)))(v_L - \hat{b}(v_L)),$$

which is clearly increasing in q_0. For $b(q_0, v_L) \geq \hat{b}(v_L)$, the payoff is

$$F_S(\phi_S(b(q_0, v_L)))F_L(\phi_L(b(q_0, v_L)))(v_L - b(q_0, v_L)),$$

which is maximized by setting $b(q_0, v_L) = b_L(v_L)$ as long as $b_L(v_L) > \hat{b}(v_L)$. Observe that

$$\phi'_L(\hat{b}(v_L))\frac{f_L(\phi_L(\hat{b}(v_L)))}{F_L(\phi_L(\hat{b}(v_L)))}(v_L - \hat{b}(v_L)) + \phi'_S(\hat{b}(v_L))\frac{f_S(\phi_S(\hat{b}(v_L)))}{F_S(\phi_S(\hat{b}(v_L)))}(v_L - \hat{b}(v_L)) > 1,$$

because the first term is one from the definition of $\hat{b}$. This implies that $b_L(v_L) > \hat{b}(v_L)$. □

[26] It may have a downward jump at q_0.

We can similarly state that equilibrium bid behavior in the Vickrey auction corresponds to equilibrium bid behavior in a second price auction with three bidders.

Lemma 3. *The Vickrey auction has an equilibrium (in weakly dominant strategies) where each large bidder bids $b_L(v) = v$ on all units and small bidders bid $b_S(v) = v$.*

We can use the connections between the DP and Vickrey auctions and their single-unit counterparts to extend the implications of Corollary 2 to the multiple large bidder setting. In particular Lebrun [1999] show that in the asymmetric first-price auction, the large bidder bids aggressively relative to the small bidder if the small bidder is stronger in the sense that $F_S \succeq_{rh} F_L$. Thus, if $F_S \succeq_{rh} F_L$, we have that the large bidder's interim expected quantity won in the DP auction exceeds her interim expected quantity in the Vickrey auction, because the Vickrey auction is efficient. The standard envelope theorem argument then shows that the large bidder always has a higher interim expected payoff in the DP auction versus the Vickrey auction if $F_S \succeq_{rh} F_L$. The same argument shows that a large bidder has the reverse preference if $F_L \succeq_{rh} F_S$.

Corollary 4. *If $F_S \succeq_{rh} F_L$, then a large bidder's interim expected payoff in the DP auction is higher than her interim expected payoff in Vickrey auction for any large bidder type $v_L \in [0, \overline{v}_L]$. Similarly, if $F_L \succeq_{rh} F_S$, then a large bidder's interim expected payoff in the Vickrey auction is higher than her interim expected payoff in DP auction for any large bidder type $v_L \in [0, \overline{v}_L]$.*

A.2.2 Equilibrium in the UP auction and an Impossibility Result

Proposition 3 and Lemma 3 show that we can characterize equilibrium bid behavior in the DP and Vickrey auctions when there are many large bidders. We characterize equilibrium bid behavior by illustrating a connection between each auction and its single-unit counterpart. In this section, we show that we are unable to tractably characterize equilibrium bid behavior in the UP auction in the same way.

By tractable equilibria we are referring to equilibria for which the choice of clearing price remains optimal for each large bidder after the opponents' private information is revealed, which is equivalent to saying that it is an ex post equilibrium in our model. If there is an ex post equilibrium of the UP auction, then the large bidder's bid curve maximizes her payoff pointwise, for any realization of her rival's type. As mentioned in Footnote 20, equilibria satisfying this property are the focus of much of the literature on UP auctions and related market games.

To understand why these equilibria are tractable, and others are not, it is helpful to consider a large bidder's problem in this environment as being to choose the clearing price on a stochastic residual supply curve that varies with her large rival's type. Suppose there is an ex post equilibrium of the UP auction and let $b(v_1, v_2)$ be the clearing price the two large bidders have values v_1 and v_2. If large bidder 2 submits a demand curve, $q_2(b, v_2)$, when her type is v_2, the residual supply available to large bidder 1 for this realization of bidder 2's type is $F_S(b) - q_2(b, v_2)$. Choosing the clearing price is a simple optimization problem given v_2, and we can consider doing this for all v_2. If the solution to this pointwise optimization problem gives *admissible* curves $q_1(b, v_1)$ and $q_2(b, v_2)$

such that the clearing price $b(v_1, v_2)$ is optimal for each bidder and each realization of (v_1, v_2), then we have found an ex post equilibrium.

More formally, consider solving a relaxed version of the problem stated below in (4) by optimizing pointwise (i.e., for each v_2) ignoring the constraint that the resulting schedule of price-quantity pairs is admissible (i.e., $q_1(\cdot, v_1)$ is a nonincreasing function for each v_1). We have written the problem assuming that $q_2(b, \cdot)$ is nondecreasing for each b, which must be true in an ex post equilibrium (see Footnote 28).[27]

$$\max_{b(v_1,v_2)} \int_0^{\overline{v}} (F_S(b) - q_2(b, v_2))(v_1 - b)\, dF(v_2) \tag{4}$$
$$\text{s.t. } v_2' > v_2 \implies F_S(b(v_1, v_2')) - q_2(b(v_1, v_2'), v_2') \leq F_S(b(v_1, v_2)) - q_2(b(v_1, v_2), v_2),$$

If a solution to this relaxed problem does not yield admissible demand curves (they could be upward sloping or not functions at all), then in a Bayesian Nash equilibrium the constraint must bind somewhere. While this observation is straightforward, knowing a priori where the constraint binds endogenously in equilibrium appears to be a very difficult problem.

