# Efficient Multi-unit Auctions for Normal Goods

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**Abstract**

I study multi-unit auction design when bidders have private values, multi-unit demands, and non-quasilinear preferences. Without quasilinearity, the Vickrey auction loses its desired incentive and efficiency properties. I give conditions under which we can design a mechanism that retains the Vickrey auction's desirable incentive and efficiency properties: (1) individual rationality, (2) dominant strategy incentive compatibility, and (3) Pareto efficiency. I show that there is a mechanism that retains the desired properties of the Vickrey auction if there are two bidders who have single-dimensional types. I also present an impossibility theorem that shows that there is no mechanism that satisfies Vickrey's desired properties and weak budget balance when bidders have multi-dimensional types.

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# 1 Introduction

## 1.1 Motivation

Understanding how to design auctions with desirable incentive and efficiency properties is a central question in mechanism design. The Vickrey-Clarke-Groves (hereafter, VCG) mechanism is celebrated as a major achievement in the field because it performs well in both respects — agents have a dominant strategy to truthfully report their private information and the mechanism implements an efficient allocation of resources. However, the VCG mechanism loses its desired incentive and efficiency properties without the quasilinearity restriction. Moreover, there are many well-studied cases where the quasilinearity restriction is violated: bidders may be risk averse, have wealth effects, face financing constraints or be budget constrained. Indeed, observed violations of quasilinearity are frequently cited as reasons for why we do not see multi-unit Vickrey auctions used in practice. For example, Ausubel and Milgrom (2006), Rothkopf (2007), and Nisan et al. (2009) all cite budgets and financing constraints as salient features of real-world auction settings that inhibits the use of the Vickrey auctions. Che and Gale (1998) note that bidders often face increasing marginal costs of expenditures when they have access to imperfect financial markets.

In this paper, I study multi-unit auctions for $K$ indivisible homogenous goods when bidders have private values, multi-unit demands, and non-quasilinear preferences. I provide conditions under which we can construct an auction that retains the desired incentive and efficiency properties of the Vickrey auction: (1) ex post individual rationality, (2) dominant strategy incentive compatibility, and (3) ex post Pareto efficiency (hereafter, efficiency). My results hold on a general preference domain. Instead of assuming that bidders have quasilinear preferences, I assume only that bidders have positive wealth effects; i.e. the goods being auctioned are normal goods. My environment nests well-studied cases where bidders are risk averse, have budgets, or face financing constraints.

My first main result shows that there is a mechanism that satisfies the desired properties of the Vickrey auction if there are two bidders and bidders have single-dimensional types (Theorem 1). The mechanism implements an (ex post Pareto) efficient outcome — i.e. an outcome where there are no ex post Pareto improving trades amongst bidders. The proof of Theorem 1 differs from proofs of positive implementation results in quasilinear settings. With quasilinearity, an efficient auction can be constructed in two steps. First, we note that there is a generically unique efficient assignment of goods. Then, we solve the efficient auction design problem by finding transfers that implement the exogenously determined assignment rule. Without quasilinearity, the space of efficient outcomes is qualitatively different because a particular assignment of units can be associated with an efficient outcome for some levels of payments, but not for others. This is because a bidder's willingness to buy/sell an additional unit to/from her rival depends on her payment. For this reason, I use a fixed point argument to determine the efficient mechanism's payment rule and assignment rule simultaneously. More precisely, I construct a transformation that maps an arbitrary mechanism to a more efficient mechanism. The transformed mechanism specifies the efficient assignment of units in the case when payments are determined according to the arbitrary mechanism's payment rule. The transformed payment rule is the payment rule that implements the transformed assignment rule. I show that a fixed point of the transformation defines an efficient mechanism and I use Schauder's fixed point theorem to show that a fixed point of the transformation exists. Thus, there is a mechanism that retains the Vickrey auction's desirable incentive and efficiency properties in the two-bidder single-dimensional types case. Furthermore, I provide a constructive proof that shows there is also a mechanism with the desired Vickrey properties when many bidders compete to win two units.

The positive implementation results for the single-dimensional types case do not carry over to multi-dimensional settings though. I obtain an impossibility result because wealth effects and multi-unit demands combine to inhibit efficient implementation.[1] These two modeling assumptions imply that in an efficient auction, a bidder's demand for later units of the good endogenously depends on her rivals' reported types, even in the private value setting. This is because a bidder's demand for her second unit of the good depends on the price she paid for her first unit, and in an efficient auction, the price a bidder paid for her first unit necessarily varies with her rivals' reported types. Thus, positive wealth effects imply that in an efficient auction, there is endogenous interdependence between a bidder's demand for later units and her rivals' types. Furthermore, the prior literature on efficient multi-unit auction design without quasilinearity has not noted this connection between private value models without quasilinearity and interdependent value models with quasilinearity (like those studied by Dasgupta and Maskin (2000), for example). I use this connection to motivate my proofs of the impossibility theorem for the multi-dimensional types case (Theorem 3).

The paper proceeds as follows. The remainder of the Introduction discusses the related literature. Section 2 presents my model for bidders with single-dimensional types. Section 3 presents the results for the single-dimensional case. Section 4 presents the impossibility theorem for bidders with multi-dimensional types. Proofs and additional results are in the appendix.

[1] The prior literature shows that we get positive implementation results if we relax either assumption. When there are no wealth effects, the Vickrey auction is efficient and dominant strategy implementable, even if bidders have multi-unit demands and multi-dimensional types. Similarly, when bidders have unit demands and non-quasilinear preferences, Saitoh and Serizawa (2008) and Morimoto and Serizawa (2015) show that the minimum price Walrasian rule is the unique mechanism that is efficient and dominant strategy implementable. See the related literature section for further discussion.

## 1.2 Related literature

Friedman (1960) proposed the uniform price auction for homogenous goods. If bidders truthfully report their demands, the uniform price auction will allocate goods efficiently. However, Ausubel et al. (2014) show that bidders have an incentive to underreport their demand in the uniform price auction. In contrast, the Vickrey-Clarke-Groves mechanism efficiently allocates goods and gives bidders a dominant strategy to truthfully reveal their private information to the mechanism designer. Holmstrom (1979) gives conditions under which VCG is the unique mechanism that satisfies these two objectives. In addition, Ausubel (2004) describes an ascending auction format, called the clinching auction, that implements the VCG allocation and payment rule.

Two crucial assumptions are needed to obtain Vickrey's positive implementation result: (1) agents have private values and (2) agents have quasilinear preferences. There is a long literature that studies how Vickrey's result generalizes without private values. In this literature, Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Jehiel et al. (2006) give impossibility results for when agents have multi-dimensional types.[2] In contrast, Bikhchandani (2006) shows that there are non-trivial social choice rules in interdependent value settings where bidders compete to win private goods. He proves the existence of a constrained efficient mechanism in a single unit auction setting where bidders have multi-dimensional types.

There is a relatively smaller literature on how Vickrey's positive implementation result generalizes without (2), the quasilinearity restriction, and that is the question I study in this paper.[3] In particular, I study how Vickrey's results extend to a multi-unit auction setting with homogenous goods where bidders have multi-unit demands and non-quasilinear preferences.

There is a literature that studies efficient multi-unit auction design in settings where bidders have unit demands and non-quasilinear preferences. Saitoh and Serizawa (2008) and Morimoto and Serizawa (2015) both show that Vickrey's positive implementation result can be extended to such settings. Saitoh and Serizawa study the case where all objects are homogenous. That is the case studied in this paper as well. Morimoto and Serizawa show that Vickrey's positive implementation result can be extended to a heterogeneous good setting

[2] Maskin (1992), Krishna (2003), and Perry and Reny (2002, 2005) give sufficient conditions for efficient auction design in single-dimensional type settings.

[3] Most of the literature on auctions without quasilinearity has studied revenue maximization and bid behavior in commonly used auctions. Maskin and Riley (1984) study revenue maximization when bidders have single-dimensional private information. Baisa (2017) studies revenue-maximizing auction design in a similar setting to this paper where bidders have positive wealth effects. There is also a literature that studies the performance of standard auction formats in certain non-quasilinear settings. Matthews (1987), Hu, Matthews, and Zou (2015), and Che and Gale (1996, 1998, 2006) study standard auctions when bidders have budgets, face financial constraints, and are risk averse.

where bidders have non-quasilinear preferences. In particular, they show that Demange and Gale's (1985) minimum price Walrasian rule is dominant strategy implementable and efficient when bidders have unit demands. Their positive implementation result holds in cases where bidders have multi-dimensional private information. My paper is different from this line of research because I study the case where goods are homogenous and bidders have multi-unit demands. My results show that the combination of multi-dimensional private information and multi-unit demands yield an impossibility result in a homogenous good setting.

Most other work on efficient multi-unit auction design without quasilinearity focuses on a particular violation of quasilinearity — bidders with hard budgets. Dobzinski, Lavi, and Nisan (2012) study efficient multi-unit auction design where bidders have multi-unit demands, constant and private marginal values for additional units, and hard budgets. They show that the clinching auction (see Ausubel (2004)) is an efficient auction if bidders have public budgets. If bidders have private budgets, then they show that there is no efficient auction. Subsequent work by Lavi and May (2012) and Goel, Mirrokni, and Paes Leme (2015, Theorem 5.11) also provide impossibility results for the case where bidders have hard budgets. In Lavi and May, bidders have two-dimensional types and a public budget; and in Goel, Mirrokni, and Paes Leme, bidders have an infinite-dimensional type and a public budget. Kazumura and Serizawa (2016) also study efficient design with multi-unit demand. Like this paper, they move beyond the hard budgets case and consider a general non-quasilinear setting. In their setting, there are heterogeneous goods and one buyer has multi-unit demands. Their setting assumes that bidders have infinite-dimensional private information because their impossibility theorem allows bidders to have any rational preference.

My paper expands on this line of research by similarly studying the efficient auction design problem. Like the papers cited in the prior paragraph, I use the taxation principle to form necessary conditions for efficient auction design that yield a proof by contradiction. My proof approach also differs. The prior work has obtained impossibility results by finding an efficient mechanism in a more restricted setting and then showing that the efficient mechanism loses incentive compatibility in their more general settings. I instead get a proof by contradiction by noting the connection between this design problem and the efficient design problem with interdependent values - the proof of my impossibility theorem (Theorem 3) does not rely on the results I used to prove the existence of an efficient mechanism (Theorems 1 and 2). In addition, my impossibility theorem for bidders with multi-dimensional types (Theorem 3) holds on a coarser type space and nests the cases where bidders have two-dimensional and infinite-dimensional types.

Maskin (2000) and Pai and Vohra (2014) study a related question of expected surplus maximizing auctions in the budget case under the weaker solution concept of Bayesian implementation for bidders with *i.i.d* types. In contrast, this paper studies Vickrey's problem of efficient auction design in dominant strategies. My results are also related to recent literature on value maximizing bidders (see Fadaei and Bichler (2016)). Value maximization is a limiting case of my model where a bidder gets arbitrarily small disutility from spending money up to their budget.

Similar to this paper, Kazumura and Serizawa (2016) study efficient design with multi-unit demand in a general non-quasilinear setting. In their setting, there are heterogeneous goods and buyers with non-quasilinear preferences, and one buyer with multi-unit demands. Their setting assumes that bidders have infinite-dimensional private information because their impossibility theorem allows bidders to have any rational preference.

Outside of the auction literature, there is some work on the scope of implementation without quasilinearity. Kazumura, Mishra, and Serizawa (2017) provide results on the scope of dominant strategy implementation in a general mechanism design setting where agents are not restricted to have quasilinear preferences. Garratt and Pycia (2014) investigate how positive wealth effects influence the possibility of efficient bilateral trade in a Myerson and Satterthwaite (1983) setting. In contrast to this paper, Garratt and Pycia show that the presence of wealth effects may help induce efficient trade when there is two-sided private information. Nöldeke and Samuelson (2018) also study implementation in principal-agent problems and two-sided matching problems without quasilinearity. They extend positive implementation results from the quasilinear domain to the non-quasilinear domain by establishing a duality between the two settings.

# 2 Model

## 2.1 Bidder preferences - the single-dimensional types case

A seller has $K \geq 2$ units of an indivisible homogenous good. There are $N \geq 2$ bidders who have private values and multi-unit demands. Bidder $i$'s preferences are described by her type $\theta_i \in [0, \overline{\theta}] := \Theta \subset \mathbb{R}_+$. If bidder $i$ wins $q \in \{0, 1, \ldots, K\} := \mathbb{K}$ units and receives $m \in \mathbb{R}$ in monetary transfers, her utility is $u(q, m, \theta_i) \in \mathbb{R}$. We assume that $u$ is commonly known and $\theta_i \in \Theta$ is bidder $i$'s private information. A bidder's utility function is continuous in her type $\theta_i$ and continuous and strictly increasing in monetary transfers $m$.[4]

[4]It is without loss of generality to assume that a bidder has an initial wealth of 0, or measure wealth in terms of deviation from initial wealth. A bidder with utility $u$ and initial wealth $w_0$, has the same preferences over units and transfers as a bidder with initial wealth 0 and utility $\hat{u}$ where we define $\hat{u}$ as $\hat{u}(q, m, \theta_i) = u(q, m + w_0, \theta_i)\ \forall q \in \{0, 1, \ldots, K\},\ m \in \mathbb{R},\ \theta_i \in \Theta$. I study deviations from initial wealth because this allows a more flexible interpretation of the model where we can also include wealth as an element of bidders' private information. For example, in Section 3.1, I provide an example of an efficient

If $\theta_i = 0$, then bidder $i$ has no demand for units,

$$u(q, m, 0) = u(q', m, 0),\ \forall q, q' \in \mathbb{K},\ m \in \mathbb{R}.$$

If $\theta_i \in (0, \overline{\theta}]$, then bidder $i$ has positive demand for units,

$$q' > q \iff u(q', m, \theta_i) > u(q, m, \theta_i),\ \forall q, q' \in \mathbb{K},\ m \in \mathbb{R},\ \theta_i \in (0, \overline{\theta}].$$

Without loss of generality, I assume that $u(0, 0, \theta_i) = 0\ \forall \theta_i \in \Theta$. Bidders have bounded demand for units of the good. Thus, I assume that there exists a $h > 0$ such that

$$0 > u(q, -h, \theta_i)\ \forall q \in \mathbb{K},\ \theta_i \in \Theta.$$

I make three additional assumptions on bidders' preferences. First, I assume that bidders have declining demand for additional units. Therefore, if a bidder is unwilling to pay $p$ for her $q^{th}$ unit, then she is unwilling to pay $p$ for her $(q+1)^{st}$ unit. This generalizes the declining marginal values assumption imposed in the benchmark quasilinear setting.

**Assumption 1.** (Declining Demand)
*Bidders have declining demand for additional units if $u$ is such that*

$$u(q-1, m, \theta_i) \geq u(q, m-p, \theta_i) \implies u(q, m, \theta_i) > u(q+1, m-p, \theta_i),$$

*for any $m \in \mathbb{R},\ q \in \{1, \ldots, K-1\}$, and $\theta_i \in \Theta$.*

Second, I assume that bidders have positive wealth effects. This means a bidder's demand does not decrease as her wealth increases. To be more concrete, suppose that bidder $i$ was faced with the choice between two bundles of goods. The first bundle provides $q_h$ units of goods for a total price of $p_h$ , and the second bundle provides $q_\ell$ units of goods for a total price of $p_\ell$, where $q_h, q_\ell \in \mathbb{K}$ and $p_h, p_\ell \in \mathbb{R}$ are such that $q_h > q_\ell$ and $p_h > p_\ell$. If bidder $i$ prefers the first bundle with more goods, then positive wealth effects state that she also prefers the first bundle with more goods if we increased her wealth prior to her purchasing decision. This is a multi-unit generalization of Cook and Graham's (1977) definition of an indivisible, normal good. I define two versions of positive wealth effects, weak and strict. I assume that bidder preferences satisfy the weak version, which nests quasilinearity, when I present the positive implementation result. When I present the impossibility theorems, I assume the strict version of positive wealth effects, because the strict version rules out the

mechanism where a bidder's wealth (her soft budget) varies with her private type.

quasilinear setting where the benchmark Vickrey auction solves the efficient auction design problem.

**Assumption 2.** (Positive Wealth Effects)
*Consider any* $q_h, q_\ell, p_h, p_\ell$ *where,* $q_h > q_\ell$, $p_h > p_\ell$, $q_h, q_\ell \in \mathbb{K}$, *and* $p_h, p_\ell \in \mathbb{R}$. *Bidders have weakly positive wealth effects if*

$$u(q_h, -p_h, \theta_i) \geq u(q_\ell, -p_\ell, \theta_i) \implies u(q_h, m - p_h, \theta_i) \geq u(q_\ell, m - p_\ell, \theta_i) \ \forall m > 0,\ \theta_i \in \Theta,$$

*and strictly positive wealth effects if*

$$u(q_h, -p_h, \theta_i) \geq u(q_\ell, -p_\ell, \theta_i) \implies u(q_h, m - p_h, \theta_i) > u(q_\ell, m - p_\ell, \theta_i) \ \forall m > 0,\ \theta_i \in \Theta.$$

Finally, I assume that bidders with higher types have greater demands.

**Assumption 3.** (Single Crossing)
*Suppose* $q_h > q_\ell$ *and* $p_h > p_\ell$ *where* $q_h, q_\ell \in \mathbb{K}$, *and* $p_h, p_\ell \in \mathbb{R}$. *Then, bidder preferences are such that*

$$u(q_h, -p_h, \theta_i) \geq u(q_\ell, -p_\ell, \theta_i) \implies u(q_h, -p_h, \theta_i') > u(q_\ell, -p_\ell, \theta_i') \ \forall \theta_i, \theta_i' \in \Theta \ s.t.\ \theta_i' > \theta_i.$$

I let $b_1(\theta_i)$ where $b_1 : \Theta \to \mathbb{R}_+$ be the amount that bidder $i$ is willing to pay for her first unit of the good. Thus, $b_1(\theta_i)$ implicitly solves

$$0 = u(1, -b_1(\theta_i), \theta_i),$$

for all $\theta_i \in \Theta$. It is without loss of generality to assume types are such that $b_1(\theta) = \theta \ \forall \theta \in \Theta$.[5] Thus, $\theta_i$ parameterizes the intercept of bidder $i$'s demand curve (assuming bidder $i$ pays no entry fee).

I similarly define $b_k(\theta_i, x)$ where $b_k : \Theta \times \mathbb{R} \to \mathbb{R}_+$ as bidder $i$'s willingness to pay for her $k^{th}$ unit, conditional on winning her first $k-1$ units for a cost of $x \in \mathbb{R}$. More precisely, $b_k(\theta_i, x)$ is implicitly defined as solving

$$u(k-1, -x, \theta_i) = u(k, -x - b_k(\theta_i, x), \theta_i),$$

for all $k \in \{2, \ldots, K\}$, $\theta_i \in \Theta$ and $x \in \mathbb{R}$. I analogously define $s_k(\theta_i, x)$ as bidder $i$'s willingness to sell her $k^{th}$ unit, conditional on having paid $x$ in total. Thus, a bidder's willingness to sell

[5] It is without loss of generality to assume that $b_1(\theta) = \theta \ \forall \theta \in \Theta$, because (1) we assume that $u(q, m, 0) = u(q', m, 0)$ which implies that $b_1(0) = 0$, and (2) single crossing implies that $b_1(\cdot)$ is strictly increasing.

her $k^{th}$ unit $s_k(\theta_i, x)$ is implicitly defined as solving

$$u(k, -x, \theta_i) = u(k-1, -x + s_k(\theta_i, x), \theta_i),$$

for all $k \in \{1, \ldots, K\}$, $\theta_i \in \Theta$ and $x \in \mathbb{R}$. Note that by construction,

$$s_k(\theta_i, x + b_k(\theta_i, x)) = b_k(\theta_i, x) \; \forall k \in \{1, \ldots, K\}, \; \theta_i \in \Theta, \; x \in \mathbb{R}.$$

In words, this means that bidder $i$ is indifferent between buying/selling her $k^{th}$ unit at price $b_k(\theta_i, x)$, given that she paid $x$ to win her first $k-1$ units.

Assumptions 1, 2, and 3 imply:

1. $b_k(\theta, x) > b_{k+1}(\theta, x)$ and $s_k(\theta, x) > s_{k+1}(\theta, x)$ for all $k \in \{1, \ldots, K-1\}, \; \theta \in \Theta, \; x \in \mathbb{R}$.
2. $b_k(\theta, x)$ and $s_k(\theta, x)$ are continuous and decreasing in the second argument $x$ for all $x \in \mathbb{R}, \; k \in \{1, \ldots, K\}, \; \theta \in \Theta$.[6]
3. $b_k(\theta, x)$ and $s_k(\theta, x)$ are continuous and strictly increasing in the first argument $\theta$ for all $\theta \in \Theta, \; k \in \{1, \ldots, K\}, \; x \in \mathbb{R}$.

The first point is implied by declining demand. The second point is implied by positive wealth effects. The final point is implied by single crossing.

## 2.2 Mechanisms

By the revelation principle, it is without loss of generality to consider direct revelation mechanisms. I restrict attention to deterministic direct revelation mechanisms. A direct revelation mechanism $\Gamma$ maps the profile of reported types to an outcome. An outcome specifies a feasible assignment of goods and payments. An assignment of goods $y \in \mathbb{K}^N$ is feasible if $\sum_{i=1}^{N} y_i \leq K$. I let $Y$ be the set of all feasible assignment. A direct revelation mechanism $\Gamma$ consists of an assignment rule $q$ and a payment rule $x$. An assignment rule $q$ maps the profile of reported types to a feasible assignment $q : \Theta^N \to Y$. I let $q_i(\theta_i, \theta_{-i})$ denote the number of units won by bidder $i$ when she reports type $\theta_i \in \Theta$ and her rivals report types $\theta_{-i} \in \Theta^{N-1}$. The payment rule maps the profile of reported types to payments $x : \Theta^N \to \mathbb{R}^N$. I let $x_i(\theta_i, \theta_{-i})$ denote the payment of bidder $i$ in mechanism $\Gamma$ when she reports type $\theta_i \in \Theta$ and her rivals report types $\theta_{-i} \in \Theta^{N-1}$.

I study direct revelation mechanisms that satisfy the following properties.

[6] $b_k$ and $s_k$ are weakly decreasing under weakly positive wealth effects and strictly decreasing under strictly positive wealth effects.

**Definition 1.** (Ex-post Individual Rationality)
A mechanism $\Gamma$ is ex-post individually rational if

$$u(q_i(\theta_i,\theta_{-i}),-x_i(\theta_i,\theta_{-i}),\theta_i)\geq 0\ \forall(\theta_i,\theta_{-i})\in\Theta^N,\ i\in\{1,\ldots,N\}.$$

Thus, a mechanism is ex-post individually rational (hereafter, individually rational) if a bidder's utility never decreases from participating in the mechanism.

I study mechanisms that are dominant strategy incentive compatible (hereafter, incentive compatible). Thus, we say that $\Gamma$ is incentive compatible, then bidder $i$'s payoff from reporting her true type $\theta_i\in\Theta$ weakly exceeds her payoff from reporting any $\theta_i'\in\Theta$, for any report by her rivals $\theta_{-i}\in\Theta^{N-1}$. This is stated in Definition 2.

**Definition 2.** (Dominant Strategy Incentive Compatibility)
A mechanism $\Gamma$ is dominant strategy incentive compatible if

$$u(q_i(\theta_i,\theta_{-i}),-x_i(\theta_i,\theta_{-i}),\theta_i)\geq u(q_i(\theta_i',\theta_{-i}),-x_i(\theta_i',\theta_{-i}),\theta_i)\ \forall\theta_i,\theta_i'\in\Theta,\ \theta_{-i}\in\Theta^{N-1},\ i\in\{1,\ldots,N\}.$$

I look at mechanisms that satisfy ex-post Pareto efficiency. This is the same efficiency notion studied by Dobzinski, Lavi, and Nisan (2012) and Morimoto and Serizawa (2015).

