

Optimal auctions with endogenous budgets

Brian Baisa* and Stanislav Rabinovich†

February 10, 2016

Abstract

We study the benchmark independent private value auction setting when bidders have endogenously determined budgets. Before bidding, a bidder decides how much money she will borrow. Bidders incur a cost to borrowing. We show that bidders are indifferent between participating in a first-price, second-price and all-pay auction. The all-pay auction gives higher revenue than the first-price auction, which gives higher revenue than the second price auction. In addition, when the distribution of values satisfies the monotone hazard rate condition, the revenue maximizing auction is implemented by an all-pay auction with a suitably chosen reserve price.

Keywords: Optimal auction; All-pay auction; Budget Constraints; Liquidity.

JEL classification: D44; D47; D82

*Corresponding author, bbaisa@amherst.edu; Amherst College, Department of Economics.

†srabinovich@amherst.edu; Amherst College, Department of Economics.

1 Introduction

The seminal papers on auction design assume that a bidder's ability to pay for a good exceeds her willingness to pay; preferences are quasilinear.¹ Yet, in many well-studied auction markets, this restriction does not hold - bidders face binding budget constraints. Authors have argued that the presence of budgets limits the applicability of auction theory in real-world settings.² In response, a literature developed that analyzed how the presence of budgets changes the auction design problem. For example, Che and Gale (1996, 1998) compare standard auctions with budgets, and Laffont and Robert (1996), Che and Gale (2000), and Pai and Vohra (2014) construct revenue-maximizing auctions when bidders have budgets.³

The above literature assumes that budgets are exogenously determined.⁴ In practice, however, bidders can choose the amount of resources devoted towards bidding in the auction. Bidding in the auction requires liquid resources, which can be obtained by borrowing or diverting resources away from alternative profitable investments. Thus, even if a bidder does not borrow from a bank, she still incurs an opportunity cost of funds. This raises the question: what is the optimal selling mechanism when buyers endogenously make such liquidity choices? This question is also important not only for the auctions literature, but also for the growing literature in monetary economics that models liquidity choices for decentralized trade.⁵

In this paper, we study auction design with endogenously determined budgets. We consider an auction for an indivisible good, where bidders have independent private values, and endogenously determine their budgets after observing their private information. Borrowing is costly, and a bidder incurs a cost of borrowing whether or not her bid wins.

We show that bidders are indifferent between competing in the first-price, second-price and all-pay auctions. However, the auctions are not revenue-equivalent. The all-pay auction has the highest expected revenue, and the second-price auction has the lowest expected revenue. This is the same revenue ranking that we see in the exogenous budget case of Che and Gale (1996), but the intuition is distinct. In Che and Gale (1996), the all-pay auction

¹See Myerson (1981) and Riley and Samuelson (1981).

²See Rothkopf (2007).

³This paper studies the case of private values, as do the previously cited works. There is additional work that studies bidding in auctions with budgets in interdependent value environments. For example, Fang and Parreiras (2003) and Kotowski and Li (2011).

⁴Two exceptions are Burkett (2015a, 2015b), where budgets are endogenously determined in a principal-agent relationship, and Rhodes-Kropf and Viswanathan (2005), who consider a variety of forms of endogenous financing in a first-price auction for a risky asset.

⁵Our paper's environment, in which agents choose their money holdings before engaging in trade, is reminiscent of the models in Lagos and Wright (2005) and Rocheteau and Wright (2005). Lagos, Rocheteau and Wright (2015) provide a recent survey. In this literature, the pricing mechanism is crucial for determining real balances, output, and efficiency. Galenianos and Kircher (2008), also introduced auctions into monetary models. However, this literature assumes that goods are sold by second-price auctions, without examining whether it is optimal.

yields higher revenues than first or second price because the budget constraint is less likely to bind in the all-pay. In our model, the all-pay auction yields higher revenues because it economizes on bidders' borrowing costs.

