Vortices in an Optically Trapped $^{87}$Rb Bose-Einstein Condensate

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Abstract

The crossed-beam optical dipole trap [1] constructed in 2006 by Daniel Guest [2] has opened up avenues of research in Bose-Einstein condensation unavailable with the magnetic or magneto-optical hybrid traps in use elsewhere. The dipole trap liberates the condensate’s spin degree of freedom and the external magnetic field, permitting experiments on spinor condensates. Furthermore, the high degree of cylindrical symmetry and tight trapping in all three dimensions afforded by the cross-beam configuration permitted the first [2] observation of sustained rotation in an all-optically confined condensate. These two results suggest a third possibility: studying the exchange of angular momentum between the condensate’s macroscopic rotation, manifested in vortex cores, and its intrinsic spin state.

In addition, our trap geometry allows us to outcouple an atom laser beam under the influence of gravity, giving rise to the possibility of studying rotational modes in an atom laser.

In this thesis, I will discuss our progress in refining the optical trap to meet the stringent conditions required for trapping vortex-laden condensates. I will also present the results of our further exploration of vortex dynamics in spinor condensates and atom lasers.
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I can hardly overstate my debt to the work of my lab partner Mike Goldman ’08. His programming skills, mathematical intuition, and love of efficiency proved invaluable assets both to the lab and to me. I have cited him in the body of this text wherever my work is particularly derivative from his, but his friendship and collaboration have shaped this thesis in more ways than a footnote can convey.

Wherever my work has led me over the past year, I have found, like Waldo, that many other thesis students have been there before. I would like to acknowledge my debt to their work in constructing and refining the apparatus and establishing the lore of its quirks, so that problems that took them days to solve were the work of minutes for me. I am particularly thankful to Jason Merrill ’06 and Daniel Guest ’07E, with whom I had the pleasure of working,
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# Contents

1 Introduction 1
   1.1 An Excursion into the History and Theory of BEC 5
      1.1.1 History 5
      1.1.2 Theory 7
   1.2 Condensation at Amherst, Condensed 11
      1.2.1 Properties of $^{87}$Rb 11
      1.2.2 Cooling and Trapping 13
      1.2.3 Optical Trap 16
      1.2.4 Introducing Vortices 17
      1.2.5 Imaging 18

2 Atoms in the Presence of Far Off-Resonant Light 20

3 Refining the Optical Trap 32
   3.1 Improvements to the Beam Path 33
   3.2 Alignment Techniques 37
      3.2.1 Finding the Trap 38
      3.2.2 Focus 42
      3.2.3 Leveling the FORT 44
      3.2.4 Waist Size 47
      3.2.5 Fine Alignment Tricks 48
4 Vortices in an Optically Trapped BEC  
4.1 Quantized Vorticity ........................................ 52  
4.2 Results ....................................................... 56  
4.2.1 Vortex Lifetimes ......................................... 56  
4.2.2 Vortex-Laden Spinor Condensate ...................... 59  
4.2.3 Vortex-Laden Atom Laser ............................... 62  

5 Conclusion .................................................... 66

A Properties of \(^{87}\text{Rb} \)  
A.1 Atomic Properties ........................................... 67  
A.2 \(D_2 \) Transition Properties ............................. 68  
A.3 \(D_1 \) Transition Properties .............................. 69

B Quantizing the Radiation Field ............................... 70
List of Figures

1.1 Optical lattice as a quantum simulator. ............................... 2
1.2 Carl Wieman and Eric Cornell with images of a BEC and a magnetic trap ................................................. 7
1.3 Condensation schematic ................................................. 9
1.4 Hyperfine energy levels for the $^{87}$Rb ground and first excited states ......................................................... 12
1.5 $^{87}$Rb hyperfine ground state with Zeeman splitting .......... 13
1.6 Optical table diagram. .................................................. 15
1.7 Schematic of vortex introduction into the magnetic trap ... 18
1.8 Vortex-laden $|1, -1\rangle$ condensate before and after transfer into the optical trap ............................................. 19
2.1 Dipole force schematic. ................................................ 22
3.1 Evidence for beam instability in a vortex decay series ....... 33
3.2 Hour-long optical trap stability ....................................... 34
3.3 Improvements in optical trap beam path. ......................... 35
3.4 Image of the cage-mounted fiber launch system ............... 36
3.5 FORT alignment optics ................................................ 38
3.6 Trap depth near the focus of a single-beam trap ............. 40
3.7 Distortion of a condensate in a single-beam FORT caused by the proximity of the other FORT ....................... 41
3.8 Ray-optics illustration of translation of the FORT focus using the fiber position ........................................ 44
3.9 Potential plots for tilted single-beam FORTs .................. 46
3.10 Trap spilling used as an alignment technique ................. 49
3.11 Vertical fine alignment signal .................................. 50
3.12 Comparison of condensates with ideal and imperfect vortex appearance ............................................... 51