**Definition 3.** (Ex-post Pareto Efficient)
An outcome $(y,x)\in Y\times\mathbb{R}^N$ is ex-post Pareto efficient if $\forall(\tilde{y},\tilde{x})\in Y\times\mathbb{R}^N$ such that

$$u(\tilde{y}_i,-\tilde{x}_i,\theta_i)>u(y_i,-x_i,\theta_i),$$

for some $i\in\{1,\ldots,N\}$, then either $\sum_{i=1}^N x_i>\sum_{i=1}^N\tilde{x}_i$, or $\exists j\in\{1,\ldots,N\}$ such that $u(y_j,-x_j,\theta_j)>u(\tilde{y}_j,-\tilde{x}_j,\theta_j)$.

Thus, an outcome is ex-post Pareto efficient, if any reallocation of resources that makes bidder $i$ strictly better off necessarily makes her rival strictly worse off, or strictly decreases revenue. I say that the mechanism $\Gamma$ is an ex-post Pareto efficient mechanism (hereafter, efficient) if $(q(\theta),x(\theta))\in Y\times\mathbb{R}^N$ is an ex-post Pareto efficient outcome $\forall\theta\in\Theta^N$.

The weak budget balance condition is an individual rationality constraint on the auctioneer.

**Definition 4.** (Weak Budget Balance)
A mechanism $\Gamma$ satisfies weak budget balance if

$$\sum_{i=1}^N x_i(\theta_i,\theta_{-i})\geq 0\ \forall(\theta_1,\ldots,\theta_N)\in\Theta^N,\ i\in\{1,\ldots,N\}.$$

A mechanism that satisfies weak budget balance always yields weakly positive revenue.

When I study the single-dimensional types setting with $N \geq 3$ bidders, I impose a stronger but related requirement — no subsidies. A mechanism provides no subsidies if it never pays a bidder a positive amount to participate. Morimoto and Serizawa (2015) impose the same condition when studying efficient auctions in a setting where bidders have unit demand.

**Definition 5.** (No Subsidies)
A mechanism $\Gamma$ satisfies no subsidies if $x_i(\theta_i, \theta_{-i}) \geq 0 \ \forall(\theta_i, \theta_{-i}) \in \Theta^N, \ i \in \{1, \ldots, N\}$.

# 3 Efficient auctions for bidders with single-dimensional types

In this section, I prove that there is a mechanism that has the Vickrey auction's desirable incentive and efficiency properties when there (1) are two bidders and $K$ units, and (2) $N$ bidders and two units. I consider the two cases separately. In addition, I discuss the challenges associated with extending my positive implementation results to the general $N \times K$ single-dimensional types setting.

## 3.1 The two bidder $K$ object case

In this subsection, I prove that there is a mechanism that has the Vickrey auction's desirable incentive and efficiency properties when there are two bidders and $K$ units. More precisely, I assume that bidder $i$'s private information is described by a single-dimensional parameter $\theta_i \in \Theta = [0, \overline{\theta}]$ and $\theta_i$ parameterizes bidder $i$'s commonly known utility function $u$, where $u$ satisfies the conditions described in Section 2.1. I show that when there are two bidders, there is a symmetric mechanism that satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) no subsidies. I use a fixed point proof to characterize the efficient mechanism. In particular, I form a transformation that maps an arbitrary mechanism to a more efficient mechanism, and I show that the fixed point of the transformation corresponds to a mechanism that retains the Vickrey auction's desirable properties.

I describe an arbitrary symmetric mechanism by cut-off rule $d : \Theta \to \Theta^K$. The $k^{th}$ dimension of the cut-off rule $d_k(\theta_j)$ gives the lowest type that bidder $i$ must report to win at least $k$ units when her rival reports type $\theta_j$.[7] Hence, a mechanism $\Gamma$ has cut-off rule $d$ if

[7] Note that if a direct revelation mechanism is such that $q_i(\theta_i, \theta_j) \geq k$, then dominant strategy incentive compatibility implies that $q_i(\theta_i', \theta_j) \geq k, \ \forall \theta_i' \geq \theta_i$.

expressions 1 and 2 hold for all $k \in \{1, \dots, K\}$, where

$$\theta_i > d_k(\theta_j) \implies q_i(\theta_i, \theta_j) \geq k, \tag{1}$$

$$d_k(\theta_j) > \theta_i \implies k > q_i(\theta_i, \theta_j). \tag{2}$$

Incentive compatibility implies that $q_i(\theta_i, \theta_j)$ is weakly increasing in $\theta_i$ for all $\theta_i, \theta_j \in \Theta$. Incentive compatibility and efficiency imply that $q_i(\theta_i, \theta_j)$ weakly decreasing in $\theta_j$ for all $\theta_i, \theta_j \in \Theta$.[8] Thus, the cut-off rule $d_k(\theta)$ is weakly increasing in $\theta$ and weakly increasing in $k$ for all $\theta \in \Theta$ and $k \in \{1, \dots, K\}$. I let $\mathcal{D} \subset \{d | d : \Theta \to \Theta^K\}$ be the set of all cut-off rules that are weakly increasing in $\theta$ and $k$. Note that a cut-off rule $d \in \mathcal{D}$ does not necessarily correspond to a feasible mechanism.

I use the taxation principle (see Rochét (1985)) to find a pricing rule that implements a mechanism described by cut-off rule $d \in \mathcal{D}$. The pricing rule $p$ is a mapping $p : \Theta \times \mathcal{D} \to \mathbb{R}^{K+1}$ that states the price a bidder pays to win each additional unit of the good is conditional on her rival's type. We say that a pricing rule $p$ implements a (symmetric) cut-off rule $d \in \mathcal{D}$ if bidder $i$ demands at least $k$ units where $k \in \{1, \dots, K\}$ if and only if her type $\theta_i \in \Theta$ exceeds the $k^{th}$ unit cut-off $d_k(\theta_j)$.

The pricing rule $p(\cdot, d)$ is such that bidder $i$ demands at least one unit ($\theta_i > p_1(\theta_j, d)$) if and only if her type exceeds the first unit cut-off ($\theta_i > d_1(\theta_j)$). Thus, the pricing rule is such that $p_1(\theta_j, d) = d_1(\theta_j)\ \forall \theta_j \in \Theta$. I proceed inductively to find the price a bidder pays to win a $k^{th}$ unit. The pricing rule is such that bidder $i$ demands at least $k$ units of the good ($b_k(\theta_i, \sum_{n=1}^{k-1} p_n(\theta_j, d)) > p_k(\theta_j, d)$) if and only if her type exceeds the $k^{th}$ unit cut-off $\theta_i > d_k(\theta_j)$. Note that the term $b_k(\theta_i, \sum_{n=1}^{k-1} p_n(\theta_j, d))$ is bidder $i$'s demand for her $k^{th}$ unit conditional on having paid $\sum_{n=1}^{k-1} p_n(\theta_j, d)$ for her first $k-1$ units. Therefore, the price of the $k^{th}$ unit is

$$p_k(\theta_j, d) = b_k(d_k(\theta_j), \sum_{n=1}^{k-1} p_n(\theta_j, d))\ \forall k \in \{1, \dots, K\},\ \theta_j \in \Theta,\ d \in \mathcal{D}.$$

This inductive construction shows that a symmetric cut-off rule $d \in \mathcal{D}$ is implemented by the pricing rule $p(\cdot, d)$ described above. Lemma 1 shows bidder $i$ pays a higher total price for $k$ units when bidder $j$ has a higher type. To condense notation, I let

$$P_k(\theta_j, d) = \sum_{n=1}^{k} p_n(\theta_j, d)$$

[8] Bidder $i$'s $q_i(\theta_i, \theta_j)$ is weakly decreasing in $\theta_j$ because efficiency implies $q_j(\theta_i, \theta_j) = K - q_i(\theta_i, \theta_j)$ and $q_j(\theta_i, \theta_j)$ is weakly increasing in $\theta_j$ for all $\theta_j \in \Theta$.

be the total amount a bidder spends to win $k$ units when she faces cut-off rule $d$.

**Lemma 1.** *$P_k(\theta_j, d)$ is weakly increasing in $\theta_j$ for all $k \in \{1, \ldots, K\}$, $\theta_j \in \Theta$, and $d \in \mathcal{D}$.*

I construct a transformation that maps an arbitrary mechanism to a more efficient mechanism. The transformed mechanism's assignment rule is such that a bidder wins at least $k$ units of the good where $k \in \{1, \ldots, K\}$ if and only if her willingness to pay for her $k^{th}$ unit ranks among the top $K$ willingness to pay of both bidders. However, the ranking of bidders' willingness to pay for additional units depends on the pricing rule because wealth effects imply that a bidder's willingness to pay for her $k^{th}$ unit varies with the amount she paid for her first $k-1$ units.[9] I obtain the ranking by calculating bidders' willingness to pay for additional units under the pricing rule that corresponds to the arbitrary mechanism. This ranking of bidders' willingness to pay determines my transformed mechanism's assignment rule. In other words, the transformed assignment rule is the efficient assignment rule if prices were determined by the untransformed mechanism's pricing rule. The transformed pricing rule is the pricing rule that implements the transformed assignment rule. I argue that a fixed point of this transformation corresponds to an efficient mechanism and I use Schauder's fixed point theorem to show that such a fixed point exists.

In order to formalize the above argument, I calculate a bidder's willingness to pay for her $k^{th}$ unit conditional on her payment for her first $k-1$ units under the untransformed pricing rule that implements cut-off rule $d \in \mathcal{D}$. This amount is

$$b_k(\theta_i, P_{k-1}(\theta_j, d)).$$

Similarly, bidder $j$'s willingness to pay for her $K-k+1^{st}$ unit conditional on her payment for her first $K-k$ units is

$$b_{K-k+1}(\theta_j, P_{K-k}(\theta_i, d)).$$

I construct the transformed assignment rule by defining a function that compares the above two quantities. In particular, I define a function $f : \{1, \ldots, K\} \times \Theta^2 \times \mathcal{D} \to \mathbb{R}$, where $f$ is such that

$$f(k, \theta_i, \theta_j, d) := b_k(\theta_i, P_{k-1}(\theta_j, d)) - b_{K-k+1}(\theta_j, P_{K-k}(\theta_i, d)),$$

for all $k \in \{1, \ldots, K\}$, $\theta_i, \theta_j \in \Theta$, $d \in \mathcal{D}$. Thus, $f(k, \theta_i, \theta_j, d)$ represents the amount that bidder $i$'s willingness to pay for her $k^{th}$ unit exceeds her rival's willingness to pay for her $K-k+1^{st}$ unit, when we evaluate bidders' willingness to pay under the pricing rule implementing

[9] This is an important difference between my model where bidders have non-quasilinear preferences and the quasilinear benchmark model. In the quasilinear setting, a bidder's willingness to pay for her $k^{th}$ unit is independent of the amount she paid to win her first $k-1$ units because there are no wealth effects.

the cut-off rule $d \in \mathcal{D}$. Bidder $i$'s willingness to pay for her $k^{th}$ unit ranks among the top $K$ willingness to pay of both bidders when $f(k, \theta_i, \theta_j, d)$ is positive.

**Lemma 2.** *The function $f(k, \theta_i, \theta_j, d)$ is (1) strictly decreasing in k, (2) strictly increasing in $\theta_i$, and (3) strictly decreasing in $\theta_j$ for all $k \in \{1, \ldots, K\},\ \theta_i, \theta_j \in \Theta,\ d \in \mathcal{D}$.*

I define the transformed cut-off rule to be such that bidder $i$'s type exceeds $k^{th}$ cut-off if and only if her willingness to pay for her $k^{th}$ unit ranks among the top $K$ willingness to pays. Formally, bidder $i$'s transformed cut-off rule is such that

$$T_k(d)(\theta_j) := \begin{cases} \inf\{\theta \in \Theta | f(k, \theta, \theta_j, d) \geq 0\} & if\ f(k, \overline{\theta}, \theta_j, d) \geq 0, \\ \overline{\theta} & if\ f(k, \overline{\theta}, \theta_j, d) < 0, \end{cases}$$

for all $k \in \{1, \ldots, K\},\ \theta_j \in \Theta,\ d \in \mathcal{D}$.

Note that when $f(k, \overline{\theta}, \theta_j, d) > 0$, then Lemma 2 implies that bidder $i$'s willingness to pay for her $k^{th}$ unit exceeds her rival's willingness to pay for her $K - k + 1^{st}$ unit when $\theta_i$ is sufficiently large. In this case, the transformed cut-off rule $T_k(d)(\theta_j)$ states the lowest type for which bidder $i$'s willingness to pay for her $k^{th}$ unit exceeds her rival's willingness to pay for her $K - k + 1^{st}$ unit. If $f(k, \overline{\theta}, \theta_j, d) < 0$, then bidder $i$'s willingness to pay for her $k^{th}$ unit is always less than her rival's willingness to pay for her $K - k + 1^{st}$ unit. In this case, the transformed assignment rule gives bidder $i$ wins fewer than $k$ units for any reported type. I calculate a bidder's willingness to pay for her $k^{th}$ unit by assuming that the price she paid for her first $k - 1$ units was determined by the pricing rule corresponding to the (untransformed) cut-off rule $d$. This is stated in Remark 1 below.

*Remark* 1. If $\theta_i, \theta_j \in \Theta$ and $d \in \mathcal{D}$ , then

$$b_k(\theta_i, P_{k-1}(\theta_j, d)) \geq b_{K-k+1}(\theta_j, P_{K-k}(\theta_i, d)) \implies \theta_i \geq T_k(d)(\theta_j),$$

and

$$b_{K-k+1}(\theta_j, P_{k-k}(\theta_i, d)) \geq b_k(\theta_i, P_{k-1}(\theta_j, d)) \implies T_k(d)(\theta_j) \geq \theta_i.$$

My transformed cut-off rule is related to the assignment rule used by Perry and Reny (2002, 2005). The papers by Perry and Reny study efficient auction design in an interdependent value setting where there are two bidders and bidders have single-dimensional types and quasilinear preferences (see Section 3 of 2002 paper, or Section 4 of the 2005 paper). In their papers, bidder $i$'s cut-off for her $k^{th}$ unit is the lowest signal such that her marginal value for her $k^{th}$ unit exceeds her rival's marginal value for her $K - k + 1^{st}$ unit. In my private value non-quasilinear setting, a bidder's willingness to pay for her $k^{th}$ unit conditional on the amount she paid for her first $k-1$ units, $b_k(\theta_i, P_{k-1}(\theta_j, d))$, takes the place of a bidder's marginal value in interdependent value settings studied by Perry and Reny.

Note that when preferences are quasilinear, there is a (generically) unique efficient assignment of goods. Thus, in Perry and Reny's quasilinear setting the efficient auction design problem is solved by finding a pricing rule that implements the exogenously-determined efficient assignment rule. However, there is not a unique efficient assignment of goods in my non-quasilinear setting. That is because without quasilinearity a bidder's willingness to pay/sell for a unit of the good depends on her level of transfers. My transformed cut-off rule is the efficient assignment rule for the case where prices are determined by the pricing rule that implements the untransformed cut-off rule. Yet the transformed assignment rule $T(d)$ is implemented by the transformed pricing rule $p(\cdot, T(d))$. Thus, if $d$ is not a fixed point, then the assignment rule that sorts units efficiently when prices are determined by the untransformed pricing rule $p(\cdot, d)$, is not the assignment rule that sorts units efficiently when prices are determined by the transformed pricing rule $p(\cdot, T(d))$. Or in other words, we construct $T$ to be such that a mechanism with an assignment rule given by $T(d)$ and pricing rule $p(\cdot, d)$ gives an efficient outcome. But if $d$ is not a fixed point of $T$, then the mechanism with allocation rule $T(d)$ and pricing rule $p(\cdot, d)$ would not be incentive compatible. Instead, we can construct a mechanism that implements that same assignment rule $T(d)$ by using the pricing rule $p(\cdot, T(d))$. However, the outcome associated with the assignment described by $T(d)$ is no longer efficient when we change the pricing rule from $p(\cdot, d)$ to $p(\cdot, T(d))$. For this reason, I use a fixed point theorem to prove the existence of an efficient (and implementable) mechanism in my non-quasilinear setting.

The above argument implies that a fixed point of the transformation $T$ defines an efficient mechanism. To see this suppose that cut-off rule $d^* \in \mathcal{D}$ is a fixed point of $T$. The corresponding pricing rule $p(\cdot, d^*)$ is such that (1) bidder $i$ demands $k$ units if and only if her rival demands $K-k$ units, and (2) bidder $i$ wins her $k^{th}$ unit if and only if her willingness to pay for her $k^{th}$ unit exceeds her rival's willingness to pay for her $K-k+1^{st}$ unit. Both points follow from the implications of Remark 1 above. Thus, there are no Pareto improving trades where bidder $i$ sells units to bidder $j$ and the auction outcome is efficient.

**Theorem. 1A.** *If $d^* \in \mathcal{D}$ is a fixed point of the mapping $T$, then $d^*$ corresponds to a feasible mechanism that satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) no subsidies.*

I use Schauder's fixed point theorem to show that the mapping $T$ has a fixed point $d^* \in \mathcal{D}$.[10] In particular, I show that (1) $d \in \mathcal{D} \implies T(d) \in \mathcal{D}$, (2) $T$ is a continuous

[10]Schauder's fixed point theorem guarantees the existence of a fixed point of my mapping $T$. However,

mapping, and (3) $\mathcal{D}$ is compact. These three conditions guarantee the existence of a fixed point according to Schauder's fixed point theorem (see Aliprantis and Border (2006), pg. 583).

**Theorem 1.** *There exists a cut-off rule $d^* \in \mathcal{D}$ that is a fixed point of the mapping $T$.*

Thus, Theorem 1 shows that in the $2 \times K$ setting, there is a mechanism that retains the desirable properties of the Vickrey auction. Furthermore, this efficient mechanism can be implemented by a multi-unit Vickrey auction with a restricted bid space.[11] To see this, consider a cut-off rule $d^*$ where $d^* = T(d^*)$. We use $d^*$ to construct a multi-unit Vickrey auction where a bidder selects from a single-dimensional family of bid curves. The bid curves are such that if bidder $i$ bids $\theta_i$ for her first unit, then she also bids

$$\beta_k(\theta_i) := p_{K-k+1}(\theta_i, d^*)$$

for her $k^{th}$ unit.

Note that if bidder $i$ submits bid curve $\beta(\theta_i)$ and bidder $j$ submits bid curve $\beta(\theta_j)$, then by construction bidder $i$ wins at least $k$ units in the Vickrey auction only if

$$\theta_i \geq d_k^*(\theta_j),$$

and bidder $i$ wins strictly fewer than $k$ units only if

$$d_k^*(\theta_j) \geq \theta_i.$$

Moreover, if bidder $i$ wins $k$ units in the Vickrey auction with restricted bid space, she pays $P_k(\theta_j, d^*)$. Thus, the multi-unit Vickrey auction with restricted bid space implements the outcome of direct revelation mechanism that corresponds to cut-off rule $d^*$.[12]

**Corollary 1.** *The Vickrey auction with restricted bid space satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) no subsidies.*

Note that the Vickrey auction without any restrictions on the bid space does not satisfy the four aforementioned properties. Baisa (2016) shows that bidders with positive wealth effects misreport their demand for later units in the multi-unit Vickrey auction. I give an

it does not guarantee a unique fixed point. Thus, we can say that there is a mechanism that implements an efficient outcome (Theorem 1), but we cannot say whether there is a unique efficient mechanism and assignment rule.

[11] See Chapter 12 of Krishna (2010) for a formal description of the standard multi-unit Vickrey auction for homogenous goods.

[12] See Section 3.4 for an example of the efficient mechanism for the soft budgets case.

explicit description of the efficient mechanism for a setting where bidders have soft budgets in Section 3.1.

Finally, note that we assume that bidders are ex-ante symmetric when we proved Theorem 1. Thus, it is natural to ask whether the positive result from Theorem 1 extends to a setting without symmetry of bidder preferences. In the Section A3 of the appendix I show that it is a straightforward exercise to extend Theorem 1 to a setting where bidders are ex-ante asymmetric. In particular, I show that we can construct a nearly identical transformation maps that arbitrary cut-off rule for bidder 1 to a more efficient cut-off rule for bidder 1. In addition, I show that any feasible mechanism that allocates all units and is described by its cut-off rule for bidder 1 could also be described by its corresponding cut-off rule for bidder 2. Thus, my transformation that maps bidder 1's arbitrary cut-off rule to a transformed cut-off rule, also implicitly maps the corresponding cut-off rule for bidder 2 to a transformed cut-off rule for bidder 2. The fixed point of this mapping defines an efficient auction in this asymmetric setting as well.

### 3.1.1 Note on restriction to deterministic mechanisms

Like the prior literature on efficient multi-unit auctions without quasilinearity, I restrict attention to deterministic mechanisms. Alternatively, one could also consider the problem of designing an efficient and dominant strategy implementable auction when we can use randomization. Garratt (1999) shows that randomization can increase the gains from trade of associated with exchanging an indivisible good. In addition, Baisa (2017) shows that the efficient stochastic mechanism sells the indivisible good like a perfectly divisible normal good in net supply one. Therefore, the problem of designing an efficient stochastic auction in a single unit setting is equivalent to studying a divisible good version of the problem studied in this subsection. We can then use Theorem 1 to show that there is an approximately efficient stochastic mechanism that sells a single indivisible good, and the approximately efficient stochastic mechanism is dominant strategy implementable. In particular, Theorem 1 shows that there is an efficient auction for $K \in \mathbb{N}$ indivisible goods when there are two bidders who have single-dimensional types. Let the $K$ indivisible homogenous goods each be a lottery ticket that provides a $\frac{1}{K}$ probability of winning the indivisible good. When $K$ is large, the mechanism is dominant strategy implementable and also approximately efficient. Thus, selling the good as discrete lottery tickets, each with an arbitrarily small probability of winning allows us to construct a mechanism that implements an approximately efficient outcome.[13]

[13]Baisa (2017) also shows that the stochastic mechanism must also structure a bidder's expected payment efficiently. An efficient payment scheme is such that any bidder that wins the good with positive probability

## 3.2 The $N$ bidder two units case

In this subsection, I show that we can characterize an efficient mechanism when there are two units. The proof is constructive. The mechanism is symmetric and its first-unit cut-off rule implicitly defines the assignment rule, because we construct the mechanism to be such that a bidder wins both units (i.e. her type exceeds her second unit cut-off) if and only if all other bidders have a type below their first unit cut-off. That simplification allows us to characterize the efficient mechanism using a single equation which describes the first unit cut-off. This differs from how results presented in the prior section, where we needed to construct multiple cut-off rules simultaneously in order to define an efficient mechanism.