We consider a general class of equilibrium demand curves, and show that there cannot exist an ex post equilibrium in this extension of our model. To generate a contradiction, we assume that there is a symmetric ex post equilibrium of the UP auction. The symmetry assumption simplifies the presentation of the proofs but is not essential to the argument that there is no ex post equilibrium. A type-v large bidder submits the demand curve, $q(b, v)$, which must be nonincreasing in b and nondecreasing in v.[28] Given that it is weakly dominated to demand any quantity at $b > v$, we assume that $q(b, v) = 0$ for all $b > v$. The monotonicity of $q(\cdot, v)$ implies that $q(\cdot, v)$ is differentiable almost everywhere [Royden, 1968]. We assume that if $q(b, v)$ is not differentiable with respect to b at b', then the left-hand partial derivative, $q_b(b'-, v)$, exists. For the remainder of this section, the term "equilibrium" refers to an equilibrium with these properties. The result we prove using two lemmas below is the following.

Proposition 4. *There does not exist a symmetric, ex post equilibrium demand curve for the large bidders in the uniform-price auction.*

We prove the above Propostion in two steps (Lemmas 4 and 5). To explain the intuition for the proof, we introduce some additional notation. Given the assumption that bids are undominated, when $v_2 = 0$ bidder 1 competes only with the small bidders and chooses a clearing price, $\hat{b}(v_1) \in \arg\max_b F_S(b)(v_1 - b)$. Note that this implies that $q(\hat{b}(v_1), v_1) = F_S(\hat{b}(v_1))$. Since bidder 2 is

[27]Note that the monotonicity of $q_2(b, \cdot)$ implies that in order to satisfy the constraint it also must be true that $v_2' > v_2 \implies b(v_1, v_2') \geq b(v_1, v_2)$.

[28]It must be nonincreasing in b due to the rules of the auction, while it must be nondecreasing in v, due to the ex post optimization problem satisfying a single-crossing property. Specifically, given v_2 if $b(v_1, v_2) \in \arg\max_{b(v_1,v_2)}(F_S(b(v_1, v_2)) - q(b(v_1, v_2), v_2))(v_1 - b(v_1, v_2))$, then due to the fact that the cross partial of the objective with respect to b and v_1 is nonnegative $b(\cdot, v_2)$ is nondecreasing for each v_2 [Milgrom and Shannon, 1994]. The fact that the residual supply curve is nondecreasing then implies that the quantity purchased by bidder 1, $q(b, v_1)$, is nondecreasing in v_1 for each v_2.

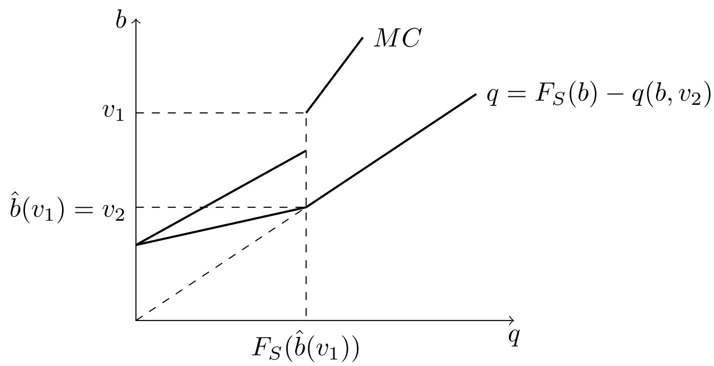


Figure 2: Illustration of Lemma 4

assumed to never bid above v_2, in an ex post equilibrium bidder 1 must win $F_S(\hat{b}(v_1))$ for all $v_2 < \hat{b}(v_1)$. The price $\hat{b}(v_1)$ plays a critical role in the first lemma.[29]

When $v_2 > 0$, if bidder 1 chooses the clearing price $b \geq \hat{b}(v_2)$ he is awarded the quantity $S(b, v_2) = F_S(b) - q(b, v_2)$,[30] which we refer to as bidder 1's residual supply. Our proof that there is no such $q(b, v)$ that forms an ex post equilibrium relies on two contradictory lemmas.

In the first, we argue that for small enough v_2, if bidder 1's residual supply curve has a kink at $b = v_2$ then an ex post equilibrium cannot exist. Figure 2 illustrates the intuition of the proof. With a kink in the supply curve at $F_S(\hat{b}(v_1))$, the associated marginal cost jumps at this point. Equilibrium requires that the type-v_1 bidder purchase the quantity $F_S(\hat{b}(v_1))$ for all $v_2 < \hat{b}(v_1)$. The jump in the marginal cost implies that there is a type v_1' for the large bidder with $v_1' < v_1$ who optimally purchases the same quantity $F_S(\hat{b}(v_1))$ when $v_2 = \hat{b}(v_1)$, but this creates a problem for the existence of an ex post equilibrium because the type-v_1' bidder purchases a smaller quantity $F_S(\hat{b}(v_1'))$ at a lower price $\hat{b}(v_1')$ when $v_2 = 0$. We therefore can't find a downward sloping demand for bidder 1 that leads to these two purchase decisions for the type-v_1' bidder. This implies that for an ex post equilibrium to exist in this environment there cannot be a kink in the residual supply curve.

In the second, we argue that bidder 1's residual supply curve must have a kink at $b = v_2$ in any ex post equilibrium. In combination with the previous lemma, we conclude that an ex post equilibrium is impossible. The idea behind the second lemma is to assume that there is no kink in the residual supply curve at a relevant v_2 (i.e., a value of v_2 that is not too high). This requires that $q_b(v_2, v_2) = 0$, which implies that for a small enough bid below v_2 the bidder purchases an arbitrarily small quantity in equilibrium. We rule out that this can be optimal by showing that the bidder would prefer to win a strictly positive amount in this case.