4.1 Theoretically predicted features of a rotating superfluid .... 55
4.2 Vortex and atom number time series data comparison between March and October .................................... 58
4.3 Vortex and atom number time series data comparison between March and October .................................... 59
4.4 Vortices in a spinor condensate in the lower hyperfine level ................................................................. 60
4.5 Freely evolved and Stern-Gerlach discriminated upper hyperfine level spinor condensate ............................. 61
4.6 Rotating and non-rotating atom laser images .................... 63
4.7 Conceptual diagram of the spatial precession of a single-vortex atom laser about its vortex core ..................... 64
4.8 Density profile series of precessing vortical atom laser ........ 65

A.1 Properties of $^{87}$Rb ........................................... 67
A.2 D$_2$ Transition Properties ......................................... 68
A.3 D$_1$ Transition Properties ......................................... 69
Chapter 1

Introduction

Bose-Einstein condensation (BEC) is a phase of matter in which identical particles accumulate in the quantum ground state. The close analogy of BEC to laser light, which is a collection of photons in the same quantum state, raises the hope that when condensates can be controlled and manipulated effectively, made stable, and prepared inexpensively, BEC will yield technological applications [3] as sweeping and revolutionary as those of lasers. For example, advances have been made on the front of precision clocks [4], which take advantage of condensates’ coherence to beat the theoretical precision ceiling of cesium atomic clocks, and atomic-scale lithography and nanofabrication using atom beams [5] that could in principle have a condensate as their source.

BEC also promises a deeper understanding of fundamental physics, both as a tool to investigate other phenomena and as an object of experimental inquiry in its own right. For example, BEC has been proposed as a tool for precise measurement of small quantities: One technique exploits the phase shifts incurred by BEC superposition states under the influence of weak forces and appropriate quantum mechanical manipulations to read out those phase shifts [6]. Another line of inquiry treats condensates in optical lattices as “quantum simulators” to answer outstanding questions in quantum mechan-
ical, condensed matter, and high-energy theory [7]. One such experiment is shown in Fig. 1.1.

Figure 1.1: Schematic of an experiment performed by Greiner et al. [8], in which a BEC confined in an optical lattice is treated as a model quantum mechanical system. The diagram represents the position (left) and momentum (right) distributions for the so-called superfluid-Mott insulator phase transition. The distributions can be seen to obey the Heisenberg uncertainty relation. The lattice potential in (a), represented by the gray topography, is in the superfluid state: its peaks are low enough that condensate atoms can move between lattice sites, resulting in a large position uncertainty (left) but a well-defined momentum (right). The lattice potential in (b) is a perfect Mott-insulator: the lattice sites are deep enough that each atom is localized to a particular site (left), but the momentum distribution is blurred. Image from Ref. [9], reproduced in Ref. [7].

In its own right, BEC is an interesting test subject for the exploration of the so-called “quantum-classical frontier,” i.e. the macroscopic and mesoscopic regimes in which the validity of quantum mechanics remains largely untested. The question of whether “normal-sized” objects are subject to such quantum quirks as tunneling, superposition states, entanglement, and interference has vexed theorists and experimentalists alike since Erwin Schrödinger proposed
his notorious cat paradox in 1935 [10]. Condensates, objects composed of up to $\sim 10^6$ atoms in the same quantum state, are promising candidates for the cat. As we learn to make condensates larger still and perfect our ability to entangle them, it will be interesting to see whether they continue to display the same “quantum weirdness.”

Dilute gas BEC is a recent development—at the time of this writing, only thirteen years have passed since its discovery. Consequently, many questions about the properties and behavior of BEC remain, and some of these may prove important to the other lines of research discussed above.

This thesis is aimed at addressing several such open questions, in particular those involving the behavior of a rotating condensate. When a BEC acquires angular momentum about its center of mass, it exhibits a strange behavior known as quantized vorticity, which looks as odd as it sounds—vortices appear as a regular array of holes in the condensate, each carrying a quantum of circulation (see Fig. 4.1). The dynamics and interactions of these vortices have not been fully explored. For instance, the real-time dynamics of vortex evolution have never been studied. Also, vortex evolution near a Feshbach resonance, in which the interatomic interaction strength is tuned to zero, has not been observed. In addition, the possibilities for angular momentum exchange between the vortex and spin degrees of freedom have not been explored under conditions in which all spin states are accessible.