I let $q_i : \Theta^N \to \{0, 1, 2\}$ be the assignment rule for bidder $i$ in the mechanism, $\Gamma$. Incentive compatibility implies that $q_i(\cdot, \theta_{-i})$ is weakly increasing $\forall \theta_{-i} \in \Theta^{N-1}$. I let $d : \Theta^{N-1} \to \Theta$ be the bidders' symmetric first unit cut-off in mechanism $\Gamma$. Specifically,

$$d(\theta_{-i}) = \begin{cases} \inf\{\theta \in \Theta | q_i(\theta_i, \theta_{-i}) \geq 1\} & if\ q_i(\overline{\theta}, \theta_{-i}) \geq 1 \\ \overline{\theta} & if\ q_i(\overline{\theta}, \theta_{-i}) = 0 \end{cases}.$$

If bidder $i$'s type is below her first unit cut-off, then she wins no units. Bidder $i$ wins at least one unit if her type exceeds the first unit cut-off, and bidder $i$ wins both units if her type exceeds the first unit cut-off and none of her rivals have a type that exceeds their first unit cut-off. Thus,

$$q_i(\theta_i, \theta_{-i}) = \begin{cases} 0 & if\ d(\theta_{-i}) > \theta_i \\ 1 & if\ \theta_i > d(\theta_{-i})\ and\ \theta_j > d(\theta_{-j})\ for\ some\ j \in \{1, \ldots, N\}\ where\ j \neq i, \\ 2 & if\ \theta_i > d(\theta_{-i})\ and\ d(\theta_{-j}) > \theta_j\ \forall j \in \{1, \ldots, N\}\ where\ j \neq i. \end{cases}$$

Mechanism $\Gamma$ has a pricing rule $p : \Theta^{N-1} \to \mathbb{R}^2$. The pricing rule states the price a bidder pays for each unit of the good given her rivals' reported types. The pricing rule $p$ implements assignment rule $q$. The pricing rule is such that bidder $i$ pays nothing if she does not win any units. In addition, bidder $i$ wins at least one unit if and only if her type type exceeds her first unit cut-off. Thus, we set the price of bidder $i$'s first unit to be her first unit cut-off. And finally, the pricing rule is such that bidder $i$ wins both units if and only if her willingness to pay for her second unit exceeds her highest demand rival's willingness to pay for her first

must have the same marginal utility of money in the win state and in the loss state. If this did not hold, the auctioneer could offer the bidder a Pareto improving insurance contract that allows her to equalize her marginal utility of income across the win state and the lose state.

unit. Thus we have that for any $\theta_{-i} \in \Theta^{N-1}$, $p : \Theta^{N-1} \to \mathbb{R}^2$ is such that

$$p_1(\theta_{-i}) = d(\theta_{-i}),$$
$$p_2(\theta_{-i}) = \max_{j \in \{1,\ldots,i-1,i+1,\ldots,N\}} \theta_j.$$

Equation 3 implicitly defines bidder $i$'s first unit cut-off $d(\theta_{-i})$. It is without loss of generality to define the first unit cut-off for bidder 1 and to assume that that $\overline{\theta} \geq \theta_2 \geq \theta_3 \geq \theta_j \geq 0 \; \forall j \in \{4, \ldots, N\}$ because mechanism $\Gamma$ is symmetric. The mechanism's cut-off rule $d$ is implicitly defined by the equation below

$$d(\theta_{-1}) = \max\{\theta_3, b_2(\theta_2, d(d(\theta_{-1}), \theta_{-1,2}))\} \; \forall \theta_{-1} \in \Theta^{N-1} \; s.t. \; \theta_2 \geq \theta_3 \geq \theta_j \; \forall j \in \{4, \ldots, N\}. \tag{3}$$

Equation 3 implies that bidder 1 wins her first unit if and only if her demand for her first unit exceeds both her highest rival's demand for her second unit and her second highest rival's demand for her first unit. The first term on the right hand side of Equation 3 is bidder 3's willingness to pay for her first unit. Bidder 3 is the second highest demand rival of bidder 1. The second term is bidder 2's willingness to pay for her second unit conditional on paying $d(d(\theta_{-1}), \theta_{-1,2})$. Recall that the price bidder 2 pays for her first unit is $p_1(\theta_{-2}) = d(\theta_{-2})$. Thus, $d(d(\theta_{-1}), \theta_{-1,2})$ is the price bidder 2 pays to win her first unit when bidder 1's type is exactly at the first unit cut-off.

Equation 3 is the analog of the demand reduction term in Section 4 of Perry and Reny's (2005) quasilinear interdependent value multi-unit auction model. In the two-unit version of their model, they find a cut-off rule by fixing a bidder's rivals type and finding the signal where the bidder's value for her first unit equals her rival's value for her second unit. My cut-off rule similarly finds the cut-off by finding the type of bidder 1 where her willingness to pay for her first unit equals the second highest willingness to pay of her rivals. In my case, the second highest willingness to pay of bidder 1's rivals is the maximum of bidder 2's willingness to pay for her second unit and bidder 3's willingness to pay for her first unit.

Lemma 3 shows that Equation 3 implicitly defines a unique and continuous cut-off rule $d : \Theta^{N-1} \to \Theta$.

**Lemma 3.** *There is a unique function $d : \Theta^{N-1} \to \Theta$ that is continuous and satisfies Equation 3.*

Lemma 3 shows that we can use the cut-off rule $d$ to construct a mechanism that satisfies Properties (1)-(4). The mechanism satisfies individual rationality and no subsidies by construction. Incentive compatibility is satisfied because the mechanism is such that a bidder does not misreport her type because she wins a unit if and only if her demand for the unit exceeds the price of a unit. The mechanism is efficient because it only assigns to a bidder if she has one of the two highest types. Moreover, one bidder wins both units if and only if her demand for both units exceeds her highest rival's demand for her first unit. Thus, the mechanism's outcome is such that there are no ex post Pareto improving trades among bidders.

**Theorem 2.** *There exists a mechanism $\Gamma$ that satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) no subsidies. The mechanism has first unit cut-off rule $d$ that is the unique solution to Equation 3 and pricing rule $p$, where for any $\theta_{-i} \in \Theta^{N-1}$,*

$$\begin{aligned} p_1(\theta_{-i}) &= d(\theta_{-i}), \\ p_2(\theta_{-i}) &= \max\nolimits_{j \neq i} \theta_j. \end{aligned}$$

While it is straightforward to extend the implications of Theorem 1 to a setting where bidders are asymmetric, that is not true when we consider Theorem 2. To see why note that we construct the efficient auction in the $N$ bidder two object setting by characterizing a bidder's first unit cut-off using Equation 3. We are able to use Equation 3 to implicitly characterize the cut-off rule by comparing a bidder's willingness to pay for her first unit with her highest demand rivals willingness to pay for her second unit, among other competing bids. We calculate the highest demand rival's willingness to pay for her second unit, conditional on the price she paid for her first unit, by assuming that the price her rival paid to win her first unit was determined using the same cut-off rule. If bidders were asymmetric, then naturally, the efficient auction should be asymmetric and have different first unit cut-offs for each bidder. In this case, we would be unable to use a single expression, like Equation 3, to characterize the efficient mechanism's cut-off rule. In the next section, I discuss similar challenges associated with extending my two positive results to the general $N \times K$ setting.

### 3.3 The $N$ bidder $K$ object case

I use different approaches to prove the existence of a mechanism with Vickrey's desired properties in the two-bidder $K \geq 2$ object case, and the $N \geq 3$ bidder 2 object case. In the latter case, I give a constructive argument. I form a single equation (Equation 3) that implicitly describes a bidder's first unit cut-off rule. I form the equation by noting that bidder $i$ wins a unit if and only if her willingness to pay for her first unit exceeds: (1) her second-highest rival's demand for her first unit, and (2) her highest rival's demand for her second unit, when bidder $i$'s type equals her first unit cut-off. The efficient auction design problem simplifies down to finding bidder $i$'s first unit cut-off, because we know her second unit cut-off will be defined to be such that she wins both units if and only if her willingness to pay for her second unit, conditional on buying the first, exceeds her highest rival's willingness to pay for her first unit.

We cannot use that same constructive approach and form a single equation that characterizes an efficient auction in the three (or more) unit case. To see this consider the three unit case. We need two cut-off rules for each bidder to define an efficient mechanism - a bidder's first unit cut-off, and her second unit cut-off; the third is implicitly defined by the case where only one bidder exceeds her first unit cut-off. However, with three or more units, there is a simultaneity problem when we determine the cut-off rules. This is because we would need to know the bidders' second unit cut-off rule to determine their first unit cut-off rule and vice versa. In particular, we determine a bidder's first unit cut-off by comparing her willingness to pay for her first unit with her highest demand rival's willingness to pay for her third unit, among other competing bids. However, bidder $i$'s highest demand rival's willingness to pay for her third unit depends on how much that rival paid to win her first and second units. We determine those two prices using the first and second unit cut-off rules. Similarly, when we determine bidder $i$'s second unit cut-off, we compare her conditional willingness to pay for her second unit with higher highest demand rival's willingness to pay for her second unit. And her highest rival's conditional willingness to pay for her second unit depends on the first unit cut-off. Thus, we need to simultaneously determine the first and second unit cut-off rules that each satisfy a version of Equation 3.

In subsection 3.1, we assumed there were only two bidders and we faced an equivalent simultaneity problem. In our efficient auction, bidder $i$'s first unit cut-off depends on the amount her rival pays to win her first $K-1$ units. And the amount her rival pays to win her earlier units depends on the earlier unit's cut-off rules. We were able to overcome this simultaneity issue in the two-bidder case by using a fixed point approach. However, the fixed point argument used in subsection 3.1 cannot be applied to a setting with $N \geq 3$ bidders because we can not construct an analog to our cut-off rule $d(\cdot)$ that is monotone (coordinate-wise) when there are at least three bidders. Recall, in the two-bidder case, we use the monotonicity $d$ to show that the space of all cut-off rules $\mathcal{D}$ is compact. The compactness of $\mathcal{D}$ is a necessary condition to use Schauder's fixed point theorem.

However, when there are at least three bidders, there is no mechanism that satisfies Vickrey's desired properties and also has a cut-off rule that is monotone in the coordinate-wise sense. More precisely, there is no Vickrey-like mechanism where a bidder wins a weakly greater number of units if her demand increases and her rivals' reported demands decrease (in the coordinate-wise sense). We define this notion of strong monotonicity below. We then present an impossibility result (Proposition 1) that illustrates why no such mechanism exists, and consequently, why we are unable to reuse the fixed point proof from subsection 3.1 for the general $N \times K$ case.

**Definition 6.** (Strong Monotonicity)

A mechanism $\Gamma$ satisfies strong monotonicity if bidder $i$'s assignment rule $q_i : \Theta^N \to \{0, 1, \ldots, K\}$ is such that for all $\theta_i^h, \theta_i^\ell \in \Theta$ and $\theta_{-i}^h, \theta_{-i}^\ell \in \Theta^{N-1}$ where $\theta_i^h \geq \theta_i^\ell,$ and $\theta_{-i}^h \geq \theta_{-i}^\ell$, then

$$q_i(\theta_i^h, \theta_{-i}^\ell) \geq q_i(\theta_i^\ell, \theta_{-i}^h) \; \forall i \in \{1, \ldots, N\}.$$

Strong monotonicity is related to other practical constraints that have been studied in mechanism design. For example, any mechanism that is non-bossy in the sense of Satterthwaite and Sonnenschein's (1981) and assigns all units also satisfies strong monotonicity.[14]

To illustrate the impossibility result lets again consider the single-dimensional types setting and suppose that $N \geq 3$ bidders compete to win two units. I show that in any such setting there is no mechanism that satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, (4) no subsidies and (5) strong monotonicity.

**Proposition 1.** *There is no mechanism that satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, (4) no subsidies, and (5) strong monotonicity.*

I prove the impossibility result by contradiction. I show that if there is a mechanism that satisfies the five properties, then there is endogenous interdependence in bidders' demands, even in my private value model. I note two important features of the implied endogenous interdependence in my proof. First, I note that the interdependence in bidder demands is only present in bidders' demands for later units. A bidder's willingness to pay for her first unit is her private type $\theta_i$, and this quantity does not vary with her rivals' types. Second, I note that the interdependence is negative. When a bidder's rivals increase their demands, the price a bidder pays for her first unit increases. Positive wealth effects then imply that the bidder has a lower demand for later units. The presence of negative interdependence leads to the violation of strong monotonicity. There is an identical tension between efficiency and strong monotonicity in a quasilinear setting where bidders' demands for later units are negatively interdependent on rival types. To be more concrete, consider a modified version of Perry and Reny's (2005) quasilinear multi-unit auction setting.[15] However, suppose that a bidder's marginal value for her first unit is independent of her rivals' types and her marginal

[14] Recall non-bossiness requires that a change in bidder $i$'s type changes one of her rival's assignment only if it changes her own assignment. Borgs et al. (2005) also study an auction design problem with a similar property that they call independence of irrelevant alternatives. In a quasilinear setting with private values, strong monotonicity is implied by efficiency. Strong monotonicity is implied by efficiency and incentive compatibility in the two-bidder case.

[15] In my notation, this would be a case where $\theta_i \in \mathbb{R}^+$, and $u_i(q, m, \theta_i, \theta_{-i}) = \sum_{j=1}^{q} v_j(\theta_i, \theta_{-i}) + m$, where $v : \Theta^N \to \mathbb{R}_+^K$.

value for later units is decreasing in her rivals' single-dimensional types. We can show that efficient auction design is incompatible with strong monotonicity in this setting as well. To see this, suppose that there are two units and three bidders where bidders 1, 2, and 3 have the highest, second highest, and lowest demands, respectively. If bidder 1's marginal value for her second unit, conditional on her rival's types exceeds bidder 2's demand for her first unit by a small amount, then efficiency implies that bidder 1 wins both units. However, an increase in bidder 3's type decreases bidder 1's demand for her second unit and flips the ranking of bidder 1's marginal value for her second unit and bidder 2's marginal value for her first unit. Hence, if bidder 3's type increases, efficiency implies that bidders 1 and 2 now each win a single unit, and that violates strong monotonicity (bidder 2 wins more units even though her rival reports a higher type). The difference between my non-quasilinear private value setting and the interdependent value quasilinear setting is that in my setting, the negative interdependence in bidder demands for later units arises endogenously in the efficient auction design problem.

The mechanism constructed in subsection 3.2 shows that we are able to construct a mechanism that satisfies the first four properties, but the mechanism necessarily violates strong monotonicity. In addition, in the Section A2 of the appendix, I construct a mechanism that gives bidders upfront subsidies and satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency and (4) strong monotonicity. The efficient auction with subsidies illustrates an important distinction between efficient auction design problems with quasilinearity and without quasilinearity. With quasilinearity, an upfront subsidy does not expand the scope of implementable social choice rules. Without quasilinearity, the designer can expand the scope of implementable social choice rules because a subsidy can change bidders' preferences endogenously. At the same time, we rarely see auctions that use subsidies in practice. As Morimoto and Serizawa (2015) state, imposing no subsidies is a useful practical constraint for the mechanism designer because "this property prevents agents who do not need objects from flocking to auctions only to sponge subsidies."

Lastly, note that when we compare that results from Sections 3.2 and A2, we see that there is not a unique assignment rule that corresponds to the efficient mechanism. This is different from the benchmark quasilinear setting where the efficient assignment rule is uniquely determined by bidder preferences. The assignment rules of the mechanisms described in Sections 3.2 and A2 differ, but both are efficient. As discussed above, this is because we can use upfront subsidies to expand the scope of dominant strategy and efficient implementation when bidders have wealth effects, because upfront subsidies endogenously shift a bidder's demand curve.

### 3.4 Numerical example for bidders with soft budgets

In this example, I study a setting with two bidders, where bidders have soft budgets and single-dimensional types. I characterize an efficient mechanism. I explicitly characterize an efficient mechanism for the soft budget setting by using a guess and verify approach.

There are two bidders 1 and 2 who compete for two homogenous units. Each bidder is described by her single-dimensional type $\theta_i \in \Theta$. I assume that a bidder with type $\theta_i$ has a soft budget of $\theta_i$, and she gets utility $\theta_i$ for her first unit of the good. In addition, each bidder has declining demand for additional units. In particular, bidder $i$ gets utility of $\theta_i$ from her first unit and marginal utility of $.9\theta_i$ from her second unit. If a bidder spends an amount $p > 0$ that exceeds her budget $\theta_i$, then the bidder must also pay interest $r \geq 0$ on her debt of $p - \theta_i$. Thus, the bidder $i$ gets disutility $r(p - \theta_i) + p$ from spending $p$. Thus, I write bidder $i$'s utility function as

$$u(q, m, \theta_i) = \theta_i V(q) + f(m, \theta_i),$$

for all $q \in \{0, 1, 2\},\ m \in \mathbb{R},\ \theta_i \in \Theta$, where

$$V(q) = \begin{cases} 0 & if\ q = 0 \\ 1 & if\ q = 1 \\ 1.9 & if\ q = 2 \end{cases},$$

and

$$f(m, \theta_i) = \begin{cases} m & if\ \ -\theta_i < m \\ m + r(\theta_i + m) & if\ \ -\theta_i \geq m. \end{cases}$$

By construction, bidder $i$ is willing to pay $\theta_i \in \Theta$ for her first unit of the good. I use the above expressions to compute bidder $i$'s willingness to pay for her second unit, conditional on paying $p$ for her first unit, which is

$$b_2(\theta_i, p) = \begin{cases} .9\theta_i & if\ p \leq .1\theta_i, \\ \frac{r(\theta_i - p) + .9\theta_i}{1+r} & if\ .1\theta_i < p < \theta_i, \\ \frac{.9\theta_i}{1+r} & if\ p \geq \theta_i. \end{cases}$$

A (first unit) cut-off rule $d : \Theta \to \Theta$ is a fixed point of the transformation $T$ in the $2 \times 2$ setting if

$$d(\theta) = b_2(\theta, d(d(\theta)))\ \forall \theta \in [0, \overline{\theta}].$$

For a given $r \geq 0$, I guess and verify that there is a linear cut-off rule $d$ that satisfies the above expression. Moreover, I show that there the linear cut-off rule that satisfies the above expression is such that $d(\theta) = g(r)\theta \ \forall \theta \in \Theta$, where $g : \mathbb{R}_+ \to (\sqrt{.1}, 1)$ is states the constant slope of the symmetric first unit cut-off rule for a bidder given the interest rate $r \geq 0$. We let, $g(r)$ be such that

$$g(r)\theta = b_2(\theta, (g(r))^2\theta) = \frac{r(\theta - (g(r))^2\theta) + .9\theta}{1+r},$$

where the final equality follows, because I assume that $g^2(r)\theta \in (.1\theta, \theta)$. By simplifying the above expression we get that

$$g(r) = \frac{-(1+r) + \sqrt{(1+r)^2 + 4r(r+.9)}}{2r}.$$

We can easily confirm that $g(r) \in [\frac{\sqrt{5}-1}{2}, .9] \subset (.1, 1) \ \forall r \geq 0$. Therefore, the cut-off rule $d(\theta) = g(r)\theta$ where $g(r)$ is defined by the fixed point of the transformation $T$ for any given interest rate $r \geq 0$. Theorem 1A then implies that $d$ is the cut-off rule for an efficient mechanism $\Gamma$ where

$$q_1(\theta_1, \theta_2) = \begin{cases} 0 & if \ \theta_1 < g(r)\theta_2, \\ 1 & if \ \theta_1 \geq g(r)\theta_2, \ and \ \theta_2 \geq g(r)\theta_1, \\ 2 & if \ \theta_1 > g(r)\theta_2, \ and \ g(r)\theta_1 > \theta_2. \end{cases}$$

and $q_2(\theta_1, \theta_2) = 2 - q_1(\theta_1, \theta_2)$. The mechanism is implemented by pricing rule $p : \Theta \to \mathbb{R}^2_+$ where

$$p_1(\theta) = g(r)\theta, \ and \ p_2(\theta) = \theta.$$

The figure below illustrates the allocation rule when we assume that bidders pay 100 percent interest ($r = 1$). In that case the efficient mechanism has first unit cut-off rule satisfying

$$d(\theta) = \frac{-2 + \sqrt{11.6}}{2}\theta \ \forall \theta \in \Theta.$$

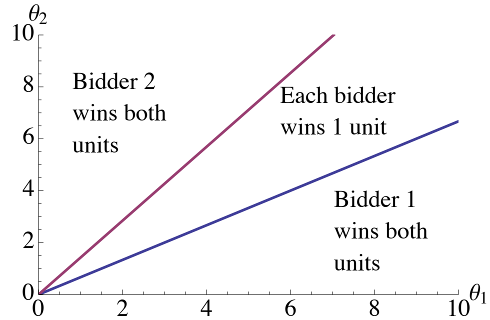


Figure 1: cut-off rule when $r = 1$.

Figure 2 below shows how the slope of the symmetric first unit cut-off $g(r)$ varies with the interest rate $r$.

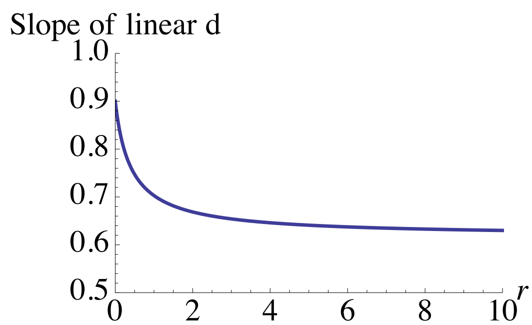


Figure 2: The slope of the first unit cut-off rule $g(r)$.

# 4 Bidders with multi-dimensional types

## 4.1 Bidder preferences

In this section, I argue that the positive result from Theorem 1 does not extend to any setting where bidders have multi-dimensional types. I study a setting where there are two bidders and two homogenous goods. A bidder's private information is described by a two-dimensional variable $\gamma_i = (\theta_i, t_i) \in [0, \overline{\theta}] \times \{\mathfrak{s}, f\}$. If bidder $i$ has type $\gamma_i = (\theta_i, t_i)$, wins $q \in \{0, 1, 2\}$ units, and receives transfer $m$, then her utility is $u(q, m, \gamma_i) \in \mathbb{R}$, where $u$ is continuous in $\theta_i$ and continuous and strictly increasing in $m$ for all $\theta_i \in \Theta$ and $m \in \mathbb{R}$. Again, I assume that bidder $i$ has no demand for units if the first dimension of her type $\theta_i = 0$,

$$u(q, -x, (0, t_i)) = u(q', -x, (0, t_i)) \; \forall q, q' \in \{0, 1, 2\}, \; x \in \mathbb{R}, \; t_i \in \{\mathfrak{s}, f\},$$

and a bidder has positive demand if $\theta_i > 0$,

$$u(q, -x, (\theta_i, t_i)) > u(q', -x, (\theta_i, t_i)),$$

for all $\theta_i \in (0, \overline{\theta}],\ q, q' \in \{0, 1, 2\},\ s.t.\ q > q',\ x \in \mathbb{R},$ and $t_i \in \{\mathfrak{s}, f\}$. The second dimension of bidder $i$'s type $t_i \in \{\mathfrak{s}, f\}$ represents the steepness of her demand curve - it can either be steep ($\mathfrak{s}$) or flat ($f$). Bidders with steeper demand curves have relatively lower demand for their second unit. Thus, I assume that

$$u(2, -x-p, (\theta_i, \mathfrak{s})) \geq u(1, -x, (\theta_i, \mathfrak{s})) \implies u(2, -x-p, (\theta_i, f)) > u(1, -x, (\theta_i, f)),$$

for all $\theta_i \in (0, \overline{\theta}],\ x, p \in \mathbb{R}_+$. Therefore, if $b_2(\gamma_i, x)$ is bidder $i$'s willingness to pay for her second unit when she has type $\gamma_i \in \Theta \times \{\mathfrak{s}, f\}$ and paid $x \in \mathbb{R}$ for her first unit, then $b_2$ is such that

$$b_2((\theta_i, f), x) > b_2((\theta_i, \mathfrak{s}), x) > 0\ \forall \theta_i \in (0, \overline{\theta}],\ x \in \mathbb{R}.$$

I assume bidder preferences satisfy (1) declining demand, (2) strictly positive wealth effects, and (3) single-crossing in $\theta$ (Assumptions 1-3, from Section 2). Again, it is without loss of generality to assume that $\theta_i$ represents bidder $i$'s willingness to pay for her first unit of the good. I refer to $\theta_i$ as bidder $i$'s intercept. Thus,

1. $\theta_i > b_2(\gamma_i, x) > 0$, and $s_1(\gamma_i, x) > s_2(\gamma_i, x),\ \forall x \in \mathbb{R}_+,\ \gamma_i = (\theta_i, t_i) \in (0, \overline{\theta}] \times \{\mathfrak{s}, f\}$.
2. $b_2(\gamma_i, x)$ is continuous and strictly decreasing in the amount a bidder has paid $x\ \forall x \in \mathbb{R},\ \gamma_i \in \Theta \times \{\mathfrak{s}, f\}$.
3. $b_2((\theta_i, t_i), x)$ is continuous and strictly increasing in $\theta_i\ \forall \theta_i \in \Theta,\ x \in \mathbb{R},\ t_i \in \{\mathfrak{s}, f\}$.

Points 1, 2, and 3 above are direct implications of Assumptions 1, 2, and 3, respectively.