Lemma 4. *If $q_b(v-, v) < 0$ for some $v < \hat{b}(1)$, $q(b, v)$ cannot be a large bidder's demand curve in an ex post equilibrium.*

[29] Standard arguments establish that $\hat{b}(v)$ is increasing [Milgrom and Shannon, 1994]. Also, $f_S(b) > 0$ for all $b < \hat{b}(v_1)$ (i.e., a single-crossing property holds) implies that $\partial/\partial b\,(F_S(b)(v_1 - b)) > 0$ for all $b < \hat{b}(v_1)$.

[30] When $b \leq \hat{b}(v_2)$, the previous paragraph implies that bidder 1 is not awarded any of the good.

Proof. Given a v_1, fix $v_2 = \hat{b}(v_1)$. The type-v_1 bidder's payoff, $\pi(b, v_1, v_2)$, for a bid b is

$$\pi(b, v_1, \hat{b}(v_1)) = (F_S(b) - q(b, \hat{b}(v_1)))(v_1 - b),$$

which must be maximized at $\hat{b}(v_1)$ since it must be that $q(\hat{b}(v_1), v_1) = F_S(\hat{b}(v_1))$ in equilibrium (i.e., the type-v_1 bidder purchases the quantity $F_S(\hat{b}(v_1))$ when the clearing price is $\hat{b}(v_1)$). Note that the optimality of $b = \hat{b}(v_1)$ requires that the left-hand derivative of the payoff at $\hat{b}(v_1)$ is nonnegative. For all $b < \hat{b}(v_1)$,

$$\pi_b(b, v_1, \hat{b}(v_1)) = f_S(b)(v_1 - b) - F_S(b) - q_b(b, \hat{b}(v_1))(v_1 - b) + q(b, \hat{b}(v_1)) > 0,$$

because the first two terms correspond to the derivative with respect to b of the payoff when $v_2 = 0$, which is positive for $b < \hat{b}(v_1)$,[31] and the second two are nonnegative because $q(b, v)$ is nonincreasing in b. Observe that $q_b(\hat{b}(v_1)-, \hat{b}(v_1)) < 0$ implies that $\pi_b(b-, v_1, \hat{b}(v_1)) > 0$ (i.e., the left-hand derivative of π with respect to b is positive). This then implies that there is a type $v_1' < v_1$ for whom $\hat{b}(v_1)$ is the preferred clearing price when $v_2 = \hat{b}(v_1)$. But since $\hat{b}(v_1') < \hat{b}(v_1)$ and hence $F_S(\hat{b}(v_1')) < F_S(\hat{b}(v_1))$, the type-$v_1'$ bidder would prefer a larger quantity at a larger price when $v_2 = \hat{b}(v_1)$ compared to when $v_2 = 0$. This cannot happen in an ex post equilibrium with downward sloping demands. □

For $q(b, v)$ to describe ex post equilibrium strategies, we therefore require that $q_b(v-, v) = 0$ for all $v < \hat{b}(1)$. However, the next lemma shows that in any ex post equilibrium we must have $q_b(v-, v) < 0$ which creates a contradiction.

Lemma 5. *Let $v_1 = \hat{b}(v_2)$ for some v_2. In any ex post equilibrium $q_b(v_1-, v_1) < 0$.*

Proof. For a contradiction, take (v_1, v_2) as in the statement of the lemma and assume that $q_b(v_1-, v_1) = 0$. Note that we must have that the clearing price is $b(v_1, v_2) = \hat{b}(v_2)$ and that $q(b(v_1, v_2), v_1) = 0$. Using the market clearing condition, the equilibrium payoff for bidder 1 when her type is v_1 can be written in two ways.

$$q(b(v_1, v_2), v_1)(v_1 - b(v_1, v_2)) = (F_S(b(v_1, v_2)) - q(b(v_1, v_2), v_2))(v_1 - b(v_1, v_2)) = 0$$

Since $q_b(v_1-, v_1) = 0$, when bidder 1 has the type $v_1 + 2\varepsilon$ bidder 1's equilibrium payoff for small ε is approximately zero since

$$q(b(v_1 + 2\varepsilon, v_2), v_1)(v_1 + 2\varepsilon - b(v_1 + 2\varepsilon, v_2)) \approx$$
$$(q(b(v_1, v_2), v_1) + q_b(b(v_1, v_2)-, v_1)b_{v_1}(v_1, v_2)2\varepsilon)(v_1 + 2\varepsilon - b(v_1, v_2) - b_{v_1}(v_1, v_2)2\varepsilon) = 0,$$

because $q(v_1, v_1) = 0$ and $q_b(v_1-, v_1) = 0$.[32] In words, the bidder with value $v_1 + 2\varepsilon$ must buy an arbitrarily small quantity in equilibrium if $q_b(v_1-, v_1) = 0$; however, we show that this bidder would

[31] See Footnote 29.

[32] This conclusion might fail to hold if $b_{v_1}(v_1, v_2)$ were not bounded. Implicitly differentiating the market clearing

prefer to buy a strictly positive quantity. Consider the clearing price $b(v_1 + 2\varepsilon, v_2) = v_1 + \varepsilon$. The payoff is larger at this price because

$$
\begin{aligned}
&(F_S(v_1 + \varepsilon) - q(v_1 + \varepsilon, v_2))(v_1 + 2\varepsilon - v_1 - \varepsilon) \approx \\
&(F_S(v_1) + f_S(v_1)\varepsilon - q(v_1, v_2) - q_b(v_1-, v_2)\varepsilon)(\varepsilon) = \\
&\qquad (f_S(v_1) - q_b(v_1-, v_2))\varepsilon^2 > 0.
\end{aligned}
$$

Note that at $v_1+\varepsilon$, bidder 1 buys a nonzero quantity implying that if $q(b, v)$ is an ex post equilibrium strategy $q_b(v_1-, v_1) < 0$, a contradiction. □

A.3 Model Extension 2: Sufficiently Small Large Bidders

In this section we study a setting where there are two symmetric large bidders. We show that a large bidder has a greater ex ante expected payoff in the DP auction than the UP auction if their capacities μ_L are sufficiently small.