For BEC in dilute atomic gases, these three experiments require conditions that we believe are uniquely met by the crossed-beam optical dipole trap at Amherst. The placement of our beams in the horizontal plane and our ability to change their intensity during the course of an experiment allows us to outcouple a beam of atoms under the influence of gravity. In a rotating condensate, we can observe the vortex cores in the outcoupled atom beam.
This may provide us with a mechanism for studying the evolution of vortices in real time.

On a Feshbach resonance, the external magnetic field is used to tune the interatomic interactions to zero. This process, which is difficult or impossible in a conventional magnetic trap, is in principle quite straightforward in an optical dipole trap, whose trapping potential is undisturbed by uniform dc magnetic fields.

Furthermore, because it is possible to create a magnetic field minimum but not a maximum in current-free space, magnetic trapping requires the atoms to be in weak-field-seeking spin states—states whose energy increases under Zeeman splitting and which are consequently attracted to magnetic field minima. In the case of our atom of choice, $^{87}$Rb, this rules out five out of eight possible hyperfine ground states and prohibits experiments on true spinor condensates, in which the spin degree of freedom is completely liberated. In an optical dipole trap, however, the electric rather than magnetic fields are responsible for trapping, so no such restriction applies.

Finally, in order to permit experiments on vortices, a trap must exhibit sufficient axial symmetry to allow the condensate to rotate freely without heating. It turns out that our cross-beam configuration has reasonably good cylindrical symmetry and can support vortex-laden condensates with lifetimes up to several seconds.

In the sections ahead, I will give some historical and theoretical background on BEC and a general overview of the procedures and apparatus used at Amherst. Then I will discuss the theory behind optical trapping and quantized vorticity. Finally, I will present the improvements made to the optical trap and our progress in realizing the vortex experiments described above.
1.1 An Excursion into the History and Theory of BEC

1.1.1 History

In 1908, the city of Leyden in the Netherlands briefly held the double distinction of being both the coldest place on earth and the home of the first BEC. Dutchman Heike Kamerlingh Onnes was awarded a Nobel Prize for liquifying $^4$He there at a record temperature of 0.9K [11], but his other record, as the first to achieve Bose-Einstein condensation, would not be recognized for another three decades [12]. There was excellent reason for this delay: not until 1924 did Satyendra Nath Bose publish the ingenious statistical scheme that was BEC’s theoretical basis [13, 14]. It took Albert Einstein’s Nobel laureate starpower to get Bose’s work recognized and his mental candlepower to recognize its application to physical systems of noninteracting boson gases [14] popular in BEC research today.

Despite the success of Bose’s statistics in modeling the blackbody spectrum [13, 14], Bose’s contemporaries received it with skepticism. The stumbling block was its startling combinatorics scheme: It essentially required that Nature count her contents blindfolded.

Suppose you were offered a hat containing a cue ball and an eight ball and asked in how many orders you could draw them out. Two, obviously: the cue ball followed by the eight ball or vice versa. Now suppose you were blindfolded and asked the same question. As far as you were concerned, the two balls would be indistinguishable, and there would be no way for you to tell in which order you drew them. You might, therefore, be tempted to answer that there is only one way: one ball followed by another ball. Of course, you know that there really are two ways—*somebody* can tell the difference between the cue ball and the eight ball, even if you can’t. Bose’s strange proposition
was that for a pair of truly indistinguishable particles, there really aren’t two ways to draw them from a hat. Furthermore, such indistinguishable particles actually exist in nature, albeit on scales far tinier than billiard balls: They include photons, electrons, atoms of the same element, identical molecules, and any particles whose intrinsic observable qualities are identical. In the words of David Griffiths, “It’s not just that we don’t happen to know which electron is which; God doesn’t know which is which, because there is no such thing as ‘this’ electron or ‘that’ electron” [10].

One consequence of this, as Einstein noticed, is that at sufficiently low temperatures, certain identical particles happily settle into the quantum ground state together, achieving a macroscopic occupancy. The wording of Einstein’s prediction proved visually evocative (See Fig. 1.2) of the phenomenon of BEC in dilute gases that would not be achieved until forty years after his death: “A separation is effected: One part condenses, the rest remains a saturated ideal gas” [14]. This separation is characteristic of a phase change, implying that BEC is more than just a mathematical oddity: It is a new phase of matter.