A mechanism $\Gamma$ satisfies incentive compatibility in a multi-dimensional type where bidder preferences are described by the utility function $u$ if

$$u(q_i(\gamma_i, \gamma_j), -x_i(\gamma_i, \gamma_j), \gamma_i) \geq u(q_i(\gamma_i', \gamma_j), -x_i(\gamma_i', \gamma_j), \gamma_i),$$

for all $\gamma_i, \gamma_i', \gamma_j \in \Theta \times \{\mathfrak{s}, f\},\ i, j \in \{1, 2\},\ i \neq j$.

## 4.2 An impossibility theorem for the multi-dimensional type case

I prove that there is no mechanism that has the Vickrey auction's desirable incentive and efficiency properties, as well as weak budget balance, in any setting where bidders have multi-dimensional types. More precisely, I assume that bidder $i$'s private information is described by the multi-dimensional parameter $\gamma_i \in \Theta \times \{\mathfrak{s}, f\}$ and I assume that $\gamma_i$ parameterizes bidder $i$'s commonly known utility function $u$, where $u$ satisfies the conditions described in Section 4.1. Theorem 2 shows that in any such case, there is no mechanism that satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) no subsidies. In other words, there is no mechanism that satisfies these four properties for any multi-dimensional type space and for any choice of utility function that satisfies the conditions described in Section 4.1.

Theorem 2 also implies that efficient auction design is impossible on any richer type space because the increase in the dimensionality of bidder private information increases the number of incentive constraints we must satisfy to solve the efficient auction design. It is relevant to note that the prior impossibility results in this literature assume richer type spaces relative to the one studied here and also make specific function form restrictions on bidder preferences.[16]

In the proof of Theorem 2, I show that if there is an efficient auction, then there is endogenous interdependence between a bidder's demand for later units and her rival's multi-dimensional type. This is because in an efficient auction, the price a bidder pays for her first unit depends on her rival's type, and positive wealth effects imply a bidder's willingness to pay for her second unit varies with the amount she paid to win her first unit. Thus, the endogenous interdependence is caused by the combination of multi-unit demands and wealth effects.[17]

**Theorem 3.** *There is no mechanism that satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) weak budget balance when bidders have multi-dimensional types.*

My proof shows that efficiency implies that a bidder wins at least one unit if and only if her willingness to pay for her first unit exceeds her rival's willingness to pay for her second unit. This necessary condition for efficiency forms a contradiction with incentive compatibility. The contradiction emerges because bidder $j$ pays more for her first unit when bidder $i$ reports a flat demand instead of a steep demand. Thus, bidder $i$ lowers her rival's willingness to pay for her second unit, and hence the price she pays for her first unit, by reporting a flat demand curve instead of a steep demand curve. This violates incentive compatibility because bidder $i$'s report changes the amount she pays for her first unit.

[16] See the discussion of Dobzinski, Lavi, and Nisan (2012), Lavi and May (2012), and Goel et al. (2015) in the related literature section. Relatedly, Kazumura and Serizawa's (2016) impossibility theorem requires that only one bidder has multi-item demand, but their type space is again rich relative to the type space studied here.

[17] If we considered a model where one of these features is absent, we would again obtain a positive implementation result, even for cases where bidder types are multi-dimensional. In an analogous model where bidders have no wealth effects, then the Vickrey auction is efficient and dominant strategy implementable. Similarly, in the unit demand case, Demange and Gale (1985) construct an efficient auction for non-quasilinear bidders.

The formal proof of Theorem 2 follows from Lemma 4, Proposition 2 and Corollary 2 which are explained below. The proof is by contradiction. Suppose there exists a mechanism $\Gamma$ that satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) weak budget balance. Mechanism $\Gamma$ has assignment rule $q$ and payment rule $x$. The taxation principle states that a change in bidder $i$'s reported type only changes her payment if it changes her assignment.

*Remark* 2. (Taxation principle) If $\Gamma$ satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) weak budget balance, then there exists pricing rules $p_1$ and $p_2$ such that

$$p_i : \Theta \times \{\mathfrak{s}, f\} \to \mathbb{R}^3 \; \forall i = 1, 2,$$

and

$$x_i(\gamma_i, \gamma_j) = \sum_{n=0}^{k} p_{i,n}(\gamma_j) \iff q_i(\gamma_i, \gamma_j) = k \; \forall k \in \{0, 1, 2\}.$$

Lemma 4 further simplifies the proof. It shows that mechanisms that satisfy Properties (1)-(4) must also satisfy the no subsidy condition. The proof of Lemma 4 shows that we violate weak budget balance if a bidder is paid a positive amount to participate in the auction $p_{i,0}(\gamma_j) < 0$. Moreover, individual rationality ensures that $p_{i,0}(\gamma_j) \leq 0$, because a bidder never regrets participating in the mechanism, even if she wins zero units. Thus, it is the case that $p_{i,0}(\gamma_j) = 0 \; \forall \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$.

**Lemma 4.** *If $\Gamma$ satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) weak budget balance, then*

$$q_i(\gamma_i, \gamma_j) = 0 \implies x_i(\gamma_i, \gamma_j) = 0 \; \forall \gamma_i, \gamma_j \in \Theta \times \{\mathfrak{s}, f\}, \; i, j = 1, 2, \; i \neq j.$$

I derive a contradiction by placing necessary conditions on a mechanism's assignment rule, and consequently on the mechanism's pricing rule. It is useful to describe a mechanism's assignment rule by cut-off rules. I let $d_{i,k}^{t_i} : \Theta \times \{\mathfrak{s}, f\} \to \Theta$ be the intercept cut-off for bidder $i$'s to win unit $k \in \{1, 2\}$ when she has steepness $t_i \in \{\mathfrak{s}, f\}$. Bidder $i$'s $n^{th}$ unit cut-off is then

$$d_{i,k}^{t_i}(\gamma_j) := \begin{cases} \inf\{\theta \in \Theta | q_i((\theta, t_i), \gamma_j) \geq k\} & if \; \exists \theta \in \Theta \; s.t. \; q_i((\theta, t_i), \gamma_j) \geq k \\ \overline{\theta} & else. \end{cases}.$$

where $k \in \{1, 2\}$, $t_i \in \{\mathfrak{s}, f\}$, and $\gamma_j \in \Theta \times \{\mathfrak{s}, f\}$. Remark 3 gives restrictions on the cut-off rules for mechanisms satisfying Properties (1)-(4).

*Remark* 3. If $\Gamma$ satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) weak budget balance, then

(1) $d^{t_i}_{i,1}(0,t_j) = d^{t_i}_{i,2}(0,t_j) = 0 \ \forall t_i, t_j \in \{\mathfrak{s}, f\}$.

(2) $d^{t_i}_{i,2}(\theta_j,t_j)$, and $d^{t_i}_{i,1}(\theta_j,t_j)$ are weakly increasing in $\theta_j$, $\forall \theta_j \in \Theta$, $t_i, t_j \in \{\mathfrak{s}, f\}$.

(3) $d^{t_i}_{i,2}(\gamma_j) \geq d^{t_i}_{i,1}(\gamma_j) \ \forall \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$, $t_i \in \{\mathfrak{s}, f\}$.

The first point states that a bidder wins both units if she reports positive demand and her rival reports no demand. The second point states that a bidder faces a greater intercept cut-off when her rival reports greater demand. The final point states that the cut-off intercept for winning both units is weakly greater than the cut-off intercept for winning a single unit. The first point follows from efficiency, and the latter two points follow from incentive compatibility.

Proposition 2 places further restrictions on the cut-off rules associated with a mechanism that satisfies Properties (1)-(4).

**Proposition 2.** *If $\Gamma$ satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) weak budget balance, then*

(1) $d^{t_i}_{i,1}(\theta_j,t_j)$ is continuous and strictly increasing in $\theta_j \ \forall t_i, t_j \in \{\mathfrak{s}, f\}$.

(2) $d^{t_i}_{i,2}(\theta_j,t_j) > d^{t_i}_{i,1}(\theta_j,t_j) \ \forall \theta_j > 0$, $t_j \in \{\mathfrak{s}, f\}$.

(3) $p_{i,1}(\gamma_j) = d^{f}_{i,1}(\gamma_j) = d^{\mathfrak{s}}_{i,1}(\gamma_j) \ \forall \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$.

(4) $d^{t_i}_{i,1}(\theta_j, f) > d^{t_i}_{i,1}(\theta_j, \mathfrak{s}) \ \forall \theta_j > 0$, $t_i \in \{\mathfrak{s}, f\}$.

The first point states that bidder $i$'s first unit cut-off intercept is continuous and strictly increasing in her rival's intercept. The second point states that bidder $i$ has a strictly greater cut-off intercept for her second unit than she does for her first unit. This follows from efficiency and declining demand.

The third point states that a bidder $i$'s first unit cut-off intercept is independent of her reported steepness. This is because bidder $i$ wins her first unit if and only if her demand for her first unit exceeds the price she pays for her first unit $\theta_i > p_{i,1}(\gamma_j)$. Thus, bidder $i$'s first unit cut-off is independent of her reported steepness as $p_{i,1}(\gamma_j) = d^{\mathfrak{s}}_{i,1}(\gamma_j) = d^{f}_{i,1}(\gamma_j)$. Given this result, I drop the superscript on a bidder's first unit cut-off for the remainder of the section. That is, I let $d_{i,1}(\gamma_j) = d^{\mathfrak{s}}_{i,1}(\gamma_j) = d^{f}_{i,1}(\gamma_j)$.

The final point of Proposition 2 states that a bidder's first unit cut-off is greater when her rival has flat demand. This is an intuitive consequence of incentive compatibility and efficiency. If bidder $j$ has a flat demand, then bidder $j$ has a relatively higher demand for her second unit. Incentive compatibility thus implies that bidder $j$ has a lower second unit cut-off when her type is flat because the infimum intercept types where $b_2((\theta_j, t_j), p_{i,1}(\gamma_i)) > p_{i,2}(\gamma_i)$ is lower when $t_j = f$ versus when $t_j = \mathfrak{s}$. A direct consequence of this observation is that bidder $i$ faces a higher first unit cut-off when her rival, bidder $j$, reports a flat demand type versus steep demand type.

**Corollary 2.** *If $\Gamma$ satisfies (1) individual rationality, (2) incentive compatibility, (3) efficiency, and (4) weak budget balance, then if $\gamma_i, \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$ are such that $\theta_i = d_{i,1}(\gamma_j)$, then*

$$\theta_i = b_2(\gamma_j, p_{i,1}(\gamma_i)) \ \forall \gamma_i, \gamma_j \in \Theta \times \{\mathfrak{s}, f\}.$$

Corollary 2 shows that if bidder $i$'s is indifferent between winning 0 and 1 units ($\theta_i = d_{i,1}(\gamma_j)$), then bidder $i$'s willingness to pay for her first unit must equal her rival's (conditional) willingness to pay for her second unit. If the two quantities were unequal, then there would be a Pareto improving trade where the bidder with the higher respective willingness to pay buys a unit from the bidder with the lower willingness to pay.

I use Corollary 2 to obtain the contradiction that proves the impossibility theorem. To see the contradiction, fix bidder $i$'s intercept type $\theta_i$ and suppose again that bidder $i$'s is indifferent between winning 0 and 1 units (i.e. $\theta_i$ is such that $\theta_i = d_{i,1}(\gamma_j)$; see point a. in Figure 1 below). Let's compare the case where bidder $i$ reports a steep demand type ($t_i = \mathfrak{s}$) with a case where bidder $i$ reports a flat demand curve ($t_i = f$). Proposition 2 shows that bidder $j$ pays more for her first unit of the good in the latter case relative to the former case (i.e. $d_{j,1}(\theta_i, f) = p_{j,1}(\theta_i, f) > p_{j,1}(\theta_i, \mathfrak{s}) = d_{j,1}(\theta_i, \mathfrak{s})$; see points b. and c. and Figure 1 below). This is intuitive, because bidder $j$ pays more for her first unit when bidder $i$ has higher demand for her second unit. Positive wealth effects then imply that bidder $j$ is willing to pay less for her second unit when bidder $i$ has a flat demand versus a steep demand,

$$b_2(\gamma_j, p_{j,1}(\theta_i, f)) < b_2(\gamma_j, p_{j,1}(\theta_i, \mathfrak{s})). \tag{4}$$

However, the above inequality contradicts the implication of Corollary 2 because

$$\theta_i = d_{i,1}(\gamma_j) \implies \theta_i = b_2(\gamma_j, p_{j,1}(\theta_i, f)) = b_2(\gamma_j, p_{j,1}(\theta_i, \mathfrak{s})). \tag{5}$$

The contradiction between expressions (3) and (4) proves the impossibility theorem.

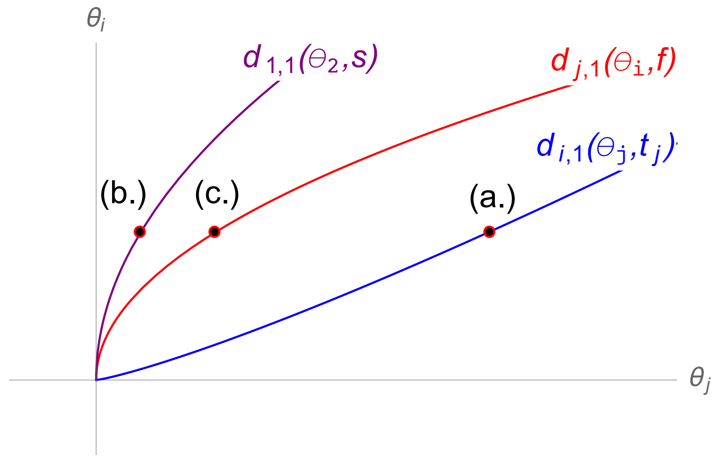


Figure 3: First Unit Cut-off rules for a fixed $t_j \in \{\mathfrak{s}, f\}$.

Thus, there is no mechanism that retains the Vickrey auction's desirable incentive and efficiency properties on any type space that satisfies the conditions given in Section 4.1. Moreover, there is no mechanism that retains the Vickrey auction's desirable incentive and efficiency properties on any richer type space — the increase in dimensionality only increases the number of incentive constraints that our mechanism must satisfy.

The proof of Theorem 3 illustrates how the combination of wealth effects and multi-unit demands inhibits efficient auction design. In contrast, in the quasilinear setting, there are no wealth effects and the Vickrey auction is the unique auction that satisfies Properties (1)-(4) (see Holmstrom (1979)). In a $2 \times 2$ quasilinear setting, a Vickrey auction is such that the price a bidder pays for her first unit equals her rival's willingness to pay for her second unit. Corollary 2 shows that this is also a necessary condition for efficient auction design in the non-quasilinear setting. Yet, in the non-quasilinear setting, the presence of wealth effects implies that the price a bidder pays for her first unit influences her demand for her second unit. By stating a high demand for her second unit, a bidder forces her rival to pay more for her first unit. This deviation can benefit a bidder in a non-quasilinear setting because when the bidder's rival pays more for her first unit, the rival has lower demand for her second unit. Moreover, a bidder pays less to win her first unit when her rival has lower demand for her second unit. Thus, no mechanism can simultaneously satisfy Properties (1)-(4) when we introduce wealth effects and multi-dimensional heterogeneity.

Finally, note we can extend the above proof to where only one bidder has multi-dimensional private information. In particular, Proposition 3 and Corollary 2 would be unchanged if we instead assume that $t_j$ was common knowledge, and thus bidder $j$'s private information is one-dimensional. Thus, we get the same contradiction between Expressions (4) and (5) when $t_j$ is common knowledge, and bidder $i$ is the only bidder with multi-dimensional private information.[18]

# Appendix

## A1: Proofs

**Proof of Lemma 1.**

*Proof.* The proof is by induction. When $k = 1$, $p_1(\theta_j, d)$ is weakly increasing in $\theta_j$ because $p_1(\theta_j, d) = d_1(\theta_j)$ and $d_1(\theta_j)$ is weakly increasing in $\theta_j$ for all $\theta_j \in \Theta$, $d \in \mathcal{D}$.

Before showing the inductive step, it is useful to note that

$$z \geq y \geq 0 \implies b_k(\theta, z) + z \geq b_k(\theta, y)) + y \ \forall k \in \{1, \dots, K\}.$$

This is because

$$z \geq y \geq 0 \implies u(k-1, -y-b_k(\theta, y), \theta) = u(k, -y, \theta) \geq u(k, -z, \theta) = u(k-1, -z-b_k(\theta, z), \theta).$$

The final inequality implies that

$$z \geq y \geq 0 \implies z + b_k(\theta, z) \geq y + b_k(\theta, y), \tag{6}$$

because $u$ is increasing in the second argument.

Returning to the proof, suppose that $P_{k-1}(\theta_j, d)$ is weakly increasing in $\theta_j \ \forall \theta_j \in \Theta,\ d \in \mathcal{D}$ and some $k \in \{1, \dots, K\}$. I show that this implies that $P_k(\theta_j, d)$ is weakly increasing in $\theta_j \ \forall \theta_j \in \Theta,\ d \in \mathcal{D}$.

Let $\overline{\theta} \geq \theta_j^h > \theta_j^\ell \geq 0$. Then,

$$P_k(\theta_j^h, d) = P_{k-1}(\theta_j^h, d) + b_k(d_k(\theta_j^h), P_{k-1}(\theta_j^h, d)) \geq P_{k-1}(\theta_j^h, d) + b_k(d_k(\theta_j^\ell), P_{k-1}(\theta_j^h, d)),$$

where the equality follows from the definition of $p_k$, and the inequality follows because $b_k$ is increasing in the first argument and $d_k(\theta_j^h) \geq d_k(\theta_j^\ell)$. Then,

$$P_{k-1}(\theta_j^h, d) + b_k(d_k(\theta_j^\ell), P_{k-1}(\theta_j^h, d)) \geq P_{k-1}(\theta_j^\ell, d) + b_k(d_k(\theta_j^\ell), P_{k-1}(\theta_j^\ell, d)) = P_k(\theta_j^\ell, d),$$

where the inequality is implied by Equation 6 where we let $z = P_{k-1}(\theta_j^h, d) \geq y = P_{k-1}(\theta_j^\ell, d) \geq 0$. The final equality holds from the definition of $p_k$. Thus if $d \in \mathcal{D}$ and $P_{k-1}(\theta_j, d)$ is weakly

[18] I credit Hongyao Ma and an anonymous referee, both of whom suggested this extension.

increasing in $\theta_j$ $\forall\theta_j \in \Theta$, then $P_k(\theta_j, d)$ is weakly increasing in $\theta_j$ $\forall\theta_j \in \Theta$. $\square$

**Proof of Lemma 2**

*Proof.* Note $f(k, \theta_i, \theta_j, d)$ is strictly increasing in $\theta_i \in \Theta$ for all $k \in \{1, \dots, K\}$, $\theta_j \in \Theta$, $d \in \mathcal{D}$, because single crossing implies that $b_k(\theta_i, P_{k-1}(\theta_j, d))$ is strictly increasing in $\theta_i$ for all $k \in \{1, \dots, K\}$, $\theta_j \in \Theta$, $d \in \mathcal{D}$. In addition, $b_{K-k+1}(\theta_j, P_{K-k}(\theta_i, d))$ is weakly decreasing in the second argument and Lemma 1 shows $P_{K-k}(\theta_i, d)$ is increasing in $\theta_i$ for all $k \in \{1, \dots, K\}$, $\theta_j \in \Theta$, $d \in \mathcal{D}$. An identical argument shows that $f$ is strictly decreasing in $\theta_j \in \Theta$ for all $k \in \{1, \dots, K\}$, $\theta_i \in \Theta$, $d \in \mathcal{D}$. Declining demand and positive wealth effects imply that $f$ is strictly decreasing in $k \in \{1, \dots, K\}$ for all $\theta_i, \theta_j \in \Theta$, $d \in \mathcal{D}$. $\square$

**Proof of Theorem 1A.**

*Proof.* I construct a mechanism $\Gamma^*$ that follows from the symmetric cut-off rule $d^* \in \mathcal{D}$. I assume ties (in terms of willingness to pays for additional units) are broken in favor of bidder 1. Thus, the mechanism $\Gamma^*$ has an assignment rule for bidder 1 where

$$q_1(\theta_1, \theta_2) = \max\{k \in \{0, 1, \dots, K\} | b_k(\theta_1, P_{k-1}(\theta_2, d^*)) \geq b_{K-k+1}(\theta_2, P_{K-k}(\theta_1, d^*))\},$$

and $q_2(\theta_1, \theta_2) = K - q_1(\theta_1, \theta_2)$ for all $\theta_1, \theta_2 \in \Theta$. The mechanism has transfer rule $x_i(\theta_1, \theta_2) = P_{q_i(\theta_i, \theta_j)}(\theta_j, d^*)$. By construction, the mechanism is feasible, satisfies no subsidies, and individual rationality. In the remainder of the proof I show that the mechanism satisfies incentive compatibility and efficiency.

**Incentive Compatibility:** I show that mechanism $\Gamma^*$ is incentive compatible by studying two cases.

Case 1: Suppose that $\theta_1, \theta_2 \in \Theta$ are such that $q_i(\theta_i, \theta_j) \geq k$ for some number $k \in \{1, \dots, K\}$. Then the construction of mechanism $\Gamma^*$ implies that

$$b_k(\theta_i, P_{k-1}(\theta_j, d^*)) \geq b_{K-k+1}(\theta_j, P_{K-k}(\theta_i, d^*)) \implies \theta_i \geq d_k^*(\theta_j),$$

where the implication follows from Remark 1. And since $b_k$ is increasing in the first argument, $\theta_i \geq d_k^*(\theta_j)$ implies that

$$b_k(\theta_i, P_{k-1}(\theta_j, d^*)) \geq b_k(d_k^*(\theta_j), P_{k-1}(\theta_j, d^*)) = p_k(\theta_j, d^*).$$

In other words, the price of bidder $i$'s $k^{th}$ unit is below her willingness to pay for her $k^{th}$ unit. Thus, bidder $i$ has no incentive to deviate by reporting a lower type and winning fewer units.

Case 2: Suppose $\theta_1, \theta_2 \in \Theta$ are such that $k > q_i(\theta_i, \theta_j)$ for number $k \in \{1, \dots, K\}$. Then, the construction of mechanism $\Gamma^*$ implies that

$$b_{K-k+1}(\theta_j, P_{K-k}(\theta_i, d^*) \geq b_k(\theta_i, P_{k-1}(\theta_j, d^*)) \implies d_k^*(\theta_j) \geq \theta_i.$$

In addition, $d_k^*(\theta_j) \geq \theta_i$ implies that

$$p_k(\theta_j, d^*) = b_k(d_k^*(\theta_j), P_{k-1}(\theta_j, d^*)) \geq b_k(\theta_i, P_{k-1}(\theta_j, d^*)).$$

Thus, the price of winning an $k^{th}$ unit where $k > q_i(\theta_i, \theta_j)$ exceeds bidder $i$'s willingness to pay for her $k^{th}$ unit, conditional on having won $k-1$ units under pricing rule $p(\theta_j, d^*)$. Therefore, bidder $i$ does not increase her utility by reporting a type $\theta_i'$ that allows her to win more units. Thus, the two cases show that the mechanism is incentive compatible.

**Efficiency:** Lastly, I show that mechanism $\Gamma^*$ is efficient. Fix $\theta_1, \theta_2 \in \Theta$. In addition, let $q_i^* \in \{0, 1, \dots, K\}$ be such that $q_i^* = q_i(\theta_1, \theta_2)$ for $i = 1, 2$. Similarly, let $x_i^* \in \mathbb{R}_+$ be such that $x_i^* = x_i(\theta_1, \theta_2)$ for $i = 1, 2$. Also, let $\rho > 0$ be such that

$$\rho := \max\{b_{q_1^*+1}(\theta_1, x_1^*), b_{q_2^*+1}(\theta_2, x_2^*)\}.$$

I show that there is no feasible outcome that Pareto dominates the outcome $\{q_i^*, x_i^*\}_{i=1}^2$. I refer to the outcome $\{q_i^*, x_i^*\}_{i=1}^2$ as the 'star' bundle. I prove this by showing that there is no bundle of the form $\{\tilde{q}_i, \tilde{x}_i\}_{i=1}^2$ (a 'tilde' bundle) — where, $\tilde{q}_1, \tilde{q}_2 \in \{0, 1, 2\}$, and $\tilde{q}_1 + \tilde{q}_2 \leq K$, and $\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}$ — that Pareto dominates outcome $\{q_i^*, x_i^*\}_{i=1}^2$. I show that the mechanism satisfies efficiency by again considering two cases.