On the design of revenue-maximizing auctions, we show that the optimal auction is implemented by an all-pay auction with a suitably chosen reserve price.⁶ This differs from an optimal auction with exogenous budgets: the latter would not necessarily sell the good to the highest-value bidder.⁷ The reason for the difference is that, with exogenous budgets, high-value bidders are unable to express high demand for the good. With endogenous budgets, high-value bidders are able to reveal that they have a higher demand by borrowing more money to place higher bids. While placing higher bids comes at a cost, the auctioneer can minimize these costs by using an all-pay payment scheme.

2 Model

2.1 Environment

A seller has one unit of an indivisible good, which she has no value for. There are $N \geq 2$ risk-neutral potential bidders. A bidder's preferences are described her valuation $v \in \mathbb{R}_+$. Bidder valuations are *i.i.d.* draw of a random variable with density f . We assume that f has full support over $[\underline{v}, \bar{v}] \subset \mathbb{R}_+$. Thus, f has an associated distribution function F which is continuous and strictly increasing over $[\underline{v}, \bar{v}]$, with $F(\underline{v}) = 0$ and $F(\bar{v}) = 1$.

A bidder determines her budget after finding out her value, but before placing her bid and observing competing bids. If a bidder borrows b , she must repay $b + c(b)$, where $c(b)$ is continuous, differentiable, strictly increasing, and weakly convex.⁸

Therefore, if bidder i wins the good, pays p and borrows b , where $b \geq p$, her utility is

$$v_i - p - c(b).$$

A bidder cannot place a bid that exceeds the amount of money that she has borrowed.

2.2 Mechanisms

By the revelation principle, we limit attention to direct revelation mechanisms. Given the profile of reported types $\mathbf{v} = (v_1, \dots, v_N)$, the direct revelation mechanism states a bidder's

⁶This requires that bidders have monotone virtual values, as in Myerson (1981).

⁷For example, in Laffont and Robert's (1996) optimal auction, all bidders with values above some threshold win with equal probability.

⁸When determining the optimal auction, we assume that c is linear.

(ex-post) probability of winning $Q_i(\mathbf{v})$, (ex-post) expected transfer $T_i(\mathbf{v})$, and borrowing $b_i(v_i)$. The amount a bidder borrows is independent of other bidders' reported types, because a bidder decides her budget before bidding. Since borrowing is costly, bidders do not borrow more than they would need to pay in the auction; hence $b_i(v) = \sup_{v_{-i}} T_i(\mathbf{v})$. Feasibility requires

$$\sum_{i=1}^N Q_i(\mathbf{v}) \leq 1.$$

Let $q_i(v) = \mathbb{E}_{v_{-i}}(Q_i(v, v_{-i}))$ be bidder i 's interim probability of winning when reporting type v . Similarly, $t_i(v) = \mathbb{E}_{v_{-i}}(T_i(v, v_{-i}))$ denotes the interim expected payment made by bidder i . Therefore, the expected utility of bidder i , if her true type is v_i and she reports type v , is

$$U_i(v, v_i) = q_i(v)v_i - t_i(v) - c(b_i(v)).$$

The direct revelation mechanism is (interim) incentive-compatible if

$$U_i(v_i, v_i) \geq U_i(v, v_i) \quad \forall v \in [\underline{v}, \bar{v}], \quad i = 1, \dots, N.$$

3 Standard Auctions

Incentive compatibility implies that $q_i(v)$ and $t_i(v)$ are weakly increasing. Thus, both functions are differentiable almost everywhere along $[\underline{v}, \bar{v}]$. Pick any point where $U_i(v, v_i)$ is differentiable with respect to v at v_i . The necessary first order condition for incentive compatibility implies

$$\frac{\partial U_i(v, v_i)}{\partial v} = q_i'(v)v_i - t_i'(v) - c'(b_i(v))b_i'(v) = 0.$$

Therefore, the total derivative of $U_i(v_i, v_i)$ with respect to v_i is

$$\frac{dU_i(v_i, v_i)}{dv_i} = q_i(v_i).$$

Since U_i is differentiable almost everywhere, this implies

$$U_i(v_i, v_i) = U_i(\underline{v}, \underline{v}) + \int_{\underline{v}}^{v_i} q_i(s) ds. \tag{1}$$

Thus, we can use standard Myersonian approach to characterize bidder i 's interim expected utility. Equation (1) implies that bidder i is indifferent between any two mechanisms that give the same interim probability of winning, and give the same expected utility to the low type. In particular, bidders are indifferent between participating in the first-price, second-price,

and all-pay auction.