In keeping with our benchmark model from Section 2, we assume that there is one unit of a divisible good. There are two large bidders that we call large bidders 1 and 2. Each large bidder has capacity $\mu_L > 0$ and we assume that large bidder values are i.i.d. draws of a random variable with continuous distribution $F_L(v_L) : [0, 1] \to [0, 1]$.

We assume that the measure of small bidders exceeds the supply of available units $\mu_S \geq 1$. This assumption implies that total demand strictly exceeds total supply for any $\mu_L > 0$. Or in other words $\mu_S + 2\mu_L > 1 \ \forall \mu_L > 0$. The continuum of small bidders distributed according to the commonly known and continuously differentiable distribution $F_S(v_S) : [0, 1] \to [0, 1]$. We let f_S be the density of small bidder values. We assume that f_S has full support over $[0, 1]$ and that f_S is continuous.

We show that when large bidder capacity is sufficiently small, large bidders get a strictly greater expected payoff in the DP auction versus the UP auction. More precisely, we show that an upper bound on a large bidder's expected payoff in the UP auction is below a lower bound on a large bidder's expected payoff in the DP auction. We obtain an upper bound on the large bidder's payoff in the UP auction by assuming that large bidders play an undominated strategy given that their small rivals bid truthfully. Or more formally, we assume that large bidder bid strategies survive two rounds of elimination of weakly dominated strategies. We first note that it is a weakly dominant strategy for a small bidder to bid truthfully and it is a weakly dominated strategy for a large bidder to over report her demand. In the second round of elimination, we show that it is always a best response for a large bidder to not shade her bid when she has a relatively high value and her small rivals bid truthfully. For a large bidder with a sufficiently high value, bid shading is never a best reply if large bidder capacity is sufficiently small and small bidders bid truthfully. We get the upper bound on a large bidder's expected payoff by finding the maximal expected payoff of the large bidder

condition, one can show that it is bounded as long as $q_{v_1}(b(v_1, v_2), v_1)$ is, which clearly must be true in any ex post equilibrium.

given that her rivals' strategies survive two rounds of iterative elimination. Thus, a large bidder's expected payoff in the undominated equilibrium of a UP auction is weakly lower than our upper bound on large bidder payoffs.

We then get a lower bound on the large bidder's expected payoff in any equilibrium of the DP auction. We obtain this lower bound on bidder payoff in the DP auction by finding a lower bound on the interim expected quantity won by a large bidder in the DP auction. We then use the envelope theorem to get a lower bound on the large bidder payoff.

In both auctions, there is a competitive region where a small bidder wins with a probability in the open interval $(0,1)$. We modify our notion of a competitive region to accommodate two large bidders. If both large bidders place sufficiently low bids (presumably because they have low values), then in both auctions all small bidders with values $v_S > v_\ell$ win with positive probability where $v_\ell := F_S^{-1}(\frac{\mu_S - 1}{\mu_S})$. If both large bidders place sufficiently high bids and each wins the amount that is her capacity μ_L, then a small bidder wins a unit if and only if her value v_S exceeds $v_h(\mu_L) : [0,1] \to [0,1]$ where $v_h(\mu_L) = F_S^{-1}(\frac{\mu_S + 2\mu_L - 1}{\mu_S})$ values $v_S < v_h$. In addition, all small bidders with values $v_S > v_m(\mu_L)$ where $v_m(\mu_L) : [0,1] \to [0,1]$ and $v_m(\mu_L) = F_S^{-1}(\frac{\mu_S + \mu_L - 1}{\mu_S})$ win whenever one large bidder wins her full capacity and the other large bidder wins no units. Note that $v_m(\mu_L)$ and $v_h(\mu_L)$ are continuous and strictly increasing in μ_L when μ_L is sufficiently small because we assume that F_S is continuous and strictly increasing in μ_L when $\mu_L > 0$ is sufficiently small. In addition, $\mu_L > 0 \iff v_\ell < v_m(\mu_L) \leq v_h(\mu_L)$.

Example 6. Suppose that both large bidders and all small bidder values are uniformly distributed over $[0,1]$. In addition, suppose that the measure of small bidders $\mu_s = 2$. In this case, if $\mu_L \leq \frac{1}{2}$, then

$$v_\ell = \frac{1}{2},\ v_m(\mu_L) = \frac{1+\mu_L}{2},\ \textit{and}\ v_h(\mu_L) = \frac{1+2\mu_L}{2}.$$

These three values will be useful in helping us to form bounds on large bidder payoffs in the UP and DP auctions.

A.3.1 Bid Behavior and Welfare Bounds in the UP Auction

We first consider the decision problem of the large bidder in the UP auction. We suppose that small bidders bid their true value because truthful bidding is a weakly dominant strategy for small bidders in the UP auction. We let $b_{Li}(q,v) : [0,1]^2 \to \mathbb{R}_+$ be a (pure) bid strategy for large bidder i. We let $\mathcal{B}_L$ be the set of all functions $b_{Li}(q,v) : [0,1]^2 \to \mathbb{R}_+$ that are nonincreasing in the first argument. Thus, $\mathcal{B}_L$ is the set of all pure bid strategies for a large bidder in the UP auction. In addition, we let $BR_{Li}(v_{Li}, b_{Lj}, \mu_L)$ be the set of best replies for large bidder i given that (1) she has value $v_{Li} \in [0,1]$, (2) her large rival bids according to some bid strategy b_{Lj}, (3) small bidders bid truthfully, and (4) each large bidder has capacity μ_L.