Case 1: First, suppose that $\tilde{q}_i \leq q_i^*$ for $i = 1, 2$. Then if the tilde bundle Pareto dominates the star bundle, it must be the case that $\tilde{x}_i \leq x_i^*$ for $i = 1, 2$, because no bidder is made strictly worse by consuming the tilde bundle. No bidder is strictly better off unless she makes a strictly lower payment. In addition, if any bidder makes a strictly lower payment, the auctioneer gets strictly lower revenue. Thus, the outcome $\{q_i^*, x_i^*\}_{i=1}^2$ is not Pareto dominated by any outcome of the form $\{\tilde{q}_i, \tilde{x}_i\}_{i=1}^2$ where $\tilde{q}_i \leq q_i^*$ for all $i = 1, 2$.

Case 2: Next, suppose that the tilde bundle is such that $q_i^* < \tilde{q}_i$ for some $i = 1, 2$. Then feasibility implies that $q_j^* < \tilde{q}_j$ where $j = 1, 2$ and $j \neq i$. In addition, it must be the case that bidder $i$ is made no worse off by consuming the tilde bundle outcome $(\tilde{q}_i, \tilde{x}_i)$. Note that bidder $i$'s willingness to pay for an additional unit when she consumes the star bundle outcome $(q_i^*, x_i^*)$ is $b_{q_i^*+1}(\theta_i, x_i^*)$ where $\rho \geq b_{q_i^*+1}(\theta_i, x_i^*)$ by construction. Thus, we have that

$$u(q_i^*, -x_i^*, \theta_i) \geq u(q_i^* + 1, -x_i^* - \rho, \theta_i) \geq u(q_i^* + k, -x_i^* - k\rho, \theta_i) \; \forall k \in \{1, \dots, K - q_i^*\},$$

where the first inequality holds because $\rho \geq b_{q_i^*+1}(\theta_i, x_i^*)$ and the second inequality holds because bidders have declining demand and positive wealth effects. Hence,

$$\rho \geq b_{q_i^*+1}(\theta_i, x_i^*) \implies \rho > b_q(\theta_i, x) \; \forall q \in \{q_i^* + 1, \ldots, K\}, \; x > x_i^*.$$

Thus, if bidder $i$ is made no worse off by the reallocation, we must have that

$$\tilde{x}_i \leq x_i^* + (\tilde{q}_i - q_i^*)\rho.$$

In other words, bidder $i$ pays less than $\rho$ for each additional unit when we move from allocation $\{q_i^*, x_i^*\}_{i=1}^2$ to allocation $\{\tilde{q}_i, \tilde{x}_i\}_{i=1}^2$. Moreover, if bidder $i$ is made strictly better off under the latter outcome, then the above expression holds with a strict inequality.

When we assume that $q_i^* < \tilde{q}_i$ for some $i = 1, 2$, then feasibility implies that $q_j^* \leq \tilde{q}_j - (\tilde{q}_i - q_i^*)$ where $j = 1, 2$ and $j \neq i$. In addition, it must be the case that bidder $j$ is made no worse off by consuming the quantity and payment outcome of $(\tilde{q}_j, \tilde{x}_j)$. Note that,

$$s_k(\theta_j, x - p + b_k(\theta_j, x - p)) - b_k(\theta_j, x - p) = 0, \; \forall k \in \{1, \ldots, K\}, \; \theta_j \in \Theta, x, p > 0.$$

Thus,

$$b_k(\theta_j, x - p) - p \geq 0 \implies s_k(\theta_j, x) - b_k(\theta_j, x - p) \geq 0, \; \forall k \in \{1, \ldots, K\}, \; \theta_j \in \Theta, x, p > 0.$$

because

$$b_k(\theta_j, x - p) - p \geq 0 \implies s_k(\theta_j, x) \geq s_k(\theta_j, x - p + b_k(\theta_j, x - p)) = b_k(\theta_j, x - p)$$

for all $k \in \{1, \ldots, K\}, \; \theta_j \in \Theta, x, p > 0$. Furthermore, recall that bidder $j$ wins $q_j^*$ units in mechanism $\Gamma$ with cut-off rule $d^* \in \mathcal{D}$ where $T(d^*) = d^*$. Remark 1 then implies that

$$b_{q_j^*}(\theta_j, x_j^* - p_{q_j^*}(\theta_j, d^*)) \geq b_{q_i^*+1}(\theta_i, x_i^*).$$

Thus,

$$b_{q_j^*}(\theta_j, x_j^* - p_{q_j^*}(\theta_j, d^*)) \geq \rho = \max\{b_{q_i^*+1}(\theta_i, x_i^*), b_{q_j^*+1}(\theta_j, x_j^*)\}.$$

This implies that

$$s_{q_j^*}(\theta_j, x_j^*) \geq \rho,$$

because we showed that $b_{q_j^*}(\theta_j, x_j^* - p_{q_j^*}(\theta_j, d^*)) \geq p_{q_j^*}(\theta_j, d^*) \implies s_{q_j^*}(\theta_j, x_j^*) \geq b_{q_j^*}(\theta_j, x_j^* -$

$p_{q_j^*}(\theta_j, d^*))$. We then have that

$$u(q_j^*, -x_j^*, \theta_j) \geq u(q_j^* - 1, -x_j^* + \rho, \theta_j) \geq u(q_j^* - k, -x^* + k\rho, \theta_j) \;\forall k \in \{1, \ldots, q_j^*\}.$$

The final inequality holds because declining demand and positive wealth effects combine to imply that

$$s_k(\theta_j, x) \geq s_{q_j^*}(\theta_j, x_j^*) \geq \rho \;\forall k \in \{1, \ldots, q_j^* - 1\},\; x \leq x_j^*.$$

In other words, bidder $j$'s utility does not increase if she sells a unit at price $\rho$. This implies that

$$u(q_j^*, -x_j^*, \theta_j) \geq u(\tilde{q}_j, -x_j^* + \rho(q_j^* - \tilde{q}_j), \theta_j).$$

Therefore, if bidder $j$ is made no worse off by winning $\tilde{q}_j$ units and paying $\tilde{x}_j$, then

$$\tilde{x}_j \leq x_j^* - \rho(q_j^* - \tilde{q}_j),$$

where the above inequality is strict if bidder $i$ is made strictly better off under the tilde outcome.

Thus, we have that

$$\tilde{x}_1 \leq x_1^* + (\tilde{q}_1 - q_1^*)\rho,\; \textit{and}\; \tilde{x}_2 \leq x_2^* - \rho(q_2^* - \tilde{q}_2),$$

which implies

$$\tilde{x}_1 + \tilde{x}_2 \leq x_1^* + x_2^*,$$

where the above holds with a strict inequality if at least one bidder is made strictly better off under the tilde outcome. Thus, $\{\tilde{q}_i, \tilde{x}_i\}_{i=1}^2$ does not Pareto dominate the outcome $\{q_i^*, x_i^*\}_{i=1}^2$ when the tilde bundle is such that $q_i^* < \tilde{q}_i$ for some $i = 1, 2$.

Therefore, our analysis of Case 1 and Case 2 shows that there is no outcome that Pareto dominates outcome $\{q_i^*, x_i^*\}_{i=1}^2$, and hence the outcome of mechanism $\Gamma^*$ is an efficient outcome for all $(\theta_1, \theta_2) \in \Theta^2$. □

**Proof of Theorem 1.**

*Proof.* I prove the theorem in three steps: **(1)** I show that if $d \in \mathcal{D}$, then $T(d) \in \mathcal{D}$; **(2)** I show that $T$ is a continuous mapping; and **(3)** I show that $\mathcal{D}$ is compact.

**(1)** In order to show that $d \in \mathcal{D} \implies T(d) \in \mathcal{D}$, I first show that $T_k(d)(\theta)$ is weakly increasing in $\theta$ for any $\theta \in \Theta,\; d \in \mathcal{D},\; k \in \{1, \ldots, K\}$. Then I show that $T_k(d)(\theta)$ is also weakly increasing in $k$ for any $k \in \{1, \ldots, K\},\; \theta \in \Theta,\; d \in \mathcal{D}$.

If $T_k(d)(\theta_j^h) = \overline{\theta}$, then $T_k(d)(\theta_j^\ell) \leq \overline{\theta}$ because $T_k(d)(\theta) \in [0, \overline{\theta}]$ $\forall \theta \in \Theta,\ d \in \mathcal{D}$. If $T_k(d)(\theta_j^h) < \overline{\theta}$, then

$$\inf\{\theta | f(k, \theta, \theta_j^h, d) > 0\} \geq \inf\{\theta | f(k, \theta, \theta_j^\ell, d) > 0\},$$

because Lemma 2 shows $f$ is strictly increasing in the first argument and thus $\theta_j^h > \theta_j^\ell$ implies that

$$f(k, \theta_i, \theta_j^\ell, d) > f(k, \theta_i, \theta_j^h, d)\ \forall k \in \{1, \ldots, K\},\ \theta_i \in \Theta,\ d \in \mathcal{D}.$$

Thus,

$$T_k(d)(\theta_j^h) \geq T_k(d)(\theta_j^\ell)\ \forall k \in \{1, \ldots, K\},\ \overline{\theta} \geq \theta_j^h > \theta_j^\ell \geq 0,\ d \in \mathcal{D}.$$

Next, I show that

$$T_{k+1}(d)(\theta_j) \geq T_k(d)(\theta_j)\ \forall k \in \{1, \ldots, K-1\},\ \theta_j \in \Theta,\ d \in \mathcal{D}.$$

If $k \in \{1, \ldots, K-1\},\ \theta_j \in \Theta,\ d \in \mathcal{D}$ are such that $T_{k+1}(d)(\theta_j) = \overline{\theta}$, then the above inequality holds because $T_k(d)(\theta) \in [0, \overline{\theta}]$ $\forall k \in \{1, \ldots, K-1\},\ \theta \in \Theta,\ d \in \mathcal{D}$.

Next suppose that $k \in \{1, \ldots, K-1\},\ \theta \in \Theta,$ and $d \in \mathcal{D}$ are such that $T_{k+1}(d)(\theta_j) < \overline{\theta}$. Note that Lemma 2 shows that

$$f(\theta_i, \theta_j, k, d) > f(\theta_i, \theta_j, k+1, d)\ \forall k \in \{1, \ldots, K-1\},\ \theta_i, \theta_j \in \Theta,\ d \in \mathcal{D}.$$

In addition $f$ is strictly increasing in the first argument. Therefore,

$$\inf\{\theta | f(\theta, \theta_j, k+1, d) > 0\} \geq \inf\{\theta | f(\theta, \theta_j, k, d) > 0\} \implies T_{k+1}(d)(\theta_j) \geq T_k(d)(\theta_j),$$

$\forall k \in \{1, \ldots, K-1\},\ \theta_j \in \Theta,\ d \in \mathcal{D}$. Thus, $T(d) \in \mathcal{D}$ $\forall d \in \mathcal{D}$, because $T_k(d)(\theta)$ is weakly increasing in $\theta$ $\forall k \in \{1, \ldots, K\},\ \theta \in \Theta$, and

$$d \in \mathcal{D} \implies T_{k+1}(d)(\theta) \geq T_k(d)(\theta)\ \forall k \in \{1, \ldots, K-1\},\ \theta \in \Theta.$$

**(2)** Next, I show that $T$ is a continuous mapping. Since $\mathcal{D}$ is a metric space (under the uniform norm), it suffices to show that if $\{d^n\}_{n=1}^\infty$ is such that $d^n \in \mathcal{D}$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} d^n = d$, then $\lim_{n\to\infty} T(d^n) = T(d)$ (see Aliprantis and Border (2006), pg. 36). More formally, assume there is a sequence $\{d^n\}_{n=1}^\infty$ such that $d^n \in \mathcal{D}$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} d^n(\theta_j) = d(\theta_j)$ $\forall \theta_j \in \Theta$ where $d \in \mathcal{D}$. I show that this implies that $T$ satisfies $\lim_{n\to\infty} T(d^n)(\theta_j) = T(d)(\theta_j)$ $\forall \theta_j \in \Theta$, where $T(d) \in \mathcal{D}$.

First, I show that

$$\lim_{n\to\infty} P_k(\theta_j, d^n) = P_k(\theta_j, d) \ \forall k \in \{1, \ldots, K\}, \ \theta_j \in \Theta.$$

The proof is by induction. The above equality is true if $k = 1$ because

$$P_1(\theta_j, d^n) = p_1(\theta_j, d^n) = d_1^n(\theta_j), \forall n \in \mathbb{N}, \ \theta_j \in \Theta.$$

Thus, $\lim_{n\to\infty} p_1(\theta_j, d^n) = \lim d_1^n(\theta_j) = d_1(\theta_j) = p_1(\theta_j, d) \ \forall \theta_j \in \Theta$.

For the inductive step of the proof, suppose that there is $k \in \{1, \ldots, K\}$ is such that

$$\lim_{n\to\infty} P_{k-1}(\theta_j, d^n) = P_{k-1}(\theta_j, d) \ \forall \theta_j \in \Theta.$$

I show that this implies that the above expression holds when $k-1$ is replaced by $k$. Note that

$$P_k(\theta_j, d^n) = b_k(d_k^n(\theta_j), P_{k-1}(\theta_j, d^n)) + P_{k-1}(\theta_j, d^n) \ \forall n \in \mathbb{N}, \ \theta_j \in \Theta.$$

Since $b_k$ is continuous in both arguments, and $\lim_{n\to\infty} d_k^n(\theta_j) \to d_k(\theta_j) \ \forall \theta_j \in \Theta$, and $\lim_{n\to\infty} P_{k-1}(\theta_j, d^n) = P_{k-1}(\theta_j, d) \ \forall \theta_j \in \Theta$, then we have that

$$\lim_{n\to\infty} b_k(d_k^n(\theta_j), P_{k-1}(\theta_j, d^n)) + P_{k-1}(\theta_j, d^n) = b_k(d_k(\theta_j), P_{k-1}(\theta_j, d)) + P_{k-1}(\theta_j, d) = P_k(\theta_j, d),$$

for all $\theta_j \in \Theta$. Thus, we have proven that

$$\lim_{n\to\infty} d^n(\theta_j) = d(\theta_j) \ \forall \theta_j \in \Theta \implies \lim_{n\to\infty} P_k(\theta_j, d^n) = P_k(\theta_j, d) \ \forall k \in \{1, \ldots, K\}, \ \theta_j \in \Theta.$$

Recall that

$$f(k, \theta_i, \theta_j, d) = b_k(\theta_i, P_{k-1}(\theta_j, d)) - b_{K-k+1}(\theta_j, P_{K-k}(\theta_i, d)) \ \forall k \in \{1, \ldots, K\}, \ \theta_i, \theta_j \in \Theta, \ d \in \mathcal{D}.$$

Since $\lim_{n\to\infty} P_k(\theta_j, d^n) = P_k(\theta_j, d) \ \forall k \in \{1, \ldots, K\}, \ \theta_j \in \Theta$ and $b_k$ is continuous in the second argument, it follows that

$$\lim_{n\to\infty} d^n(\theta_j) = d(\theta_j) \ \forall \theta_j \implies \lim_{n\to\infty} f(k, \theta_i, \theta_j, d^n) = f(k, \theta_i, \theta_j, d) \ \forall k \in \{1, \ldots, K\}, \ \theta_i, \theta_j \in \Theta.$$

I use the above expression to show that it is also the case that $\lim_{n\to\infty} T(d^n)(\theta_j) = T(d)(\theta_j) \ \forall \theta_j$.

I separate the remainder of the proof that $T$ is continuous into two cases. First, I show that if $\theta_j \in \Theta$ and $k \in \{1, \ldots, K\}$ are such that $\lim_{n\to\infty} f(k, \overline{\theta}, \theta_j, d^n) = f(k, \overline{\theta}, \theta_j, d) \leq 0$, then

$$\lim_{n\to\infty} T_k(d^n)(\theta_j) = T_k(d)(\theta_j) = \overline{\theta}.$$

Then I show that if $\theta_j \in \Theta$ and $k \in \{1, \ldots, K\}$ are such that $\lim_{n\to\infty} f(k, \overline{\theta}, \theta_j, d^n) = f(k, \overline{\theta}, \theta_j, d) \leq 0$, then

$$\lim_{n\to\infty} T_k(d^n)(\theta_j) = T_k(d)(\theta_j) \leq \overline{\theta}.$$

For the first case, if $\theta_j \in \Theta$ and $k \in \{1, \ldots, K\}$ are such that $\lim_{n\to\infty} f(k, \overline{\theta}, \theta_j, d^n) = f(k, \overline{\theta}, \theta_j, d) \leq 0$, then for any $\epsilon > 0$, there exists an $n^* \in \mathbb{N}$ such that for all $n > n^*$,

$$f(k, \overline{\theta} - \epsilon, \theta_j, d^n) < 0 \implies \overline{\theta} - \epsilon \geq T_k(d^n)(\theta_j) \implies \lim_{n\to\infty} T_k(d^n)(\theta_j) \geq \overline{\theta} - \epsilon.$$

where the first inequality holds because $f$ is strictly decreasing in the second argument. Since $\epsilon > 0$ is arbitrary, this implies that $\lim_{n\to\infty} T_k(d^n)(\theta_j) = \overline{\theta}$ when $\theta_j \in \Theta$ and $k \in \{1, \ldots, K\}$ are such that $\lim_{n\to\infty} f(k, \overline{\theta}, \theta_j, d^n) = f(k, \overline{\theta}, \theta_j, d) \leq 0$.

If $\theta_j \in \Theta$ and $k \in \{1, \ldots, K\}$ are such that $f(k, \overline{\theta}, \theta_j, d) > 0$, then $T_k(d)(\theta_j) < \overline{\theta}$, and

$$\lim_{n\to\infty} T_k(d^n)(\theta_j) = \lim \inf\{\theta | f(k, \theta, \theta_j, d^n) > 0\} = \inf\{\theta | f(k, \theta, \theta_j, d) > 0\} = T_k(d)(\theta_j)$$

where the second equality holds because (1) $f$ is strictly increasing in the second argument and (2) $f(k, \theta, \theta_j, d^n) \to f(k, \theta, \theta_j, d)\ \forall \theta_j \in \Theta$. Thus, we conclude that $T$ is a continuous mapping over the domain of $\mathcal{D}$ because

$$\lim_{n\to\infty} d^n(\theta_j) = d(\theta_j)\ \forall \theta_j \in \Theta \implies \lim_{n\to\infty} T(d^n)(\theta_j) = T(d)(\theta_j) \forall \theta_j \in \Theta.$$

**(3)** Finally, I show that $\mathcal{D}$ is compact. Or equivalently, I show that $\mathcal{D}$ is complete and totally bounded. The set $\mathcal{D}$ is complete because every Cauchy sequence $\{d^n\}_{n=1}^{\infty}$ converges to an element $d \in \mathcal{D}$ when I use the $L^1$ norm as our metric.

In addition, the set $\mathcal{D}$ is totally bounded. This is because under the $L^1$ norm any weakly increasing and bounded function can be approximated by a sequence of simple functions and $\mathcal{D}$ is a subset of the set of all weakly increasing and bounded functions. Thus, for any $\epsilon > 0$, I can construct a finite set of simple functions $\{d_1, \ldots, d_n\}$, where for any $d \in \mathcal{D}$, there is an $i$ such that $|d - d_i| < \epsilon$ according to the $L^1$ norm. Thus the set of admissible cut-off rules $\mathcal{D}$ is covered by a finite number of $\epsilon$ measure balls. Thus, $\mathcal{D}$ is compact (see Theorem 3.28 in Aliprantis and Border (2006)).

Thus, I have shown that $T : \mathcal{D} \to \mathcal{D}$ is a continuous mapping from a compact space $\mathcal{D}$ into itself. Schauder's fixed point theorem then states that the mapping $T$ has a fixed point $d^* \in \mathcal{D}$. $\square$

**Proof of Lemma 3**

*Proof.* Recall we consider the decision problem of bidder 1 and suppose that $\overline{\theta} \geq \theta_2 \geq \theta_3 \geq \theta_j \geq 0 \; \forall j \neq 1, 2, 3$. We show there is a unique $d(\cdot)$ where

$$d(\theta_{-1}) = \max\{\theta_3, b_2(\theta_2, d(d(\theta_{-1}), \theta_{-1,2}))\} \; \forall \theta_{-1} \in \Theta^{N-1} \; s.t. \; \theta_2 \geq \theta_3 \geq \theta_j \; \forall j \in \{4, \ldots, N\}. \tag{7}$$

When $b_2(\theta_2, \theta_3) \leq \theta_3$, then Equation 7 implies that

$$d(\theta_{-1}) = \theta_3 = \max\{\theta_3, b_2(\theta_2, d(d(\theta_{-1}), \theta_{-1,2}))\},$$

because

$$\theta_3 \geq b_2(\theta_2, \theta_3) \geq b_2(\theta_2, d(d(\theta_{-1}), \theta_{-1,2})).$$

Thus, there is a $\tilde{\theta}_2 \in \Theta$ where $\tilde{\theta}_2 > \theta_3$ and a unique $d(\tilde{\theta}_2, \theta_{-1,2})$ that solves Equation 7 . Let $\theta^* \geq \tilde{\theta}_2$ be the supremum $\tilde{\theta}_2$ such that $d$ is uniquely defined by Equation 7 over the interval $[\theta_3, \theta^*) \subset \Theta$.

I separate the remainder proof into three steps. **(1)** I show that the cut-off rule $d$ defined by Equation 7 is weakly in $\theta_2$ for all $\theta_2$ in the interval $(\theta_3, \theta^*) \subset \Theta$. **(2)** I show that the cut-off rule $d$ is continuous in $\theta_2$ over the interval $(\theta_3, \theta^*) \subset \Theta$. (3) I show that we can set $\theta^* = \overline{\theta}$. Thus, the cut-off rule $d$ defined by Equation 7 is continuous and weakly increasing for all $\theta_2 \in (\theta_3, \overline{\theta}]$.

**(1)** I show that the cut-off rule $d$ that is defined by Equation 7 is weakly increasing in $\theta_2$ for all $\theta_2 \in [\theta_3, \theta^*)$. I prove this by contradiction. Suppose $d$ was not weakly increasing in $\theta_2$ when $\theta_2 \in [\theta_3, \theta^*)$. Then $\exists \hat{\theta} \in [\theta_3, \theta^*)$ such that

$$\hat{\theta} = \inf\{\theta | \exists \theta' > \theta \; s.t. \; d(\theta', \theta_{-1,2}) < d(\theta, \theta_{-1,2})\}.$$

Thus, for any $\epsilon > 0$ there exists a $\theta_\ell, \theta_h \in \Theta$ such that $\theta_\ell \leq \hat{\theta} \leq \theta_h$, $\theta_h, \theta_\ell \in (\hat{\theta} - \epsilon, \hat{\theta} + \epsilon) \subset [\theta_3, \theta^*)$, and $d(\theta_\ell, \theta_{-1,2}) > d(\theta_h, \theta_{-1,2}) \geq \theta_3$. Moreover,

$$d(\theta_\ell, \theta_{-1,2}) > \theta_3 \implies \theta_\ell > d(\theta_\ell, \theta_{-1,2}) = b_2(\theta_\ell, d(d(\theta_\ell, \theta_{-1,2}), \theta_{-1,2})) > \theta_3.$$

In addition, $d(\theta_\ell, \theta_{-1,2}) > d(\theta_h, \theta_{-1,2})$ implies that

$$b_2(\theta_h, d(d(\theta_h, \theta_{-1,2}), \theta_{-1,2}) < b_2(\theta_\ell, d(d(\theta_\ell, \theta_{-1,2}), \theta_{-1,2})).$$

Since $b_2$ is increasing in the first argument and $\theta_h > \theta_\ell$, then it must be the case that

$$d(d(\theta_h, \theta_{-1,2}), \theta_{-1,2}) > d(d(\theta_\ell, \theta_{-1,2}), \theta_{-1,2}).$$

However, the above inequality can not hold because

$$d(\theta_h, \theta_{-1,2}) < d(\theta_\ell, \theta_{-1,2}) < \theta_\ell \leq \hat{\theta} \implies d(d(\theta_h, \theta_{-1,2}), \theta_{-1,2}) \leq d(d(\theta_\ell, \theta_{-1,2}), \theta_{-1,2}),$$

where the final inequality holds because $d$ is weakly increasing when $\theta < \hat{\theta}$. Thus, we have a contradiction that shows $d$ is weakly increasing.