We use equation (1) to establish a revenue ranking of the three standard auctions. We consider symmetric Bayes Nash Equilibria.

Proposition 1. (*Revenue ranking of standard auctions*)

The all-pay auction has strictly greater expected revenue than the first price auction. The first price auction has strictly greater expected revenue than the second price auction.

Proof. Since bids are strictly increasing in each auction, $q(v) = F(v)^{N-1}$, and $U(\underline{v}, \underline{v}) = 0$. Let $\beta^f(v_i)$, $\beta^s(v_i)$, and $\beta^a(v_i)$ be the symmetric equilibrium bid functions in the first price auction, the second price auction, and the all-pay auction, respectively.

In each auction, a bidder never pays in excess of her bid. Thus, a bidder borrows exactly the amount she bids: $\beta^j(v_i) = b^j(v_i)$ for $j = f, s, a$.

First, we show that $\beta^f(v_i) > \beta^a(v_i) \forall v_i \in (\underline{v}, \bar{v})$. Equation (1) implies that

$$F(v_i)^{N-1} (v_i - \beta^f(v_i)) - c(\beta^f(v_i)) = F(v_i)^{N-1} v_i - \beta^a(v_i) - c(\beta^a(v_i)),$$

and therefore

$$F(v_i)^{N-1} \beta^f(v_i) + c(\beta^f(v_i)) = \beta^a(v_i) + c(\beta^a(v_i)).$$

Since $F(v_i)^{N-1} \in (0, 1) \forall v_i \in (\underline{v}, \bar{v})$, it follows that $\beta^f(v_i) > \beta^a(v_i) \forall v_i \in (\underline{v}, \bar{v})$. But then, $c(\beta^f(v_i)) > c(\beta^a(v_i))$, and so the above equality immediately implies $\beta^a(v_i) > F(v_i)^{N-1} \beta^f(v_i)$ for all $v_i \in (\underline{v}, \bar{v})$. Thus, bidder i makes a greater expected payment in the all pay auction versus the first price auction conditional on being type v_i .

The proof that the first price has higher revenue than the second price is similar. In the first price auction, a bidder's utility when she is type v_i is

$$F(v_i)^{N-1} (v_i - \beta^f(v_i)) - c(\beta^f(v_i)).$$

In the second price auction, a bidder's utility when she is type v_i is

$$F(v_i)^{N-1} \left(v_i - \mathbb{E}(\max_{j \neq i} \beta^s(v_j) | v_i \geq \max_{j \neq i} v_j) \right) - c(\beta^s(v_i)).$$

Equation (1) implies that these utilities must be equal, and so

$$F(v_i)^{N-1} \beta^f(v_i) + c(\beta^f(v_i)) = F(v_i)^{N-1} \mathbb{E}(\max_{j \neq i} \beta^s(v_j) | v_i \geq \max_{j \neq i} v_j) + c(\beta^s(v_i)).$$

Since $\beta^s(v_i) > \mathbb{E}(\max_{j \neq i} \beta^s(v_j) | v_i \geq \max_{j \neq i} v_j) \forall v_i > \underline{v}$, the above equality implies $\beta^s(v_i) > \beta^f(v_i) \forall v_i > \underline{v}$. This, in turn, implies $c(\beta^s(v_i)) > c(\beta^f(v_i))$, and so the above equality immediately implies $F(v_i)^{N-1} \beta^f(v_i) \geq F(v_i)^{N-1} \mathbb{E}(\max_{j \neq i} \beta^s(v_j) | v_i \geq \max_{j \neq i} v_j)$ for all $v_i \in (\underline{v}, \bar{v})$. Thus, bidder i makes a higher expected payment conditional on winning in the first price auction than in the second price auction. Since both auctions assign the good to the bidder with the highest v_i , this implies that the expected total payment made in the first price auction exceeds the expected total payment made in the second price auction. \square