The following Lemma shows that for any $\epsilon > 0$, if a large bidder i has value $v_{Li} \geq v_h(\mu_L) + \epsilon$, then in the UP auction a large bidder bids at least $v_h(\mu_L)$ on a measure μ_L of units when her capacity $\mu_L > 0$ is sufficiently small. This is because bidding at least $v_h(\mu_L)$ for all μ_L units is always best response, for any strategy used by the large bidder's large rival. In addition, bidding below $v_h(\mu_L)$ is not always a best response to your large rival. Or in other words, the large bidder has no incentive to engage in bid shading when her value v strictly exceeds the value of the highest small bidder in the competitive region and large bidder capacity is sufficiently small.

Lemma 6. *For any $\epsilon > 0$, then there exists a $\delta > 0$ such that for all $\mu_L \leq \delta$ and $v \geq v_h + \epsilon$,*

$$b_{Li}(q, v) \in BR_{Li}(v, b_{Lj}, \mu_L) \ \forall b_{Lj} \in B_L,$$

if and only if

$$b_{Li}(q, v) \geq v_h \ \forall q < \mu_L, \ i, j = 1, 2 \text{ where } i \neq j.$$

Proof. First, we show that for any $v > v_h(\mu_L)$, there is a sufficiently small $\mu_L^* > 0$ such that for all $\mu_L \in (0, \mu_L^*)$ we have that

$$\mu_L = \arg \max_{q \in [0, \mu_L]} q \left(v - F_s^{-1} \left(\frac{\mu_s - 1 + q + q_j}{\mu_s} \right) \right) \ \forall q_j \in [0, \mu_L].$$

Or in other words, a large bidder i with value v best responds by submitting a bid that ensures that she wins μ_L units if (1) she her value v strictly exceeds $v_h(\mu_L)$ and (2) her large rival wins q_j units invariant of her bid. We prove this by showing that the derivative of the above expression with respect to q is positive for all $q \in [0, \mu_L]$ when $\mu_L > 0$ is sufficiently small. The derivative with respect to q of the objective is

$$\left(v - F_S^{-1} \left(\frac{\mu_S - 1 + q + q_j}{\mu_S} \right) \right) - q \left(\frac{1}{f_S(F_S^{-1}(\frac{\mu_S - 1 + q + q_j}{\mu_S}))} \frac{1}{\mu_S} \right). \tag{5}$$

Note that

$$F_S^{-1} \left(\frac{\mu_S - 1 + q + q_j}{\mu_S} \right) \leq F_s^{-1} \left(\frac{\mu_S - 1 + 2\mu_L}{\mu_S} \right) = v_h(\mu_L), \tag{6}$$

and

$$\overline{x} := \max_{q \in [0, \mu_L]} \left(\frac{1}{f_S(F_S^{-1}(\frac{\mu_s - 1 + q + q_j}{\mu_s}))} \frac{1}{\mu_S} \right) \geq \left(\frac{1}{f_S(F_S^{-1}(\frac{\mu_s - 1 + q + q_j}{\mu_s}))} \frac{1}{\mu_S} \right), \tag{7}$$

where $\overline{x} < \infty$ because f_S is finite for all $v \in [0, 1]$. Thus, expressions (6) and (7) imply that we can bound (5) by

$$\left(v - F_s^{-1} \left(\frac{\mu_s - 1 + q + q_j}{\mu_s} \right) \right) - q \left(\frac{1}{f(F_s^{-1} \left(\frac{\mu_s - 1 + q + q_j}{\mu_s} \right))} \frac{1}{\mu_s} \right) \geq (v - v_h) - \mu_L \overline{x} \ \forall q \leq \mu_L.$$

Moreover, if $v > v_h$, then when $\mu_L > 0$ is sufficiently small

$$(v - v_h) - \mu_L \overline{x} > 0.$$

Thus $\mu_L = \arg\max_{q\in[0,\mu_L]} q(v - F_s^{-1}((\mu_s - 1 + q + q_j)/\mu_s)\ \forall q_j \in [0, \mu_L]$.

Next, fix $v > v_h$ and suppose that μ_L is such that $\mu_L = \arg\max_{q\in[0,\mu_L]} q(v - F_s^{-1}((\mu_s - 1 + q + q_j)/\mu_s)\ \forall q_j \in [0, \mu_L]$. We have already shown that this occurs if $\mu_L > 0$ is sufficiently small. Then, bidder wins μ_L units with certainty if she bids $b_{Li}(q, v) \geq v_h$ whenever $v > v_h + \epsilon$ and $q < \mu_L$.

Fix bidder j's type v_j. Suppose that bidder i deviates to another strategy where she wins $q < \mu_L$ units when her rival is type v_j. We show that bidder i gets a lower payoff from this deviation for any strategy used by her large rival, bidder j. Thus, it follows that the deviation is not a best reply.

Suppose that bidder j wins $\tilde{q}_j \in [0, \mu_L]$ units if bidder i bids $b_{Li}(q, v) \geq v_h$ whenever $v > v_h + \epsilon$ and $q < \mu_L$. Moreover, bidder j wins $q'_j \in [0, \mu_L]$ units when bidder i deviates. Since bidder i wins fewer units by deviating, and hence lowers the price, then it must be the case that bidder j wins weakly more units when bidder i deviates, $q'_j \geq \tilde{q}_j$, because she must submit a nonincreasing demand curve. The following chain of inequalities shows that bidder i's deviation is unprofitable.

$$\begin{aligned}\mu_L\left(v - F_s^{-1}\left(\frac{\mu_s - 1 + \mu_L + \tilde{q}_j}{\mu_s}\right)\right) &\geq q\left(v - F_s^{-1}\left(\frac{\mu_s - 1 + q + \tilde{q}_j}{\mu_s}\right)\right)\\ &\geq q\left(v - F_s^{-1}\left(\frac{\mu_s - 1 + q + q'_j}{\mu_s}\right)\right)\end{aligned}$$

The first expression is the payoff from the prescribed strategy, and the last expression is the payoff from the deviation. The first inequality was shown to hold above when μ_L is sufficiently small. The second follows from the facts that $\tilde{q}_j \leq q'_j$ and F_s^{-1} is increasing. Thus, there is no strategy that gives a greater payoff than bidding $b_{Li}(q, v) \geq v_h$ when $v > v_h + \epsilon$ and $q < \mu_L$. □

As a corollary to this lemma, we can place an upper bound on a large bidder's expected payoff given that both she and her large rival play a strategy that is not weakly dominated when her small rivals bid truthfully.