**(2)** A similar proof by contradiction shows that $d$ is continuous in $\theta_2 \in \Theta$ over $(\theta_3, \theta^*) \subset \Theta$. If $d$ is not continuous over this interval, then there is a $\hat{\theta} \in (\theta_3, \theta^*)$ that is the first discontinuity in $d$. By construction $d$ is continuous when $\theta$ is such that $b_2(\theta, 0) < \theta_3$. Thus, $\lim_{\theta \to^+ \hat{\theta}} d(\theta, \theta_{-1,2}) > \theta_3$. Yet, $d$ is continuous in $\theta_2$ when $\theta_2 < \hat{\theta}$. Thus, when $\epsilon$ is sufficiently small, $d(\hat{\theta} - \epsilon, \theta_{-1,2}) \approx d(\hat{\theta} + \epsilon, \theta_{-1,2})$ because $d(d(\hat{\theta} - \epsilon, \theta_{-1,2}), \theta_{-1,2}) \approx d(d(\hat{\theta} + \epsilon, \theta_{-1,2}), \theta_{-1,2}) \leq d(\hat{\theta} - \epsilon, \theta_{-1,2})$. Since $b_2$ is continuous in both arguments, this implies that $d(\hat{\theta} + \epsilon, \theta_{-1,2}) \approx d(\hat{\theta} - \epsilon, \theta_{-1,2})$, which contradicts our assumption that $d$ is discontinuous at $\hat{\theta}$.

**(3)** I show that $\theta^* = \overline{\theta}$ by contradiction. Suppose that $\theta^* \in (\theta_3, \overline{\theta})$. Thus, for any $\epsilon > 0$ there exists a $\tilde{\theta} \in [\theta^*, \theta^* + \epsilon)$ such that $d(\tilde{\theta}, \theta_{-1,2})$ is not uniquely defined by Equation 7. Note that $\hat{\theta} - b_2(\tilde{\theta}, d(x, \theta_{-1,2}))$ is strictly increasing in $\hat{\theta}$ when $\hat{\theta} \in [\theta_3, \theta^*) \subset \Theta$.

If

$$\hat{\theta} - b_2(\tilde{\theta}, d(\hat{\theta}, \theta_{-1,2})) \geq 0 \textit{ when } \hat{\theta} = \theta_3,$$

then Equation 7 implies that $d(\hat{\theta}, \theta_{-1,2}) = \theta_3$, because

$$b_2(\tilde{\theta}, d(\hat{\theta}, \theta_{-1,2})) = b_2(\tilde{\theta}, \theta_3) < \theta_3.$$

Yet this contradicts our assumption that $\theta^* < \overline{\theta}$. Thus, it must be the case that

$$\hat{\theta} - b_2(\tilde{\theta}, d(\hat{\theta}, \theta_{-1,2})) < 0 \textit{ when } \hat{\theta} = \theta_3.$$

In addition

$$\hat{\theta} - b_2(\tilde{\theta}, d(\hat{\theta}, \theta_{-1,2})) > 0$$

when $\hat{\theta} = \theta^* - \epsilon$ where $\epsilon > 0$ is sufficiently small. This is because by construction $\tilde{\theta} - \theta^* < 2\epsilon$ and when $\epsilon$ is sufficiently small,

$$\tilde{\theta} \approx \theta^* > b_2(\tilde{\theta}, 0) > b_2(\tilde{\theta}, d(\hat{\theta}, \theta_{-1,2})).$$

Since $\hat{\theta} - b_2(\tilde{\theta}, d(x, \theta_{-1,2}))$ is strictly increasing and continuous in $\hat{\theta}$, then there exists a unique $\hat{\theta} \in (\theta_3, \theta^*)$ such that

$$\hat{\theta} - b_2(\tilde{\theta}, d(x^*, \theta_{-1,2})).$$

In addition, if we let $\hat{\theta} = d(\tilde{\theta}, \theta_{-1,2})$, then $d(\tilde{\theta}, \theta_{-1,2})$ satisfies Equation 7. □

**Proof of Theorem 2**

*Proof.* Because mechanism $\Gamma$ is symmetric, it is without loss of generality to assume that $\overline{\theta} \geq \theta_2 \geq \theta_3 \geq \theta_j \geq 0 \ \forall j \neq 1, 2, 3$ and I study the problem from the perspective of bidder 1. By construction, mechanism $\Gamma$ satisfies (1) IR and (2) no subsidies.

Next, I show that the mechanism is incentive compatible. If $(\theta_1, \ldots, \theta_N) \in \Theta^N$ are such that $d(\theta_{-1}) > \theta_1$, then $q_1(\theta_1, \theta_{-1}) = x_1(\theta_1, \theta_{-1}) = 0$. Bidder 1 does not have a profitable deviation in reporting her type because the price of one unit exceeds bidder 1's demand for her first unit $p_1(\theta_{-1}) > \theta_1$. Moreover, the price of the second unit exceeds the price of the first unit.

**Incentive compatible:** I consider two cases to prove incentive compatibility.

Case 1: If $(\theta_1, \ldots, \theta_N) \in \Theta^N$ are such that $\theta_1 > d(\theta_{-1})$ and $\theta_2 > d(\theta_{-2})$, then $q_1(\theta_1, \theta_{-1}) = 1$. Bidder 1 has no incentive to report a lower type that does not win any units because her willingness to pay for the first unit $\theta_1$ weakly exceeds the price she pays for the first unit $d(\theta_{-1}) = p_1(\theta_{-1})$. In addition, $\theta_2 > d(\theta_{-2})$ implies

$$p_2(\theta_{-1}) = \theta_2 > d(\theta_{-2}) \geq b_2(\theta_1, d(d(\theta_{-2}), \theta_{-1,2})) \geq b_2(\theta_1, d(\theta_2, \theta_{-1,2})) = b_2(\theta_1, p_1(\theta_{-1})),$$

where the first equality holds from the definition of $p_2$, the first inequality holds by assumption, the second inequality holds by the construction of $d$, the third inequality follows because $d$ is weakly increasing in the first argument and $b_2$ is decreasing in the second argument, and the final equality holds by the construction of $p_1$. Thus, we see that bidder 1's willingness to pay for her second unit is below the price she must pay to win a second unit. Thus, bidder 1 does not gain by over-reporting her type and winning an additional unit. Moreover, we can see that the mechanism satisfies feasibility because bidder 1 and bidder 2 each wins and demands exactly one unit under the mechanism's pricing rule. All other bidders win no units and demand no units.

Case 2: If $(\theta_1, \ldots, \theta_N) \in \Theta^N$ are such that $\theta_1 > d(\theta_{-1})$ and $d(\theta_{-2}) > \theta_2$, then $q_1(\theta_1, \theta_{-1}) = 2$. Bidder 1 has no incentive to report a lower type that does not win any units because her willingness to pay for the first unit $\theta_1$ weakly exceeds the price she pays for the first unit $d(\theta_{-1}) = p_1(\theta_{-1})$. Bidder 1 has no incentive to report a lower type that wins only one unit because $d(\theta_{-2}) > \theta_2$ implies that

$$b_2(\theta_1, p_1(\theta_{-1})) = b_2(\theta_1, d(\theta_2, \theta_{-1,2})) \geq b_2(\theta_1, d(d(\theta_{-2}), \theta_{-1,2}) = d(\theta_{-2}) > \theta_2 = p_2(\theta_{-1}),$$

where the first equality holds from the definition of $p_1$. The first inequality holds because $d$ is weakly increasing in the first argument and $b_2$ is decreasing in the second argument. The second equality holds because $d(\theta_{-2}) > \theta_2 \geq \theta_3 \implies d(\theta_{-2}) = b_1(\theta_1, d(d(\theta_{-2}), \theta_{-1,2}))$. Thus, bidder 1's conditional willingness to pay for her second unit exceeds the price of her second unit, and therefore, bidder 1 does not want to deviate and report a type that ensures that she only wins one unit. In addition, $q_j(\theta_1, \theta_{-1}) = 0 \; \forall j \neq 1$ if $\theta_1 \geq d(\theta_{-1})$ and $d(\theta_{-2}) > \theta_2$. This holds for bidder 2 by assumption. This holds for bidders $j \neq 1, 2$ because $d(\theta_{-j}) \geq \theta_3 \geq \theta_j$. Thus, the above construction specifies a mechanism that is feasible and incentive compatible.

**Efficiency:** I consider two cases to prove efficiency.

Case 1: Consider an outcome that is such that two bidders each win one unit. That is, $(\theta_1, \ldots, \theta_N) \in \Theta^N$ is such that $q_i(\theta_i, \theta_{-i}) \neq 2 \; \forall i \in \{1, \ldots, N\}$. Again, it is without loss of generality to assume the two bidders are bidders 1 and 2 and that $\theta_1, \theta_2 \geq \theta_3 \geq \theta_j \forall j \neq 1, 2, 3$. Because bidders 1 and 2 each win exactly one unit, we know that $\theta_1 \geq d(\theta_{-1})$ and $\theta_2 \geq d(\theta_{-2})$. There are no Pareto improving trades between a winning bidder (without loss of generality, bidder 1) and a losing bidder (without loss of generality, bidder 3) because

$$s_1(\theta_1, p_1(\theta_{-1})) \geq s_1(\theta_1, \theta_1) = \theta_1 \geq \theta_3,$$

where the first inequality holds because $\theta_1 \geq d(\theta_{-1}) = p_1(\theta_{-1})$ and a bidder's willingness to sell her first unit $s_1$ is decreasing in the second argument (her payment) by positive wealth effects. The first equality holds from the definition of $s_1$. Thus, there are no ex post Pareto improving trades between a winning bidder and a losing bidder because the winning bidder's willingness to sell exceeds the losing bidder's willingness to pay. There are no ex post Pareto improving trades where bidder 2 buys a unit from bidder 1 because

$$s_1(\theta_1, p_1(\theta_{-1})) \geq \theta_1 = p_2(\theta_{-2}) \geq b_2(\theta_2, p_1(\theta_{-2})),$$

where the first inequality was shown above, and the second inequality is because the mechanism is incentive compatible, and hence the price bidder 2 pays for her second unit exceeds her willingness to pay for her second unit when she wins one unit. Thus, bidder 1's willingness to sell a unit exceeds bidder 2's willingness to pay for a unit and there is no ex post Pareto improving trades where bidder 1 sells a unit to bidder 2. A symmetric argument shows there are no ex post Pareto improving trades where bidder 2 sells a unit to bidder 1. Thus, there are no ex post Pareto improving trades when two bidders each wins one unit.

Case 2: Consider an outcome where one bidder (without loss of generality, bidder 1) wins both units. That is, $(\theta_1, \ldots, \theta_N) \in \Theta^N$ is such that $q_1(\theta_1, \theta_{-1}) = 2$ I show that there are no ex post Pareto improving trades where bidder 1 sells a unit to a losing bidder. Incentive compatibility implies that bidder 1's conditional willingness to pay for her second unit exceeds the price of her second unit. If we continue to assume that $\overline{\theta} \geq \theta_2 \geq \theta_j \geq 0 \ \forall j \neq 1, 2$ this implies that

$$b_2(\theta_1, p_1(\theta_{-1})) \geq p_2(\theta_{-1}) = \theta_2.$$

From the above expression, we then have that

$$s_2(\theta_1, p_1(\theta_{-1}) + p_2(\theta_{-1})) \geq s_2(\theta_1, p_1(\theta_{-1}) + b_2(\theta_1, p_1(\theta_{-1}))) = b_2(\theta_1, p_1(\theta_{-1})) \geq \theta_2 \geq \theta_j \ \forall j \neq 1, 2,$$

where the first inequality holds because positive wealth effects imply that $s_2$ is decreasing in the second argument. The first equality holds from the definition of $s_2$ and $b_2$. The second inequality holds from incentive compatibility. Thus, there are no ex post Pareto improving trades between bidder 1 and bidder $j \neq 1$ because bidder 1's willingness to sell her second unit exceeds any of her rival's willingness to pay for a single unit. □

## Proof of Proposition 1

Without loss of generality, I construct the proof by placing necessary restrictions on the assignment rule of bidder 1 when her rivals have types $\theta_{-1} \in \Theta^{N-1}$ where $\theta_{-1}$ is such that $\overline{\theta} \geq \theta_2 \geq \theta_3 \geq \theta_j \geq 0 \ \forall j \neq 1, 2, 3$. The proof of Proposition 1 is by contradiction. I assume that there exists a mechanism that satisfies Properties (1)-(5), and then obtain a contradiction. I obtain Lemmas A1-A3 under this assumption. I then use these three lemmas to draw a contradiction.

Lemma A1 shows that bidder 1 wins a unit only if her demand is among the two highest demands reported.

**Lemma. A1.** *Bidder 1's first unit cut-off rule is such that* $d_{1,1}(\theta_{-1}) \geq \theta_3$.

*Proof.* I show that $d_{1,1}(\theta_{-1}) \geq \theta_3$. The proof is by contradiction. Suppose that there exist $\theta_2, \theta_3$ such that $\theta_3 > d_{1,1}(\theta_{-1})$. This implies that if $\theta_1 = d_{1,1}(\theta_{-1}) + \epsilon$, then $q_1(\theta_1, \theta_{-1}) \geq 1$ and if $\theta_1 = d_{1,1}(\theta_{-1}) - \epsilon$, then $q_1(\theta_1, \theta_{-1}) = 0$. Thus, as $\theta_1$ approaches $d_{1,1}(\theta_{-1})$ from above, bidder 1 is willing to sell one of her units for at most $\theta_1$ (if bidder 1 wins 2 units when $\theta_1 > d_{1,1}(\theta_{-1})$, then her willingness to sell an additional unit is lower). Thus, there is a Pareto improving trade where bidder 1 sells one unit to bidder 3 for a price in the interval $(\theta_1, \theta_3)$. □

**Lemma. A2.** *If* $\theta_3 \geq b_2(\theta_2, 0)$, *then bidder 1's first unit cut-off rule is* $d_{1,1}(\theta_{-1}) = \theta_3$.

*Proof.* The proof is by contradiction. Suppose $d_{1,1}(\theta_{-1}) > \theta_3$. Let $\theta_1$ be such that $\theta_1 \in (\theta_3, d_{1,1}(\theta_{-1}))$. Then $q_2(\theta_1, \theta_2, \theta_3, \theta_{-1,2,3}) = 2$. This holds because (1) $q_1(\theta_1, \theta_{-1}) = 0$ since $d_{1,1}(\theta_{-1}) > \theta_1$ and (2) $q_i(\theta_i, \theta_{-i}) = 0 \forall i \neq 1, 2$ because Lemma A1 shows $d_{i,1}(\theta_{-i}) \geq \min\{\theta_1, \theta_2\}$ and $\min\{\theta_1, \theta_2\} > \theta_i$. Thus,

$$\theta_2 \geq d_{2,2}(\theta_{-2}) \geq d_{2,1}(\theta_{-2}) \geq \theta_3.$$

Let $\tilde{\theta}_2 = d_{2,2}(\theta_{-2}) + \epsilon$, and $\hat{\theta}_2 = d_{2,2}(\theta_{-2}) - \epsilon$. Incentive compatibility and continuity of bidder 2's preferences imply that when $\epsilon > 0$ is sufficiently small,

$$s_2(\tilde{\theta}_2, x_2(\theta_1, \theta_3)) \approx b_2(\hat{\theta}_2, d_{2,1}(\theta_1, \theta_3)) \leq b_2(\theta_2, 0) < \theta_1.$$

Thus, $s_2(\tilde{\theta}_2, x_2(\theta_1, \theta_3) < \theta_1$. This implies that there is a Pareto improving trade when bidder 2 is type $\tilde{\theta}_2$. Namely, bidder 2 sells one unit to bidder 1 for a price in the interval $\left(s_2\left(\tilde{\theta}_2, x_2(\theta_2, \theta_{-2})\right), \theta_1\right)$. □

**Lemma. A3.** *If $\theta_2, \theta_3 \in \Theta$ are such that $b_2(\theta_2, 0) < \theta_3 < \theta_2$, it follows that*

$$d_{1,2}(\theta_{-1}) = \theta_1^*$$

*where $\theta_1^* \in \Theta$ is defined as solving*

$$b_2(\theta_1^*, \theta_3) = \theta_2.$$

*Proof.* The proof is by contradiction. Suppose that there exist a mechanism $\Gamma$ satisfying Properties (1)-(5) and $\theta_2, \theta_3 \in \Theta$ with $b_2(\theta_2, 0) < \theta_3 < \theta_2$, and $d_{1,2}(\theta_{-1}) \neq \theta_1^*$. I separate the proof into two cases.

**Case 1:** Suppose that $d_{1,2}(\theta_{-1}) > \theta_1^*$. Then

$$q_1(\theta_1, \theta_{-1}) = 1 \; if \; \theta_1 \in (\theta_1^*, d_{1,2}(\theta_{-1})),$$

because $\theta_1 > \theta_1^* > \theta_2 > \theta_3 = d_{1,1}(\theta_{-1})$ where the final equality holds because Lemma A2 shows that $\theta_3 = d_{1,1}(\theta_{-1})$ if $\theta_3 \in (b_2(\theta_2, 0), \theta_2)$. In addition, $q_2(\theta_1, \theta_2, \theta_3, \theta_{-1,2,3}) = 1$ because both units are sold and bidder $i \neq 1, 2$ wins zero units when her type is not among the two highest types reported. Thus, $\theta_2 \geq d_{2,1}(\theta_{-2})$.

Let $\tilde{\theta}_2 = \min\{\theta_2, d_{2,1}(\theta_{-2}) + \epsilon\}$ where $\epsilon > 0$ is small. Note that $\tilde{\theta}_2 > \theta_3$ because $d_{2,1}(\theta_{-1}) \geq \theta_3$ and $\theta_2 > \theta_3$. Thus, $\theta_2 \geq \tilde{\theta}_2 > \theta_3 \implies \theta_3 \in (b_2(\tilde{\theta}_2, 0), \tilde{\theta}_2)$, which follows because I assume $\theta_3 \in (b_2(\theta_2, 0), \theta_2)$. Thus, Lemma A2 shows $d_{1,1}(\tilde{\theta}_2, \theta_{-1,2}) = \theta_3$, and bidder 1 is willing to pay

$b_2(\theta_1, \theta_3)$ for an additional unit. Note that

$$b_2(\theta_1, \theta_3) > b_2(\theta_1^*, \theta_3) = \theta_2 \geq \tilde{\theta}_2,$$

where the first inequality holds because $\theta_1 > \theta_1^*$ and the equality holds from the definition of $\theta_1^*$.

In addition, since $\tilde{\theta}_2 - 2\epsilon < d_{2,1}(\theta_{-2}) \leq \tilde{\theta}_2$, where $\epsilon > 0$ is arbitrarily small, incentive compatibility implies that bidder 2's willingness to sell her first unit is such that $s_1(\tilde{\theta}_2, d_{2,1}(\theta_{-2})) \approx d_{2,1}(\theta_{-2})$ because $\tilde{\theta}_2 \approx d_{2,1}(\theta_{-2})$. Yet $q_2(\tilde{\theta}_2, \theta_{-2}) = 1$ by construction. Thus, there is a Pareto improving trade where bidder 1 buys the unit from bidder 2 for a price in the interval $(\tilde{\theta}_2, b_2(\theta_1, \theta_3))$. Thus, if $d_{1,2}(\theta_{-1}) > \theta_1^*$, there exists a Pareto improving trade and the mechanism does not satisfy Properties (1)-(5).

**Case 2:** Suppose that $\theta_1^* > d_{1,2}(\theta_{-1})$. Then, $d_{1,2}(\theta_{-1}) \geq d_{1,1}(\theta_{-1}) = \theta_3$, where the final inequality holds by Lemma A2. Let $\tilde{\theta}_1 = d_{1,2}(\theta_{-1}) + \epsilon$, where $\epsilon > 0$ is sufficiently small. Thus, $q_1(\tilde{\theta}_1, \theta_{-1}) = 2$. Incentive compatibility implies that bidder 1 is approximately indifferent between buying her second unit when her type is near $d_{1,2}(\theta_{-1})$. Thus, bidder 1 is willing to sell her second unit for approximately $b_2(\tilde{\theta}_1, \theta_3)$ (this follows from Remark A2). In addition, bidder 2 is willing to pay $\theta_2$ for her first unit and $\theta_2 > b_2(\tilde{\theta}_1, \theta_3)$ because I assumed that $\theta_1^* > \tilde{\theta}_1 \approx d_{1,2}(\theta_{-1})$. Thus, there is a Pareto improving trade where bidder 1 sells her second unit to bidder 2 for a price in the interval $\Big(b_2(\tilde{\theta}_1, \theta_3), \theta_2\Big)$. $\square$

To complete the proof Proposition 1, note that strong monotonicity implies that $q_i(\theta_i, \theta_{-i})$ is weakly decreasing in $\theta_{-i}$ $\forall \theta_{-i} \in \Theta^{N-1}$. Thus, $d_{i,k}(\theta_{-i})$ is weakly increasing in $\theta_{-i}$ $\forall \theta_{-i} \in \Theta,\ i \in \{1, \ldots, N\},\ k \in \{1, 2\}$. Suppose that $(\theta_1, \ldots, \theta_N) \in \Theta^N$ is such that $\overline{\theta} > \theta_1 > \theta_2 > \theta_3 > \theta_j$ $\forall j \neq 1, 2, 3$. In addition, suppose that, $\theta_1, \theta_3$ are such that

$$b_2(\theta_1, \theta_3) \in (\theta_3, \theta_3 + \epsilon),$$

where $\epsilon > 0$ is sufficiently small. I show that this implies that $d_{2,1}(\theta_{-2}) = b_2(\theta_1, \theta_3)$. To show $d_{2,1}(\theta_{-2}) = b_2(\theta_1, \theta_3)$ note that if $\theta_2 \in (\theta_3, b_2(\theta_1, \theta_3))$ then $\theta_1 > \theta_1^*$ where $\theta_1^*$ is such that

$$\theta_2 = b_1(\theta_1^*, \theta_3).$$

Thus Lemma A3 implies that $q_1(\theta_1, \theta_{-1}) = 2 \implies q_2(\theta_2, \theta_{-2}) = 0$. If $\theta_2 \in (b_2(\theta_1, \theta_3), \theta_3 + \epsilon)$, then $\theta_1 < \theta_1^*$ and Lemma A3 implies that $q_1(\theta_1, \theta_2, \theta_3, \theta_{-1,2,3}) \leq 1$. In addition, $q_j(\theta_j, \theta_{-j}) = 0$ for all $j = 3, \ldots, N$ by Lemma A1, because $\theta_1, \theta_2 > \theta_j$. Since $\sum_{i=1}^{N} q_i(\theta_i, \theta_{-i}) = 2$, then $q_2(\theta_2, \theta_{-2}) \geq 1$. Since $q_2(\cdot, \theta_{-2})$ is weakly increasing $\forall \theta_{-2} \in \Theta^{N-1}$ by incentive compatibility, I then have that $d_{2,1}(\theta_{-2}) = b_2(\theta_1, \theta_3)$.