Although Proposition 1 mirrors results from Che and Gale (1996, 2006) with exogenous budgets, the intuition is different. With exogenous budgets, Che and Gale argue that the all-pay auction gives higher revenues because the budget constraint is less likely to bind in an all-pay auction. In our setting, the all-pay auction gives greater revenues because bidders incur lower bid preparation costs. Since the equilibrium bid distribution is higher in the first-price or second-price auction, bidders spend more money to finance their bids. It may seem that bidders should then prefer the all-pay auction to the first or second-price auctions. However, equation (1) shows us that the gains in surplus generated by borrowing less are competed away in the auction environment. Since the bidders borrow less money, yet get the same surplus, it then must be the case that the extra surplus goes to the auctioneer.

4 Revenue maximizing auction

In this section, we assume that $c(b) = rb$: there is a fixed interest rate $r > 0$. We first show that the optimal mechanism has an all-pay payment structure. For ease of notation, we write $U_i(v, v)$ as $U_i(v)$.

Lemma 1. *Consider any two incentive-compatible direct revelation mechanisms α, γ where $U_i^\alpha(\underline{v}) = U_i^\gamma(\underline{v})$ and $q_i^\alpha(v) = q_i^\gamma(v) \forall v$. In addition, suppose that $b_i^\alpha(v) = t_i^\alpha(v) \forall v$. Then, $t_i^\alpha(v) \geq t_i^\gamma(v) \forall v$.*

Proof. Let $q_i(v) = q_i^\alpha(v) = q_i^\gamma(v)$. By equation (1) we have

$$q_i(v)v - (1+r)t_i^\alpha(v) = \int_{\underline{v}}^v q_i(s)ds = q_i(v)v - t_i^\gamma(v) - rb_i^\gamma(v),$$

and so

$$(1+r)t_i^\alpha(v) = t_i^\gamma(v) + rb_i^\gamma(v).$$

From the restriction $b_i(v) \geq \sup_{v_{-i}} T_i(\mathbf{v})$, it follows that $b_i^\gamma(v) \geq t_i^\gamma(v)$. This implies that $(1+r)t_i^\alpha(v) = t_i^\gamma(v) + rb_i^\gamma(v) \geq (1+r)t_i^\gamma(v)$. Therefore $t_i^\alpha(v) \geq t_i^\gamma(v)$. \square

Thus, when considering revenue-maximizing auctions, it suffices to consider only all-pay mechanisms, i.e. mechanisms in which $b_i(v) = \sup_{v_{-i}} T_i(\mathbf{v}) = t_i(v) \forall v$. From equation (1), we then get that for any incentive compatible allocation function q_i , where $U(v) = 0$,

$$U_i(v) = \int_{\underline{v}}^v q_i(s) ds = q_i(v)v - (1+r)t_i(v).$$

Bidder i 's expected payment can therefore be written as a function of the allocation function,

$$t_i(v) = \frac{1}{1+r} \left(q_i(v)v - \int_{\underline{v}}^v q_i(s) ds \right).$$

Thus, expected revenue is

$$\frac{1}{1+r} \sum_{i=1}^N \int_{\underline{v}}^{\bar{v}} \left(q_i(v)v - \int_{\underline{v}}^v q_i(s) ds \right) f(v) dv.$$

This is equivalent to the problem solved by Myerson (1981) and Riley and Samuelson (1981). Using their results, we can rewrite the above expression in terms of virtual values. Let $\phi(v) = v - 1 - F(v)/f(v)$ be the virtual value of a bidder with type v . Expected revenue is then equal to

$$\frac{1}{1+r} \sum_{i=1}^N \mathbb{E}(q_i(v)\phi(v)).$$

Maximizing pointwise subject to feasibility shows that, if $\phi(v)$ is weakly increasing,

$$Q_i(v_i, v_{-i}) = \begin{cases} 1 & \text{if } \phi(v) \geq 0 \text{ and } v_i > \max v_{-i} \\ 0 & \text{if } \phi(v) < 0 \text{ or } \max v_{-i} > v_i \end{cases}.$$