Corollary 5. *For any $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\mu_L \leq \delta$, the large bidder's per unit expected payoff is bounded above by*

$$\begin{aligned}\frac{\overline{U}_{UP}(\mu_L)}{\mu_L} = &\int_{v_\ell}^{1} v f_L(v)dv - (F_L(v_h(\mu_L) + \epsilon) - F_L(v_\ell))v_\ell\\ &- (1 - F_L(v_h(\mu_L) + \epsilon))\left\{F_L(v_h(\mu_L) + \epsilon)v_m(\mu_L) + (1 - F_L(v_h(\mu_L) + \epsilon))\, v_h(\mu_L)\right\}.\end{aligned}$$

The upper bound follows from the following observations: (i) the large bidder wins at most μ_L when her value exceeds v_ℓ; (ii) she pays at least v_ℓ when her value is between v_ℓ and $v_h(\mu_L) + \epsilon$; (iii) she pays at least $v_m(\mu_L)$ when her value is greater than $v_h(\mu_L) + \epsilon$ and the opposing large bidder has a value less than $v_h(\mu_L) + \epsilon$; and (iii) she pays at least $v_h(\mu_L)$ when both large bidders' values exceed $v_h(\mu_L) + \epsilon$.

A.3.2 Bid Behavior and Welfare Bounds in the DP Auction

We can also get a lower bound on a large bidder's payoff in an equilibrium of the DP auction. Let $q_S^D(v_S, \mu_L) : [0,1]^2 \to [0,1]$ be a small bidder's interim win probability in an undominated pure strategy equilibrium of the DP auction where large bidder capacity is μ_L.

Remark 2. *In any pure strategy equilibrium of the DP auction where bidders play undominated strategies $q_S^D(\cdot, \mu_L)$ is continuous and increasing over the interval $[v_\ell, v_h(\mu_L)]$ for all $\mu_L > 0$. In addition, each small bidder has interim expected payoff of $U_S(v_S, \mu_L) : [0,1]^2 \to \mathbb{R}_+$, where*

$$U_S(v_S, \mu_L) = \int_0^{v_S} q_S^D(x, \mu_L) dx.$$

We can obtain a lower bound on a small bidder's payoff in any undominated equilibrium of the DP auction. In order to obtain the lower bound, we assume that both large bidder's do not shade their bids. We use the lower bound on the small bidders' payoff to get an upper bound on the highest bid placed by any small bidder in the auction.

Lemma 7. *In any pure strategy equilibrium of the DP auction where bidders play undominated strategies, for any $\mu_L \in [0,1]$ $q_S^D(v, \mu_L)$ is bounded below by $\underline{q}_S^D(v, \mu_L)$ where*

$$\underline{q}_S^D(v, \mu_L) = \begin{cases} 0 & \text{if } v < v_\ell \\ F_L(v_\ell)^2 & \text{if } v \in (v_\ell, v_m(\mu_L)) \\ F_L(v_\ell)(2 - F_L(v_\ell)) & \text{if } v \in (v_m(\mu_L), v_h(\mu_L)) \\ 1 & \text{if } v > v_h(\mu_L). \end{cases}$$

The proof of the lemma is intuitive. We get the lower bound on the small bidders win probability by assuming that the each large bidder does not shade her bid. Thus, in any undominated equilibrium, the large bidder bids less aggressively, and a small bidder's win probability exceeds the lower bound constructed her.[33]

If a small bidder has type $v_S < v_\ell$, then she wins with zero probability with certainty, because the quantity demand from her small rivals that have higher demand already exceeds the market supply, even in the absence of large bidders. If a small bidder has type $v_S \in (v_\ell, v_m(\mu_L))$, then in any undominated pure strategy equilibrium, a small bidder wins a unit if (but not only if) her bid exceeds both of her large rivals' values. In addition, we know that her bid weakly exceeds v_ℓ, because a small bidder wins with positive probability if and only if her value exceeds v_ℓ. Thus, a small bidder with type $v_S \in (v_\ell, v_m(\mu_L))$ wins if both large bidders have values below v_ℓ. Indeed, a small bidder may win in equilibrium even if this condition does not hold, but we only need a lower bound on this probability. Similarly, a small bidder with type $v_S \in (v_m(\mu_L), v_h(\mu_L))$ wins if at least one large bidder has value below v_ℓ. This is because a small bidder in this range wins a unit if her

[33] In other words, we are considering a lower bound on a small bidders interim win probability, assuming that the large bidder is playing a minimax strategy that is within the set of undominated strategies.

bid exceeds the bid of at least one large bidder. The small bidder places a bid weakly above v_ℓ. Thus, a lower bound on the small bidder's win probability is the probability that at least one large bidder has a value below v_ℓ.

We use the lower bound on small bidder interim win probabilities to place an upper bound on the highest bid submitted in the DP auction.