Now suppose bidder 3 increases her report to $\theta_3'$ where $\theta_3' > \theta_3$ is such that

$$b_2(\theta_1, \theta_3') \in (\theta_3', \theta_3' + \epsilon).$$

Again, the same argument shows that $d_{2,1}(\theta_{-2}') = b_2(\theta_1, \theta_3')$ where $\theta_{-2}' = (\theta_1, \theta_3', \ldots, \theta_N) \in \Theta^{N-1}$. In addition, $d_{2,1}(\theta_{-2}') = b_2(\theta_1, \theta_3') < b_2(\theta_1, \theta_3) = d_{2,1}(\theta_{-2})$ because $\theta_3' > \theta_3$ and bidders have strictly positive wealth effects. Yet $\theta_{-2} \leq \theta_{-2}'$ in the coordinate-wise sense. This contradicts with strong monotonicity because strong monotonicity implies that $d_{2,1}(\theta_{-2})$ is weakly increasing in $\theta_{-i}$. Thus, there is no mechanism $\Gamma$ that satisfies Properties (1)-(5).

**Proof of Lemma 4**

*Proof.* Individual rationality implies that if $\gamma_i, \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$ are such that $q_i(\gamma_i, \gamma_j) = 0$, then $x_i(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) \leq 0$. When $\gamma_i, \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$ are such that $q_i(\gamma_i, \gamma_j) = 0$, individual rationality implies that

$$u(2, -x_j(\gamma_i, \gamma_j), \gamma_j) \geq u(0, 0, \gamma_j).$$

The above expression gives us that

$$x_j(\gamma_i, \gamma_j) \leq \theta_j + b_2(\gamma_j, \theta_j) < 2\theta_j \ \forall \gamma_i, \gamma_j \in \Theta \times \{\mathfrak{s}, f\} \ s.t. \ q_i(\gamma_i, \gamma_j) = 0. \tag{8}$$

The first inequality in Equation 8 holds because $q_i(\gamma_i, \gamma_j) = 0 \implies q_j(\gamma_i, \gamma_j) \leq 2$, and hence individual rationality gives us that

$$u(0, 0, \gamma_j) = u(1, -\theta_j, \gamma_j) = u(2, -\theta_j - b_2(\gamma_j, \theta_j), \gamma_j) \leq u(2, -x_j(\gamma_i, \gamma_j), \gamma_j).$$

The second inequality in Equation 8 holds because of declining demand and positive wealth effects.

If $\gamma_i = (0, t_i) \in \Theta \times \{\mathfrak{s}, f\}$ and $\gamma_j = (\theta_j, t_j) \in \Theta \times \{\mathfrak{s}, f\}$ is such that $\theta_j > 0$, then efficiency requires that $q_j(\gamma_i, \gamma_j) = 2$. In addition, the Equation 8 shows that

$$\gamma_i = (0, t_i) \in \Theta \times \{\mathfrak{s}, f\} \implies x_j(\gamma_i, \gamma_j) = p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i) < 2\theta_j,$$

for all $\gamma_j \in \Theta \times \{\mathfrak{s}, f\}$ $s.t.$ $\theta_j > 0$. Since the above expression must hold for arbitrarily small $\theta_j > 0$, we have that

$$p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i) \leq \lim_{\theta_j \to^+ 0} 2\theta_j = 0.$$

Thus, if $\gamma_i = (0, t_i)$ and $\gamma_j = (\theta_j, t_j)$ where $\theta_j > 0$, then weak budget balance implies

$$x_i(\gamma_i, \gamma_j) + x_j(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) + (p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i)) \geq 0.$$

However I have already shown that $(p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i)) \leq 0$ and $p_{i,0}(\gamma_j) \leq 0$. Thus,

$$p_{i,0}(\gamma_j) + (p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i)) \geq 0 \implies p_{i,0}(\gamma_j) = 0 \ if \ \theta_j > 0.$$

Thus, the price bidder $i$ pays to win no units is zero because $p_{i,0}(\gamma_j) = 0 \forall \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$ *s.t.* $\theta_j > 0$. We combine this with the taxation principle to get the result. If $\gamma_i, \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$ and $\gamma_i$ and $\gamma_j$ are such that $q_i(\gamma_i, \gamma_j) = 0$, then the taxation principle implies $x_i(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) = 0$, where the final equality follows by the above argument. $\square$

**Proof of Proposition 2**

When I prove the first two bullet points of Proposition 2, I proceed with an abuse of notation by dropping $t_i$ and $t_j$ from the description of bidder types. I study the incentives that bidders have to truthfully report their steepness, given that mechanism $\Gamma$ provides the bidders with an incentive to truthfully report their steepness type. Thus, I fix $t_i, t_j \in \{\mathfrak{s}, f\}$ and suppose that a bidder truthfully reports her steepness type. I then find necessary conditions on mechanism $\Gamma$ that ensure that a bidder truthfully reports her intercept type under the assumption that she truthfully reports her steepness type. Thus, for simplicity, when I prove the first two bullet points of Proposition 2, the domain of bidder $i$'s assignment rule $q_i$ is $\Theta^2$, because I only study bidder incentives to report their intercept type. Thus $q_i(\theta_i, \theta_j)$ is bidder $i$'s assignment in mechanism $\Gamma$ that satisfies Properties (1)-(4) when we take as given that bidders $i$ and $j$ truthfully reported their steepness type. I similarly write the cut-off rules $d_{i,1}^{t_i}$ and $d_{i,2}^{t_i}$ as $d_{i,1}$ and $d_{i,2}$ to condense notation. Remark A1 below gives necessary conditions that a mechanism $\Gamma$ must satisfy if $\Gamma$ satisfies Properties (1)-(4).

*Remark.* A1. Suppose that mechanism $\Gamma$ satisfies Properties (1)-(4). Then,

$$q_i(\theta_i, \theta_j) = 2 \iff s_2(\theta_i, x_i(\theta_i, \theta_j)) \geq \theta_j,$$

and

$$q_i(\theta_i, \theta_j) = 1 \iff b_2(\theta_i, x_i(\theta_i, \theta_j)) \leq s_1(\theta_j, x_j(\theta_i, \theta_j)).$$

If $d_{i,2}(\theta_j) > d_{i,1}(\theta_j)$, then

$$p_{i,1}(\theta_j) = d_{i,1}(\theta_j)$$

and

$$d_{i,1}(\theta_j) = \lim_{\theta_i \to^- d_{i,1}(\theta_j)} s_1(\theta_i, p_{i,1}(\theta_j)).$$

The first two statements in Remark A1 are direct implications of Lemma 4. Lemma 4 implies that if bidder $j$ wins zero units, she makes zero payment. Thus, bidder $j$ is willing to pay $\theta_j \in \Theta$ for an additional unit if she wins zero units in mechanism $\Gamma$ if mechanism $\Gamma$ satisfies Properties (1)-(4). Efficiency then implies that bidder $i$ willingness to sell her final unit must weakly exceed bidder $j$'s willingness to pay for her first unit. Similarly, if bidder $i$ wins exactly one unit in mechanism $\Gamma$, then efficiency requires that the outcome of mechanism $\Gamma$ is such that bidder $i$'s conditional willingness to pay for an additional (second) unit is below her rival's willingness to sell her first unit.

The latter two statements in Remark A1 hold if the $\theta_j \in \Theta$ is such that mechanism $\Gamma$ has a cut-off rule where bidder $i$ has a strictly higher cut-off for her second unit than she does for her first unit $d_{i,2}(\theta_j) > d_{i,1}(\theta_j)$. We will later prove that this is a necessary condition for mechanism $\Gamma$'s assignment rule if and only if $\theta_j > 0$. Here I show that if this condition holds, then the price bidder $i$ pays for her first unit is her intercept cut-off. This is a direct consequence of incentive compatibility. The final point states that incentive compatibility implies that bidder $i$ is indifferent between buying and selling her first unit of the good when her intercept type equals her first unit cut-off type.

Lemma A4 below proves that if mechanism $\Gamma$ is such that bidder $i$'s intercept type approximately equals her second unit cut-off type given her rival's intercept $\theta_j \in \Theta$, then bidder $i$'s willingness to sell her second unit if she wins both units (and pays for both units) equals her willingness to pay for her second unit conditional on buying a single unit for price $d_{i,1}(\theta_j)$. If this condition does not hold, then mechanism $\Gamma$ violates incentive compatibility. This is because incentive compatibility implies that bidder $i$ is approximately indifferent between buying or selling her second unit for a price of $b_2(d_{i,2}(\theta_j), d_{i,1}(\theta_j))$ when her intercept type is approximately $d_{i,2}(\theta_j)$ and bidder $i$ paid $d_{i,1}(\theta_j)$ to win her first unit. If this indifference does not hold, then bidder $i$ has a strict incentive to misreport her intercept type when her intercept type is near $d_{i,2}(\theta_j)$.

**Lemma. A4.** *Suppose that mechanism $\Gamma$ satisfies properties (1)-(4). Then,*

$$\lim_{\theta_i \to^+ d_{i,2}(\theta_j)} s_2(\theta_i, x_i(\theta_i, \theta_j)) = \lim_{\theta_i \to^- d_{i,2}(\theta_j)} b_2(\theta_i, d_{i,1}(\theta_j)),$$

*for all $\theta_j \in \Theta$.*

*Proof.* Fix $\theta_j \in \Theta$. Let $\theta_1^* := d_{i,1}(\theta_j) \in \Theta$, $\theta_2^* := d_{i,2}(\theta_j) \in \Theta$. Thus, $\theta_2^* \geq \theta_1^*$. Similarly, let $x_1^*, x_2^* \in \mathbb{R}_+$ be such that $x_1^* := x_i(\theta_i, \theta_j)$ $\forall \theta_i$ *s.t.* $q_i(\theta_i, \theta_j) = 1$ and $x_2^* := x_i(\theta_i, \theta_j) \in \mathbb{R}_+$

$\forall \theta_i \in \Theta$ *s.t.* $q_i(\theta_i, \theta_j) = 2$. Thus $x_2^* \geq x_1^*$.

I consider two cases. First suppose that mechanism $\Gamma$ is such that $\theta_2^* = \theta_1^*$. Then bidder $i$ gets utility $u(2, -x_2^*, \theta_i)$ in mechanism $\Gamma$ if her intercept type $\theta_i \in \Theta$ is such that $\theta_i > \theta_2^*$. If bidder $i$'s intercept type is instead such that $\theta_2^* > \theta_i$, then bidder $i$ has utility $u(0, 0, \theta_i)$. Incentive compatibility implies that bidder $i$ utility is continuous and increasing in her type $\theta_i$ because $u$ is continuous in the third argument. Thus,

$$\theta_2^* = \theta_1^* \implies u(2, -x_2^*, \theta_2^*) = \lim_{\theta_i \to^+ \theta_2^*} u(2, -x_2^*, \theta_i) = \lim_{\theta_i \to^- \theta_2^*} u(0, 0, \theta_i) = u(0, 0, \theta_2^*).$$

Furthermore, $u(2, -x_2^*, \theta_2^*) = u(0, 0, \theta_2^*)$ implies that $x_2^* = \theta_2^* + b_2(\theta_2^*, \theta_2^*)$ because

$$u(0, 0, \theta_2^*) = u(1, -\theta_1^*, \theta_2^*) = u(2, -\left(\theta_1^* + b_2(\theta_2^*, \theta_1^*)\right), \theta_2^*).$$

Thus,

$$\theta_2^* = \theta_1^* \implies \lim_{\theta_i \to^+ d_{i,2}(\theta_j)} s_2(\theta_i, -x_i(\theta_i, \theta_j)) = s_2(\theta_i^*, -x_i^*) = b_2(\theta_2^*, \theta_1^*) = \lim_{\theta_i \to^- d_{i,2}(\theta_j)} b_2(\theta_i, d_{i,1}(\theta_j)).$$

For the second case, suppose that mechanism $\Gamma$ is such that $\theta_2^* > \theta_1^*$. Recall, incentive compatibility implies that a bidder's utility is continuous in her type,

$$u(2, -x_2^*, \theta_2^*) = \lim_{\theta_i \to^+ d_{i,2}(\theta_j)} u(2, -x_i(\theta_i, \theta_j), \theta_i) = \lim_{\theta_i \to^- d_{i,2}(\theta_j)} u(1, -x_i(\theta_i, \theta_j), \theta_i) = u(1, -x_1^*, \theta_2^*).$$

Thus,

$$u(2, -x_2^*, \theta_2^*) = u(1, -x_1^*, \theta_2^*) \implies s_2(\theta_2^*, x_2^*) = x_2^* - x_1^* = b_2(\theta_2^*, x_1^*).$$

□

Remark A2 follows from combining the implications of Remark A1 and Lemma A4.

*Remark.* A2. Suppose that mechanism $\Gamma$ satisfies Properties (1)-(4). Remark A1 and Lemma A4 show

$$b_2(d_{i,2}(\theta_j), d_{i,1}(\theta_j)) = \lim_{\theta_i \to^- d_{i,2}(\theta_j)} b_2(\theta_i, d_{i,1}(\theta_j)) = \lim_{\theta_i \to^+ d_{i,2}(\theta_j)} s_2(\theta_i, x_i(\theta_i, \theta_j)) \geq \theta_j.$$

Lemma A5 proves the first bullet point in Proposition 2.

**Lemma. A5.** *Suppose that mechanism* $\Gamma$ *satisfies Properties (1)-(4). Then,* $d_{i,2}(\theta_j) > d_{i,1}(\theta_j)$ *for all* $\theta_j > 0$.

*Proof.* The proof of Lemma A5 is by contradiction. Suppose that there is a mechanism $\Gamma$

such that $d_{i,2}(\theta_j') = d_{i,1}(\theta_j')$ for some $\overline{\theta} \geq \theta_j' > 0$. Let $\tilde{\theta}_i := d_{i,2}(\theta_j') = d_{i,1}(\theta_j')$. Then,

$$q_i(\theta_i, \theta_j') = \begin{cases} 2 & if\ \theta_i > \tilde{\theta}_i \\ 0 & if\ \theta_i < \tilde{\theta}_i \end{cases}.$$

Thus, Remark A2 implies

$$b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \theta_j'\ \forall \theta_j'\ s.t.\ d_{i,1}(\theta_j') = d_{i,2}(\theta_j') = \tilde{\theta}_i. \tag{9}$$

Let $\theta_j^* \in (0, \overline{\theta})$ be such that $\theta_j^* := \inf\{\theta_j : d_{i,1}(\theta_j) = d_{i,2}(\theta_j) = \tilde{\theta}_i\}$. Then, $d_{i,1}(\theta_j) < \tilde{\theta}_i$ for all $\theta_j < \theta_j^*$, because $d_{i,1}$ and $d_{i,2}$ are weakly increasing in $\theta_j$ $\forall \theta_j \in \Theta$. Thus, for any $\epsilon > 0$, the construction of $\theta_i^*$ implies that if $\theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i)$ then,

$$q_i(\theta_i, \theta_j) = \begin{cases} \geq 1 & if\ \theta_j \leq \theta_j^* - \epsilon \\ 0 & if\ \theta_j > \theta_j^* \end{cases}.$$

Thus,

$$q_j(\theta_i, \theta_j) = 2 - q_i(\theta_i, \theta_j) = \begin{cases} \leq 1 & if\ \theta_j < \theta_j^* - \epsilon \\ 2 & if\ \theta_j > \theta_j^* \end{cases}.$$

Hence we get that $d_{j,2}(\theta_i) \in [\theta_j^* - \epsilon, \theta_j^*]$ if $\theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i)$. Then, Remark A2 implies that

$$b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \theta_i.$$

Recall that Equation 9 implies that

$$b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \theta_j' \geq \theta_j^*\ \forall \theta_j'\ s.t.\ d_{i,1}(\theta_j') = d_{i,2}(\theta_j'),$$

where the final inequality follows from the definition of $\theta_j^*$. Combining the above two expressions gives

$$b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \theta_j^* \geq d_{j,2}(\theta_i) \geq b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \theta_i\ \forall \theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i),$$

where the second inequality holds because $d_{j,2}(\theta_i) \in [\theta_j^* - \epsilon, \theta_j^*]$ if $\theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i)$, and the third inequality holds because of declining demand. Thus,

$$b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \theta_i\ \forall \theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i) \implies b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \tilde{\theta}_i.$$

Yet $b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \tilde{\theta}_i$, and this contradicts our declining demand assumption. Thus we have shown that if $\Gamma$ satisfies Properties (1)-(4), then $d_{i,2}(\theta_j) > d_{i,1}(\theta_j)$ $\forall \theta_j \in (0, \overline{\theta}]$. $\square$

Lemma A6 below uses a nearly identical proof to the one above to show the second assertion in Proposition 2.

**Lemma. A6.** *Suppose that mechanism* $\Gamma$ *satisfies Properties (1)-(4). Then* $d_{i,1}(\theta_j)$ *is continuous and strictly increasing in* $\theta_j$ $\forall \theta_j \in \Theta$.

*Proof.* **Continuity proof:** The proof is by contradiction. Incentive compatibility implies that $d_{i,1}(\theta_j)$ is weakly increasing. Thus, if $d_{i,1}(\theta_j)$ is discontinuous, then there exists a $\theta_j^* > 0$ such that

$$\lim_{\theta_j \to^- \theta_j^*} d_{i,1}(\theta_j) < \lim_{\theta_j \to^+ \theta_j^*} d_{i,1}(\theta_j).$$

Let $\theta_i^\ell, \theta_i^h \in \Theta$ be such that $\theta_i^\ell := \lim_{\theta_j \to^- \theta_j^*} d_{i,1}(\theta_j)$ and $\theta_i^h := \lim_{\theta_j \to^+ \theta_j^*} d_{i,1}(\theta_j)$. Thus, $\theta_i \in (\theta_i^\ell, \theta_i^h)$ implies that

$$q_j(\theta_i, \theta_j) = 2 - q_i(\theta_i, \theta_j) = \begin{cases} \leq 1 & if\ \theta_j < \theta_j^* \\ 2 & if\ \theta_j > \theta_j^* \end{cases}.$$

Therefore, $d_{j,2}(\theta_i) = \theta_j^*$ $\forall \theta_i \in (\theta_i^\ell, \theta_i^h)$, and Remark A2 shows

$$b_2(\theta_j^*, d_{j,1}(\theta_i)) \geq \theta_i \ \forall \theta_i \in (\theta_i^\ell, \theta_i^h) \implies \lim_{\theta_i \to^- \theta_i^h} b_2(\theta_j^*, d_{j,1}(\theta_i)) \geq \theta_i^h. \tag{10}$$

Similarly, Lemmas A4 and Remark A1 show that,

$$\lim_{\theta_i \to^+ d_{i,1}(\theta_j)} b_2(\theta_j, d_{j,1}(\theta_i)) \leq \lim_{\theta_i \to^+ d_{i,1}(\theta_j)} s_1(\theta_i, d_{i,1}(\theta_j)) = d_{i,1}(\theta_j) \leq \lim_{\theta_j \to^- \theta_j^*} d_{i,1}(\theta_j) = \theta_i^\ell \ \forall \theta_j < \theta_j^*, \tag{11}$$

where the final inequality follows because $d_{i,1}(\theta_j)$ is weakly increasing. Thus, positive wealth effects imply that

$$b_2(\theta_j, d_{j,1}(\theta_i^\ell)) \leq \lim_{\theta_i \to^+ d_{i,1}(\theta_j)} b_2(\theta_j, d_{j,1}(\theta_i)) \leq \theta_i^\ell \ \forall \theta_j < \theta_j^*, \tag{12}$$

where the final inequality from Equation 11. Combining Equations 10 and 11 gives

$$\lim_{\theta_i \to^- \theta_i^h} b_2(\theta_j^*, d_{j,1}(\theta_i)) \geq \theta_i^h > \theta_i^\ell \geq b_2(\theta_j^*, d_{j,1}(\theta_i^\ell)).$$

This yields a contradiction because $\lim_{\theta_i \to^- \theta_i^h} d_{j,1}(\theta_i) \geq d_{j,1}(\theta_i^\ell)$ and positive wealth effects imply

$$b_2(\theta_j^*, d_{j,1}(\theta_i^\ell)) \geq \lim_{\theta_i \to^- \theta_i^h} b_2(\theta_j^*, d_{j,1}(\theta_i)).$$

Thus, if $\Gamma$ satisfies Properties (1)-(4) then $d_{i,1}(\theta_j)$ is continuous in $\theta_j$ $\forall \theta_j \in \Theta$.

**Strictly increasing:** I prove that $d_{i,1}(\theta_j)$ is strictly increasing in $\theta_j$ $\forall \theta_j \in \Theta$. Again, the proof is by contradiction. Incentive compatibility requires that $d_{i,1}(\theta_j)$ is weakly increasing $\theta_j$ $\forall \theta_j \in \Theta$. If $d_{i,1}(\theta_j)$ is not strictly increasing, there exists an interval $(\theta_j^\ell, \theta_j^h)$ such that $d_{i,1}(\theta_j') = d_{i,1}(\theta_j'')$ $\forall \theta_j', \theta_j'' \in (\theta_j^\ell, \theta_j^h)$. Let $\tilde{\theta}_i := d_{i,1}(\theta_j)$ $\forall \theta_j \in (\theta_j^\ell, \theta_j^h)$, $\theta_j^\ell := \inf\{\theta_j : \tilde{\theta}_i = d_{i,1}(\theta_j)\}$, and $\theta_j^h := \sup\{\theta_j : \tilde{\theta}_i = d_{i,1}(\theta_j)\}$. If $\theta_j \in (\theta_j^\ell, \theta_j^h)$

$$\tilde{\theta}_i = d_{i,1}(\theta_j) = \lim_{\theta_i \to^+ d_{i,1}(\theta_j)} s_1(\theta_i, d_{i,1}(\theta_j)) \geq \lim_{\theta_i \to^+ d_{i,1}(\theta_j)} b_2(\theta_j, d_{j,1}(\theta_i)) = b_2(\theta_j, d_{j,1}(\tilde{\theta}_i)),$$

where the second equality and the inequality holds from Remark A1, and final equality holds because we showed that $d_{j,1}$ is continuous. Using the above expression we see that

$$\tilde{\theta}_i \geq b_2(\theta_j, d_{j,1}(\tilde{\theta}_i)) \ \forall \theta_j \in (\theta_j^\ell, \theta_j^h) \implies \tilde{\theta}_i \geq b_2(\theta_j^h, d_{j,1}(\tilde{\theta}_i)). \tag{13}$$

In addition, if $\theta_j > \theta_j^\ell$, then $d_{i,1}(\theta_j) \geq \tilde{\theta}$. Thus, if $\theta_j > \theta_j^\ell$ and $\theta_i < \tilde{\theta}_i$, then $q_i(\theta_i, \theta_j) = 0 \implies q_j(\theta_i, \theta_j) = 2$. Thus, if $\theta_i < \tilde{\theta}_i$, then $d_{j,2}(\theta_i) \leq \theta_j^\ell$ and Remark A2 implies

$$b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \theta_i \ \forall \theta_i < \tilde{\theta}_i \implies \lim_{\theta_i \to^+ \tilde{\theta}_i} b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \lim_{\theta_i \to^+ \tilde{\theta}_i} \theta_i.$$

Recall that $d_{j,1}(\theta_i)$ is continuous and $d_{j,2}(\theta_i) \leq \theta_j^\ell$ $\forall \theta_i < \tilde{\theta}_i$. As such,

$$\lim_{\theta_i \to^+ \tilde{\theta}_i} b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \lim_{\theta_i \to^+ \tilde{\theta}_i} \theta_i \implies b_2(\theta_j^\ell, d_{j,1}(\tilde{\theta}_i)) \geq \tilde{\theta}_i.$$

I combine this with Equation 13 to show that

$$b_2(\theta_j^\ell, d_{j,1}(\tilde{\theta}_i)) \geq \tilde{\theta}_i \geq b_2(\theta_j^h, d_{j,1}(\tilde{\theta}_i)) \implies \theta_j^\ell \geq \theta_j^h.$$

However, this contradicts the fact that $\theta_j^h > \theta_j^\ell$. Thus, if $\Gamma$ satisfies Properties (1)-(4) then $d_{i,1}(\theta_j)$ is strictly increasing in $\theta_j$ $\forall \theta_j \in \Theta$. □

The first two implications of Proposition 2 relate to a bidder's incentive to truthfully report the intercept dimension of her type, given that the bidder truthfully reports her steepness. The final two points of Proposition 2 relate to a bidder's incentive to truthfully report her steepness dimension in a mechanism $\Gamma$ that satisfies Properties (1)-(4). Now that I study a bidder's incentive to report the second dimension of her type, I again write bidder

$i$'s multi-dimensional type as $\gamma_i$ where $\gamma_i \in \Theta \times \{\mathfrak{s}, f\}$.