Thus, if $\phi(v)$ is weakly increasing, the optimal mechanism can be implemented by an all-pay auction with a suitably chosen reserve price \underline{b} .⁹ The reserve price makes a bidder with type v^* , where $\phi(v^*) = 0$, indifferent between bidding and not bidding. That is, \underline{b} is such that

$$F(v^*)^{N-1}v^* - (1+r)\underline{b} = 0.$$

We have thus established the following result:

Proposition 2. *Suppose that $\phi(v)$ is weakly increasing. The optimal mechanism can be implemented by an all-pay auction with minimum bid \underline{b} . The minimum bid \underline{b} satisfies*

⁹If F satisfies the monotone hazard rate condition, then ϕ is weakly increasing in v .

$$\underline{b} = \frac{F(v^*)^{N-1}v^*}{1+r},$$

where v^* is such that

$$v^* = \frac{1 - F(v^*)}{f(v^*)}.$$

The result differs from results on auctions with exogenous budgets, e.g., Laffont and Robert (1996) and Pai and Vohra (2014). Both papers suggest using a mechanism with an all-pay payment scheme, but a different winning rule. Specifically, they show that the bidder with the highest valuation does not necessarily win the object with probability 1, even if all bidders have the same budget. Here we show that, when we endogenize the budget constraint, we get a different result, which looks closer to that of Myerson (1981). When bidders have weakly increasing virtual values, the optimal mechanism is a standard all-pay auction with a reserve price, which allocates the good to the highest-value bidder.

Acknowledgements

We thank an anonymous referee, Justin Burkett, Jun Ishii, and Mallesh Pai for helpful comments.

References

- [1] Justin Burkett. Endogenous budget constraints in auctions. *Journal of Economic Theory*, Forthcoming, 2015.
- [2] Justin Burkett. Optimally constraining a bidder using a simple budget. *Theoretical Economics*, Forthcoming, 2015.
- [3] Yeon-Koo Che and Ian Gale. Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints. *Economic Letters*, 50:373–379, 1996.
- [4] Yeon-Koo Che and Ian Gale. Standard auctions with financially constrained bidders. *Review of Economic Studies*, 65:1–21, 1998.
- [5] Yeon-Koo Che and Ian Gale. The optimal mechanism for selling to a budget-constrained buyer. *Journal of Economic Theory*, 92(2):198–233, June 2000.
- [6] Hanming Fang and Sergio O. Parreiras. On the failure of the linkage principle with financially constrained bidders. *Journal of Economic Theory*, 110:374–392, 2003.
- [7] Manolis Galenianos and Philipp Kircher. A model of money with multilateral matching. *Journal of Monetary Economics*, 55(6):1054–1066, September 2008.
- [8] Maciej Kotowski and Fei Li. On the continuous equilibria of affiliated-value, all-pay auctions with private budget constraints. *Games and Economic Behavior*, 85:84–108, 2014.
- [9] Jean-Jacques Laffont and Jacques Robert. Optimal auction with financially constrained buyers. *Economic Letters*, 52:181–186, 1996.
- [10] Ricardo Lagos, Guillaume Rocheteau, and Randall Wright. Liquidity: a new monetarist perspective. *Journal of Economic Literature*, forthcoming, 2015.
- [11] Ricardo Lagos and Randall Wright. A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113(31):463–484, 2005.
- [12] Eric Maskin. Auctions, development and privatization: Efficient auctions with liquidity-constrained buyers. *European Economic Review*, 44:667–681, 2000.
- [13] Roger Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, Feb. 1981.
- [14] Malleesh M. Pai and Rakesh Vohra. Optimal auctions with financially constrained buyers. *Journal of Economic Theory*, pages 383–425, March 2014.
- [15] Matthew Rhodes-Kropf and S. Viswanathan. Financing auction bids. *RAND Journal of Economics*, 36(4):789–815, 2005.
- [16] Guillaume Rocheteau and Randall Wright. Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73(1):175–202, 2005.
- [17] Michael H. Rothkopf. Thirteen reasons why the vickrey-clarke-groves process is not practical. *Operations Research*, 55(2):191–197, 2007.