Lemma 8. *If $b_S(v, \mu_L) : [0,1]^2 \to \mathbb{R}_+$ is the pure bid strategy of small bidders in any undominated pure strategy equilibrium of the DP auction where large bidders have capacity μ_L. Then $b_S(v_h(\mu_L), \mu_L) \leq \bar{b}(\mu_L)$ where*

$$\begin{aligned}\bar{b}(\mu_L) &:= v_h(\mu_L) - \int_0^{v_h(\mu_L)} \underline{q}_S^D(x, \mu_L)dx \\ &= v_h(\mu_L) - (v_m(\mu_L) - v_\ell)F_L(v_\ell)^2 - (v_h(\mu_L) - v_m(\mu_L))F_L(v_\ell)(2 - F_L(v_\ell)).\end{aligned}$$

A corollary to this lemma is that we can get a lower bound on the expected payoff of any large bidder in the DP auction. We give large bidder i's expected payoff if she bids $\bar{b}(\mu_L)$ if and only if her value exceeds $v_h(\mu_L)$. Bidder i bids zero otherwise. This is not a best response, but we can find bidder i's per unit utility from this strategy, and her payoff is independent of her large rival's strategy, because we know that bidder i wins μ_L units if and only if her value is above $v_h(\mu_L)$.

$$\frac{\underline{U}_{DP}(\mu_L)}{\mu_L} = \int_{v_h(\mu_L)}^1 v f_L(v)dv - \bar{b}(\mu_L)(1 - F_L(v_h(\mu_L)))$$

A.3.3 Welfare Comparison

In this subsection we prove that the lower bound on a large bidder's (ex ante) expected payoff in the DP auction exceeds the upper bound on the large bidder's expected payoff in the UP auction when μ_L is sufficiently small.

Proposition 5. *The lower bound on a large bidder's expected payoff in the DP auction $\underline{U}_{DP}(\mu_L)$ exceeds the upper bound on the large bidder's expected payoff in the UP auction $\overline{U}_{UP}(\mu_L)$ when $\mu_L > 0$ is sufficiently small.*

Proof. First note that

$$\lim_{\mu_L \to 0+} \frac{\overline{U}_{UP}(\mu_L)}{\mu_L} = \lim_{\mu_L \to 0+} \frac{\underline{U}_{DP}(\mu_L)}{\mu_L} = \int_{v_\ell}^1 v f_L(v)\, dv - (1 - F_L(v_\ell))v_\ell.$$

We prove the proposition by showing that

$$\left.\frac{d}{d\mu_L}\frac{\underline{U}_{DP}(\mu_L)}{\mu_L}\right|_{\mu_L=0} > \left.\frac{d}{d\mu_L}\frac{\overline{U}_{UP}(\mu_L)}{\mu_L}\right|_{\mu_L=0}. \tag{8}$$

With these two facts, a first-order Taylor approximation of $\overline{U}_{UP}(\mu_L)/\mu_L$ and $\underline{U}_{DP}(\mu_L)/\mu_L$ around $\mu_L = 0$ implies that the former is smaller than the latter when μ_L is sufficiently small.

For the UP auction we calculate that

$$\frac{d}{d\mu_L}\frac{\overline{U}_{UP}(\mu_L)}{\mu_L}\bigg|_{\mu_L=0} = -(1-F_L(v_\ell+\epsilon))\left\{(1-F_L(v_\ell+\epsilon))v_h'(0)+F_L(v_\ell+\epsilon)v_m'(0)\right\}.$$

For the DP auction,

$$\frac{d}{d\mu_L}\frac{\underline{U}_{DP}(\mu_L)}{\mu_L}\bigg|_{\mu_L=0} = -\overline{b}'(0)(1-F_L(v_\ell)),$$

where

$$\begin{aligned}\overline{b}'(0) &= v_h'(0)-v_m'(0)F_L(v_\ell)^2-(v_h'(0)-v_m'(0))F_L(v_\ell)(2-F_L(v_\ell))\\ &= v_h'(0)(1-F_L(v_\ell))^2+v_m'(0)2F_L(v_\ell)(1-F_L(v_\ell)).\end{aligned}$$

Thus, we find that

$$\frac{d}{d\mu_L}\frac{\underline{U}_{DP}(\mu_L)}{\mu_L}\bigg|_{\mu_L=0} = -(1-F_L(v_\ell))^2\left\{(1-F_L(v_\ell))v_h'(0)+2F_L(v_\ell)v_m'(0)\right\}.$$

Hence, the inequality in (8) holds if and only if

$$\begin{aligned}&-(1-F_L(v_\ell))^2\left\{(1-F_L(v_\ell))v_h'(0)+2F_L(v_\ell)v_m'(0)\right\} >\\ &-(1-F_L(v_\ell+\epsilon))\left\{(1-F_L(v_\ell+\epsilon))v_h'(0)+F_L(v_\ell+\epsilon)v_m'(0)\right\}.\end{aligned}$$

Since the right-hand side is continuous in ϵ when $\epsilon \geq 0$, we have that the above expression holds for a sufficiently small $\epsilon > 0$ if and only if

$$\begin{aligned}(1-F_L(v_\ell))\left\{(1-F_L(v_\ell))v_h'(0)+2F_L(v_\ell)v_m'(0)\right\} &< (1-F_L(v_\ell))v_h'(0)+F_L(v_\ell)v_m'(0)\\ (1-2F_L(v_\ell))v_m'(0) &< (1-F_L(v_\ell))v_h'(0).\end{aligned}$$

The last inequality holds because $v_h'(0) > v_m'(0)$. Thus, the inequality in (8) holds proving the proposition. □