The final two implications of Proposition 2 follow as Corollaries of the first two implications proven above.

**Corollary. A1.** *If $\Gamma$ satisfies Properties (1)-(4), then*

$$d^f_{i,1}(\gamma_j) = d^{\mathfrak{s}}_{i,1}(\gamma_j) = p_{i,1}(\gamma_j) \; \forall \gamma_j \in \Theta \times \{\mathfrak{s}, f\}.$$

*Proof.* Note that $d^f_{i,1}(\gamma_j) = d^{\mathfrak{s}}_{i,1}(\gamma_j) = 0$ for all $\gamma_j \in \Theta \times \{\mathfrak{s}, f\}$ where $\gamma_j = (0, t_j)$.

Lemma A5 shows that $d^{t_i}_{i,2}(\gamma_j) > d^{t_i}_{i,1}(\gamma_j)$ for all $\gamma_j \in \Theta \times \{s, f\}$ where $\theta_j > 0$. Thus, if $\theta_j > 0$ and $\theta_i \in (d^{t_i}_{i,1}(\gamma_j), d^{t_i}_{i,2}(\gamma_j))$, then

$$q_i((\theta_i, t_i), (\theta_j, t_j)) = 1.$$

The taxation principle states that for all $\gamma_i, \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$,

$$q_i((\theta_i, t_i), (\theta_j, t_j)) \geq 1 \implies \theta_i \geq p_{i,1}(\gamma_j).$$

Similarly,

$$q_i((\theta_i, t_i), (\theta_j, t_j)) = 0 \implies p_{i,1}(\gamma_j) \geq \theta_i.$$

Thus, bidder $i$ wins at least one unit if $\gamma_i$ and $\gamma_j$ are such that $\theta_i > p_{i,1}(\gamma_j)$, and only if $\theta_i \geq p_{i,1}(\gamma_j)$. This implies that bidder $i$'s first unit cut-off equals $p_{i,1}(\gamma_j) \; \forall \gamma_j \in \Theta \times \{\mathfrak{s}, f\}$. □

Corollary A2 shows the final implication of Proposition 2.

**Corollary. A2.** *If mechanism $\Gamma$ satisfies Properties (1)-(4), then*

$$d_{i,1}(\theta_j, f) > d_{i,1}(\theta_j, \mathfrak{s}) \; \forall \theta_j > 0.$$

*Proof.* The proof is by contradiction. Suppose that there exists $\theta^*_j \in \Theta$ such that $\theta^*_j > 0$ and

$$d_{i,1}(\theta^*_j, f) \leq d_{i,1}(\theta^*_j, \mathfrak{s}).$$

Suppose that $\tilde{\theta}_i \in \Theta$ is such that $\tilde{\theta}_i \in [d_{i,1}(\theta^*_j, f), d_{i,1}(\theta^*_j, \mathfrak{s})]$. Then, we have that

$$d^f_{j,2}(\tilde{\theta}_i, t_i) \geq \tilde{\theta}_i \geq d^s_{j,2}(\tilde{\theta}_i, t_i)$$

because $\tilde{\theta}_i \in [d_{i,1}(\theta^*_j, f), d_{i,1}(\theta^*_j, \mathfrak{s})]$ and $q_j(\gamma_i, \gamma_j) = 2 - q_i(\gamma_i, \gamma_j) \; \forall \gamma_i, \gamma_j \in \Theta \times \{s, f\}$ implies that

$$q_j((\tilde{\theta}_i, t_i), (\theta_j, f)) \leq 1 \; if \; \theta_i > \tilde{\theta}_i,$$

and

$$q_j((\tilde{\theta}_i, t_i), (\theta_j, \mathfrak{s})) = 2 \; if \; \tilde{\theta}_i > \theta_i.$$

The taxation principle implies that if bidder $j$ has type $(\theta_j, t_j) \in \Theta \times \{\mathfrak{s}, f\}$ where $(\theta_j, t_j)$ is such that $\theta_j = d^{t_j}_{j,2}(\tilde{\theta}_i, t_i)$, then

$$b_2((\theta_j, t_j), p_{i,1}(\tilde{\theta}_i, t_i)) = p_{i,2}(\tilde{\theta}_i, t_i). \tag{14}$$

This yields a contradiction because

$$p_{i,2}(\tilde{\theta}_i, t_i) = b_2((d^{s}_{j,2}(\tilde{\theta}_i, t_i), \mathfrak{s}), p_{i,1}(\tilde{\theta}_i, t_i)) < b_2((d^{f}_{j,2}(\tilde{\theta}_i, t_i), f), p_{i,1}(\tilde{\theta}_i, t_i)) = p_{i,2}(\tilde{\theta}_i, t_i).$$

The first and last equalities follow from Equation 14 above. The inequality follows because (1) we showed that $d^{f}_{j,2}(\tilde{\theta}_i, t_i) \geq d^{s}_{j,2}(\tilde{\theta}_i, t_i)$ and (2) by construction $b_2((\theta_j, \mathfrak{s}), x) < b_2((\theta_j, f), x) \; \forall \theta_j \in (0, \overline{\theta}], \; x \in \mathbb{R}$. Thus we have that

$$d_{i,1}(\theta_j, f) > d_{i,1}(\theta_j, \mathfrak{s}) \; \forall \theta_j > 0,$$

if mechanism $\Gamma$ satisfies Properties (1)-(4). $\square$

## A2: An efficient mechanism with subsidies

In this section, I consider a setting where there are two homogenous goods and $N \geq 3$ bidders with single-dimensional types. I present a mechanism $\Gamma^{sub}$ that satisfies (1) IR, (2) incentive compatibility, (3) efficiency, and (4) strong monotonicity (for the remainder of this subsection, Properties (1)-(4)). The example shows that we can derive a mechanism that satisfies properties (1)-(4), but the mechanism violates the no subsidies condition.

Recall, in Subsection 3.2 we constructed a mechanism $\Gamma$ that satisfied (1) IR, (2) IC, (3) efficiency, and (4) no subsidies. As Proposition 1 implies, the mechanism violates strong monotonicity. To see one example of a strong monotonicity violation, consider the mechanism $\Gamma$ and suppose that $\overline{\theta} \geq \theta_1 > \theta_2 > \theta_3 \geq \theta_j \geq 0 \; \forall j \in \{4, \dots, N\}$. In addition, suppose that $d(\theta_{-2}) = \theta_2 + \epsilon^2 = \theta_3 + 2\epsilon$, where $\epsilon > 0$ is sufficiently small. Thus, we are considering an example where bidder 2's type is $\epsilon^2$ below her first unit cut-off. Moreover, bidder 2's rival, bidder 3 has a type that is just below her type. Bidder 1 wins both units because bidder 2 wins no units when bidder 2's type is below her first unit cut-off. Thus $q_1(\theta_1, \theta_{-1}) = 2$. We construct $\Gamma^{nm}$ to be such that bidder 1 pays $p_1(\theta_{-1}) = \theta_3$ for her first unit and $p_2(\theta_{-1}) = \theta_2$ for her second unit. Thus, if $\theta_3$ increases by a small amount $\epsilon > 0$, then bidder 1 pays more to win her first unit and thus she is willing to pay less for her second unit because of positive wealth effects. Thus, the increase in bidder 3's type implies that bidder 2 now wins one unit. This is because bidder 2's willingness to pay for her first unit is greater than bidder 1's (now lower) willingness to pay for her second unit. Therefore, we see that mechanism $\Gamma$ violates strong monotonicity because bidder 2 wins strictly more units even though her rival bidder 3 increased her type.

The violation of strong monotonicity occurs in mechanism $\Gamma$ because there is interdependence between bidder 1's willingness to pay for her second unit and bidder 3's type. The increase in bidder 3's type causes a drop in bidder 1's willingness to pay for her second unit, but not bidder 2's willingness to pay for her first unit. Thus, the two quantities can reverse in rank, and this reversal means that bidder 2 wins more units (she goes from winning zero units to winning one unit) when bidder 3 increases her type.

In this section, I show that we can remedy the above violation of strong monotonicity by giving bidders upfront subsidies that depend on their rivals' types. The upfront subsidies are constructed to be such that a bidder's willingness to pay for her second unit conditional on winning her first unit depends only on her demand and her highest rival's demand. In the context of the above example, this would imply that the increase in bidder 3's demand would increase the subsidy given to bidder 1. The increase in bidder 3's demand increases the price bidder 1 pays to win her first unit. The increase in the price of bidder 1's first unit is offset by an increase in her subsidy. In other words, the subsidy is constructed to be such that bidder 1's demand for her second unit is unchanged by the change in bidder 3's demand. This avoids the violation of strong monotonicity described above.

The mechanism $\Gamma^{sub}$ is symmetric. The assignment rule is such that a bidder wins a unit only if her demand type is one of the top two demands of all bidders. The top two bidders are given the same assignment that they are given in the two-bidder mechanism, which we call $\Gamma^2$. Recall mechanism $\Gamma^2$ is the two-bidder version of mechanism $\Gamma$ that was defined in Subsection 3.2. Note that we show that the mechanism $\Gamma$ violates strong monotonicity if and only if $N \geq 3$. Because mechanism $\Gamma^{sub}$ is symmetric, it is without loss of generality to present the mechanism from the perspective of bidder 1. Furthermore, it is without loss of generality to assume that $\overline{\theta} \geq \theta_2 \geq \theta_3 \geq \theta_j \geq 0 \; \forall j \in \{4, \ldots, N\}$. I let $d_1$ and $d_2$ be the first and second unit cut-offs in mechanism $\Gamma^2$, where $d_1, d_2 : \Theta \to \Theta$. In other words $d_1(\theta_2)$ and $d_2(\theta_2)$ would be the first and second unit cut-offs for bidder 1 if she competed in an auction with only one rival, bidder 2. This implies that assignment rule for bidder 1 is such that

$$q_1(\theta_1, \theta_{-1}) = \begin{cases} 0 & if \;\; \theta_1 < \max\{\theta_3, d_1(\theta_2)\}, \\ 1 & if \;\; \max\{\theta_3, d_1(\theta_2)\} < \theta_1 < d_2(\theta_2), \\ 2 & if \;\; d_2(\theta_2) < \theta_1. \end{cases}$$

Therefore, bidder 1 wins both units if and only if she wins both units in mechanism $\Gamma^2$ where her highest demand rival, bidder 2, is her only rival. In addition, bidder 1 wins at least one unit if both (1) bidder 1 is among the two highest demand bidders and (2) bidder 1's demand exceeds her first unit cut-off in mechanism $\Gamma^2$ where bidder 2 is the only rival.

I implement the mechanism with pricing rule $p : \Theta^{N-1} \to \mathbb{R}^3$ where $p$ is implicitly described by the three equations below

$$p_0(\theta_{-1}) = d_1(\theta_2) - p_1(\theta_{-1}),$$

$$p_1(\theta_{-1}) = b_1(\max\{\theta_3, d_1(\theta_2)\}, p_0(\theta_{-1})),$$

$$p_2(\theta_{-1}) = b_2(d_2(\theta_2), p_0(\theta_{-1}) + p_1(\theta_{-1})) = b_2(d_2(\theta_2), d_1(\theta_2)).$$

Note that the subsidy is constructed to be such that a bidder's demand for her second only varies with $\theta_2$. Thus, bidder 1's demand for her second unit conditional on buying her first unit is $b_2(\theta_1, d_1(\theta_2))$.

**Proposition. A1.** *There is a mechanism $\Gamma^{sub}$ that satisfies (1) IR, (2) IC, (3) efficiency, and (4) strong monotonicity.*

By construction, the mechanism satisfies IR, IC. The subsidy allows us to avoid the violation of strong monotonicity seen in the prior section. This is because there is no interdependence between a winning bidder's demand for later units and any of her rivals who have sufficiently low demand. The mechanism is efficient because the mechanism only assigns goods to the bidders with the two highest willingness to pays. Moreover, we show in the proof that the payment rule is such that bidder $i$ wins both units if and only if her demand for her second unit exceeds any of her rivals' willingness to pay for her first unit, conditional on receiving an upfront subsidy.

### Proof of Proposition A1

*Proof.* Because mechanism $\Gamma^{sub}$ is symmetric, it is without loss of generality to continue to study the decision problem of bidder 1 where $\theta_{-1} \in \Theta^{N-1}$ is such that $\theta_2 \geq \theta_3 \geq \theta_j \ \forall j \neq 1, 2, 3$. I assume this inequality holds for the remainder of the proof.

**IR:** To show that the mechanism satisfies IR, it suffices to show that $p_0(\theta_2, \theta_3) \leq 0$. If $\theta_3 \leq d_1(\theta_2)$, then $p_0(\theta_2, \theta_3) = 0$, because

$$p_1(\theta_2, \theta_3) = b_1(d_1(\theta_2), 0) = d_1(\theta_2) \implies p_0(\theta_2, \theta_3) = 0.$$

If $\theta_3 > d_1(\theta_2)$, then we find that $p_0(\theta_2, \theta_3)$ is the $p_0$ that solves

$$p_0 = d_1(\theta_2) - b_1(\theta_3, p_0) \implies p_0 + b_1(\theta_3, p_0) = d_1(\theta_2).$$

In the proof of Theorem 2 we show that $x + b_k(\theta, x)$ is strictly increasing for all $x \in \mathbb{R},\ k \in \{1, \dots, K\},\ \theta \in \Theta$. Thus, the left hand side of the above equation is strictly increasing in $p_0$. Moreover, when $p_0 = 0$, $\theta_3 > d_1(\theta_2)$ implies that

$$b_1(\theta_3, 0) + p_0 = \theta_3 > d_1(\theta_2) \implies p_0 < 0.$$

Hence the mechanism satisfies IR because $p_0(\theta_2, \theta_3) \leq 0$.

**IC:** The mechanism is incentive compatible because

$$u(q_1(\theta_1, \theta_{-1}), -\sum_{n=0}^{q_1(\theta_1,\theta_{-1})} p_n(\theta_{-1}), \theta_1) \geq u(q_1(\theta_1', \theta_{-1}), -\sum_{n=0}^{q_1(\theta_1',\theta_{-1})} p_n(\theta_{-1}), \theta_1)\ \forall \theta_i, \theta_i', \theta_{-i}.$$

This is shown below the expressions below:

$$q_1(\theta_1, \theta_{-1}) = 0 \implies \theta_1 \leq \max\{d_1(\theta_2), \theta_3\} \iff b_1(\theta_1, p_0(\theta_2, \theta_3)) < p_1(\theta_2, \theta_3).$$

$$q_1(\theta_1, \theta_{-1}) \geq 1 \implies \theta_1 \geq \max\{\theta_3, d(\theta_2)\} \iff b_1(\theta_1, p_0(\theta_2, \theta_3)) \geq p_1(\theta_2, \theta_3),$$

$$q_1(\theta_1, \theta_{-1}) = 2 \implies \theta_1 \geq d_2(\theta_2) \iff b_2(\theta_1, p_0(\theta_2) + p_1(\theta_2)) = b_2(\theta_1, d_1(\theta_2)) \geq p_2(\theta_2, \theta_3),$$

and lastly, $q_1(\theta_1, \theta_{-1}) = 1 \implies \theta_1 \leq d_2(\theta_2)$ and

$$\theta_1 \leq d_2(\theta_2) \iff b_2(\theta_1, p_0(\theta_{-1}) + p_1(\theta_{-1})) = b_2(\theta_1, d_1(\theta_2)) \leq b_2(d_2(\theta_2), d_1(\theta_2)) = p_2(\theta_2, \theta_3).$$

Each of the above four expressions follow from the construction of $\Gamma^{Sub}$.

**Strong Monotonicity:** The mechanism satisfies strong monotonicity because the construction is such that

$$q_1(\theta_1^h, \theta_{-1}^\ell) \geq q_1(\theta_1^\ell, \theta_{-1}^h) \forall \theta_1^h > \theta_1^\ell,\ \theta_{-1}^h \geq \theta_{-1}^\ell,$$

because bidder 1's first and second unit cut-off types are weakly increasing in $\theta_2$ and $\theta_3$.

**Efficiency:** I consider two cases.

Case 1: Suppose that bidders 1 and 2 each win one unit. First, I show that there are no Pareto improving trades between bidders 1 and 2. Recall that the outcome of the mechanism $\Gamma^{sub}$ is such that bidder one wins one unit and pays $p_0(\theta_{-1}) + p_1(\theta_{-1}) = d_1(\theta_2)$ in total.

This is the same as the outcome for in the efficient mechanism $\Gamma^2$ where there are only two bidders, namely bidders 1 and 2 with types $\theta_1$ and $\theta_2$. Hence, there are no ex post Pareto improving trades between bidders 1 and 2, because there are no ex post Pareto improving trades between bidders 1 and 2 under the efficient outcome implemented by mechanism $\Gamma^2$.

Next, I show that there are no ex post Pareto improving trades between a winning bidder and a losing bidder. Without loss of generality, suppose that $\theta_1 \geq \theta_2$. I show that there are no Pareto improving trades between bidder 2 and a losing bidder whom we assume to be bidder 3. By assumption $\theta_2 \geq \theta_3$. If $\theta_2 = \theta_3$, then incentive compatibility implies that both players are indifferent between winning and losing because their type equals the first unit cut-off. Thus, efficiency implies that bidder 2's willingness to sell her first unit equals her rival's willingness to pay when $\theta_2 = \theta_3$,

$$s_1(\theta_2, d_1(\theta_1)) = s_1(\theta_2, p_0(\theta_1, \theta_3) + p_1(\theta_1, \theta_3)) = b_1(\theta_3, p_0(\theta_1, \theta_2)).$$

In addition, if bidder 3's type falls to $\theta_3' < \theta_2$, then bidder 2 willingness to sell her unit is unchanged and bidder 3's willingness to pay falls. Thus, there are no Pareto improving trades between bidders 2 and 3. There are no Pareto improving trades between bidder 1 and bidder 3 because bidder 1's willingness to sell exceeds bidder 2's as

$$\theta_1 \geq \theta_2,\ d_1(\theta_1) \geq d_1(\theta_2) \implies s_1(\theta_1, d_1(\theta_2)) \geq s_1(\theta_2, d_1(\theta_1))$$

where the implication follows because $s_1$ is increasing in the first argument and decreasing in the second.

Case 2: Suppose that bidder 1 wins both units. I show that there are no Pareto improving trades between bidder 1 and all other bidders. Note that $p_0(\theta_1, \theta_2) = p_0(\theta_1, \theta_3) = 0$ because $d_1(\theta_1) \geq \theta_2 \geq \theta_3$. Thus, no losing bidder receives a subsidy. The losing bidder with the highest willingness to pay is bidder 2 who is willing to pay $\theta_2$ for her first unit. The outcome for bidders 1 and 2 is equivalent to the outcome for bidders 1 and 2 in the efficient mechanism $\Gamma^2$ where there are only two bidders, namely bidders 1 and 2. Thus, there are no ex post Pareto improving trades between bidder 1 and bidder 2 in mechanism $\Gamma^{sub}$ because the outcome for bidders 1 and 2 is the same as the outcome for bidders 1 and 2 in the efficient mechanism $\Gamma^2$. $\square$

## A3: Asymmetric Bidders

In the proof of Theorem 1 I construct a transformation that maps an arbitrary cut-off rule $d \in \mathcal{D}$ to a more efficient cut-off rule $T(d) \in \mathcal{D}$. In this subsection, I show that the proof of Theorem 1 outlined in Section 3 can be extended to an asymmetric bidder setting in a straightforward way.

In the asymmetric setting, I can similarly start with an arbitrary cut-off rule $d^i \in \mathcal{D}^i$ for bidder $i$. The set $\mathcal{D}^i$ is the set of all weakly increasing mappings of the form $d^i : \Theta_j \to \Theta_i^k$ where

$$d^i_{m+1}(\theta_j) \geq d^i_m(\theta_j) \ \forall m \in \{1, \ldots, k-1\}, \ \theta_j \in \Theta_j.$$

Given bidder 1's cut-off rule $d^1 \in \mathcal{D}^1$, there is a corresponding cut-off rule for bidder 2 which I call $d^2(d^1)$ and $d^2(d^1) \in \mathcal{D}^2 \ \forall d^1 \in \mathcal{D}^1$. More formally, if a mechanism has cut-off rule $d^1 \in \mathcal{D}^1$ for bidder 1 then we say that $d^1$ corresponds to an assignment rule that is such that,

$$\theta_1 \geq d^1_m(\theta_2) \implies q_1(\theta_1, \theta_2) \geq m,$$

and

$$d^1_m(\theta_2) > \theta_1 \implies m > q_1(\theta_1, \theta_2).$$

Note that it is without loss of generality to break ties in favor of bidder 1. Thus, bidder 1's cut-off rule $d^1 \in \mathcal{D}^1$ determines the assignment rule for bidder 1 $q_1(\theta_1, \theta_2) \forall (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$. We then define bidder 2's assignment rule as being the assignment rule that assigns all remaining units to bidder 2, $q_2(\theta_1, \theta_2) = k - q_1(\theta_1, \theta_2) \ \forall (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$. I then let $d^2(d^1)$ be the cut-off rule that corresponds to the allocation rule that corresponds to cut-off rule $d^1 \in \mathcal{D}^1$. Note that $d^1 \in \mathcal{D}^1 \implies d^2(d^1) \in \mathcal{D}^2$.

I can also analogously define the pricing rule $p^i : \Theta^j \times \mathcal{D}^i \to \mathbb{R}^{k+1}$, that implements a given cut-off rule for bidder $i$. In keeping with the argument given in Section 3, I let $p^i_0$ be such that

$$p^i_0(\theta_j, d^i) = 0 \ \forall \theta_j \in \Theta_j, \ d^i \in \mathcal{D}.$$

The price bidder $i$ pays to win her first unit equals the first unit cut-off

$$p^i_1(\theta_j, d^i) = d^i(\theta_j) \ \forall \theta_j \in \Theta_j, \ d^i \in \mathcal{D}.$$

I define the price that bidder $i$ pays for her $m^{th}$ unit given her rivals type and the cut-off rule $d^i$ inductively,

$$p^i_m(\theta_j, d^i) = b_m(d^i_m(\theta_j), \sum_{n=1}^{m-1} p^i_n(\theta_j, d^i)) \ \forall m \in \{1, \ldots, k\}, \ \theta_j \in \Theta_j, \ d^i \in \mathcal{D}.$$

I also define an analogous function $f$ that states the difference between bidder 1's willingness to pay for her $m^{th}$ unit compared to bidder 2's willingness to pay for her $k-m+1^{st}$ unit.

$$f(m,\theta_1,\theta_2,d^1) := b^1_m(\theta_1,\sum_{a=1}^{m-1} p^1_a(\theta_2,d^1)) - b^2_{k-m+1}(\theta_2,\sum_{a=1}^{k-m} p^2_a(\theta_1,d^2(d^1)))\ \forall m \in \{1,\ldots,k\},\ (\theta_1,\theta_2) \in \Theta_1 \times \Theta_2,$$

An identical argument to the one presented in Lemma 2 shows that the function $f(m,\theta_i,\theta_j,d^1)$ is (1) strictly decreasing in $m$ $\forall m \in \{1,\ldots k\}$, (2) strictly increasing in $\theta_1$ $\forall \theta_1 \in \Theta_1$, and (3) strictly decreasing in $\theta_2$ $\forall \theta_2 \in \Theta^2$. The remainder of the proof then proceeds identically. I use the function $f$ to define my mapping $T$. The domain and range of $T$ is now $\mathcal{D}^1$ instead of $\mathcal{D}$ as it was in the symmetric case. The fixed point of $T$ corresponds to a mechanism that has an efficient assignment rule, and we can use Schauder's fixed point theorem to show that the mapping has a fixed point.

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