A.4 Model Extension 3: Informed Large Bidders

In the final extension, we consider a situation in which each large bidder knows the opposing large bidder's type. In this case, we show there exists equilibria of the DP and UP auctions in which the large bidders use similar strategies. As with our results in the body of the paper, the fact that small bidders shade in the DP auction — because they are not informed of the large bidders' types — implies that the large bidders prefer the DP auction. In this equilibrium the large bidders submit flat bid curves. Suppose that the large bidder with the lower type (e.g., bidder 2 with type $v_{L2} < v_{L1}$) submits a flat bid curve for all units equal to her marginal value. The higher type large bidder then faces a residual supply curve that is perfectly elastic for all units up to $F_S(v_{L2})$. A clearing price, p, exceeding v_{L2} yields $1-(1-F_S(p)) = F_S(p)$ units. The best reply of bidder 1 is to purchase exactly

$F_S(v_{L2})$ units if his type is not too high and $F_S(\hat{b}_U(v_{L1}))$ where $\hat{b}_U(v) \in \arg\max_b F_S(b)(v-b)$ otherwise.[34] More precisely, bidder 1's best reply is to place the bid

$$b_U(q, v_{L1}, v_{L2}) = \begin{cases} v_{L1} & v_{L1} \leq v_{L2} \\ v_{L1}\mathbf{1}\{q \leq F_S(v_{L2})\} & v_{L1} > v_{L2} \geq \hat{b}_U(v_{L1}) \\ \hat{b}_U(v_{L1}) & v_{L1} > \hat{b}_U(v_{L1}) > v_{L2}. \end{cases} \tag{9}$$

where $\mathbf{1}\{\cdot\}$ is an indicator function (the first and third cases indicate flat bids for all units). This bid specifies that the higher type of large bidder purchases the quantity $F_S(v_{L2})$ at a price v_{L2} or the quantity $F_S(\hat{b}_U(v_{L1}))$ at price $\hat{b}_U(v_{L1})$. It is important to note that this is not the only equilibrium of the UP auction. This fact makes it difficult to draw strong conclusions about the UP auction with informed large bidders. However, this equilibrium in the UP auction has a similar counterpart in the DP auction. For the DP auction, suppose again that the bidder with the lower type submits a flat bid for all units equal to her marginal value. The other large bidder again faces a residual supply curve that is flat for some initial amount of units and upward sloping after that. The small bidders shade their bids now though. The small bidders are not informed in this model and hence do not know with certainty what the clearing price will be, so they may gain in expectation by reducing their bid below their value. In a monotone pure strategy equilibrium let the inverse of their bid function be $\phi_S(b)$ and define $\hat{b}_D(v) \in \arg\max_b F_S(\phi_S(b))(v-b)$ to be the large bidder's optimal bid when the other large bidder is absent, and consider the strategy defined by

$$b_D(q, v_{L1}, v_{L2}) = \begin{cases} v_{L1} & v_{L1} \leq v_{L2} \\ v_{L1}\mathbf{1}\{q \leq F_S(\phi_s(v_{L2}))\} & v_{L1} > v_{L2} \geq \hat{b}_D(v_{L1}) \\ \hat{b}_D(v_{L1}) & v_{L1} > \hat{b}_D(v_{L1}) > v_{L2}, \end{cases} \tag{10}$$

which specifies that the higher type of large bidder either purchases $F_S(\phi_S(v_{L2}))$ units at price v_{L2} or the quantity that would be awarded at the price $\hat{b}_D(v_{L1})$.

Proposition 6. *If two large bidders are informed of each other's type, then it is an equilibrium of the UP auction for small bidders to bid at their value and large bidder 1 (and symmetrically for large bidder 2) to bid according to $b_U(q, v_{L1}, v_{L2})$.*

In a monotone pure-strategy equilibrium, let the inverse bid function used by the small bidders in the DP auction be $\phi_S(b)$. Using weak dominance, $\phi_S(b) \geq b$ for all equilibrium bids. The large bidders' best replies are given by $b_D(q, v_{L1}, v_{L2})$.

The large bidders prefer the DP auction to the UP auction, when these are the equilibrium strategies.

Proof. That small bidders do not bid above (and generally below) their values in the DP auction is

[34] To see this, note that the residual supply curve has a kink at the quantity $F_S(v_{L2})$ where the marginal cost has an upward jump. Only after the bidder's type is large enough does she begin purchasing more than $F_S(v_{L2})$. The bidder is indifferent between purchasing $F_S(v_{L2})$ and increasing her bid when $v_{L2} = \hat{b}_U(v_{L1})$.

clear. The logic for the large bidder's preference is analogous to that used in Proposition 1 because for any realization of large bidder types the large bidder with the higher type faces a residual supply curve that is weakly lower in the DP auction due to the small bidder shading. The large bidder with the lower type is indifferent between the two auctions because they purchase no units. □

To show that the equilibrium in the UP auction is not unique and that other equilibria generate distinct results we briefly sketch the details of an alternative.

Example 7. Let $F_S(b) = b$. Suppose large bidders have the types v_{L1} and v_{L2} and that each demands a quantity q_i by bidding the amount $\overline{v}$ for units $q \leq q_i$ and zero afterwards. The following "quantity demands" form an equilibrium.

$$q_i(v_{Li}, v_{Lj}) = \begin{cases} \frac{v_{Li}}{2} & v_{Li} \geq 2v_{Lj} \\ \frac{2}{3}v_{Li} - \frac{1}{3}v_{Lj} & 2v_{Lj} > v_{Li} \geq \frac{v_{Lj}}{2} \\ 0 & v_{Lj} > 2v_{Li} \end{cases}$$

Notice in that in this example the large bidder with the lower type may have a strictly positive payoff. That is the lower type large bidder weakly prefers this equilibrium. The large bidder with the higher type has an ambiguous preference.

This equilibrium is reminiscent of the one described in Back and Zender [1993], in which bidders submit demand curves with maximum bids followed by zero bids. The Back and Zender [1993] equilibrium remains an equilibrium when bidders have private information, but the one in our previous example does not in our environment. The essential reason is that the quantity that must be demanded depends carefully on the type of the other large bidder.

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