Liquid helium technically contains a condensate since it achieves macroscopic ground state occupancy; however, Einstein derived his result for the simpler case of an ideal atomic gas [14]. The more interesting and less theoretically fraught case of condensation was therefore considered to be dilute gas BEC, which better approximates the noninteracting regime. Developments in low-temperature physics, especially in magneto-optical trapping and evaporative and laser cooling, led to a race for this form of BEC in the 1990’s. It was narrowly won by Eric Cornell and Carl Wieman at the University of Colorado, who produced a BEC in $^{87}\text{Rb}$ in June of 1995, seventy years after Einstein predicted the phenomenon. Later that summer, Hulet’s group at Rice University produced evidence for Bose-Einstein condensation in $^7\text{Li}$. Wolfgang Ketterle’s
group at MIT followed shortly thereafter with a sodium condensate two orders of magnitude larger than the Colorado group’s and produced in $1/40^{th}$ of the time [13, 14]. Cornell, Wieman, and Ketterle shared the 2001 Nobel Prize in Physics for their achievement.

![Carl Wieman (left) and Eric Cornell holding images of a BEC and a magneto-optical trap, respectively. Note that in the BEC image, in which height represents density, the condensate appears as a tall, localized spike visually distinguishable from the surrounding flattened thermal cloud. This is evidence of the phase transition Einstein predicted.](Figure 1.2)

1.1.2 Theory

Daniel Schroeder [16] offers an elegant qualitative argument for why Bose-Einstein condensation occurs in a system of identical bosons. The crux of the phenomenon is the counting oddity discussed above. For a system of $N$ distinguishable particles, the number of macrostates without a significant ground state occupancy is enormous compared to the number of macrostates
with a significant ground state occupancy: There is only one microstate in which all the particles are in the ground state, but there are \( N \) microstates in which one particle is in the first excited state, \( \binom{N}{2} \) microstates in which two particles are in the first excited state, etc. The sheer number of excited states is sufficient to overwhelm the Boltzmann factor pressure toward lower energy macrostates under any reasonable conditions. On the contrary, for a system of \( N \) indistinguishable particles, there is only one microstate for each of the aforementioned macrostates because it does not matter which of the particles is excited. As long as the particles are bosons—and consequently not subject to the Pauli exclusion principle forbidding the sharing of states—this system will find it energetically favorable to achieve macroscopic occupancy of the ground state at sufficiently low temperatures.

By means of a simple dimensional argument combined with some basic quantum mechanics and thermodynamics, we can get some interesting order-of-magnitude results about BEC. Consider a gas of \( N \) identical bosonic atoms confined in a three-dimensional harmonic potential with natural frequency \( \omega_0 \). Ideal gas theory, which treats gas particles as billiard balls, works in the high temperature, low density regime, but at sufficiently low temperatures and high densities, quantum effects become apparent. As a gas cools, the average momentum \( p \) decreases, so the average thermal de Broglie wavelength, given by

\[
\lambda_B = \frac{h}{p},
\]

where \( h \) is Planck’s constant, increases. Roughly speaking, Bose-Einstein condensation occurs when the particles’ wavefunctions overlap, that is, when the gas particles are cold and dense enough that their spatial extent, which is

\[\text{In this case, as Schroeder explains, “reasonable” means that } Z_1, \text{ the single-particle partition function, is much greater than 1. In other words, many single-particle states are available.} \ [16]\]
of order the thermal de Broglie wavelength, is approximately equal to the interparticle spacing [16, 17]. Fig. 1.3 shows this process schematically.

Figure 1.3: Conceptual diagram showing the behavior of a cooled bosonic gas. On the hottest temperature scales (a), the atoms behave like billiard balls, forming an ideal gas. When they are cooled, their thermal de Broglie wavelength $\lambda_B$ increases (b) until the atoms begin to behave as “wavicles.” At the critical temperature (c), the matter-waves begin to overlap, and Bose-Einstein condensation occurs. As the temperature approaches absolute zero (d), the de Broglie wavelength increases without bound until all the atoms have condensed. Based on Ref. [13].

From the Equipartition Theorem, we know that each quadratic degree of freedom of a system carries an average of $1/2k_B T$ units of energy, where $T$ is the temperature and $k_B$ is Boltzmann’s constant. A gas of atoms in a harmonic trap has six quadratic degrees of freedom: the three spatial coordinates and the three components of momentum along each spatial axis. Only the latter three contribute to the kinetic energy, however, so the total average kinetic energy per particle is:

$$K = \frac{3}{2}k_B T.$$  \hspace{1cm} (1.2)

Since $p = \sqrt{2mK}$, where $m$ is the particle mass, we can write the average
momentum as:
\[ p = \sqrt{3mk_B T}. \]  
(1.3)

Substituting this into Eq. (1.1), we obtain the average thermal de Broglie wavelength in terms of \( T, m, \) and fundamental constants [16]:
\[ \lambda_B = \frac{\hbar}{\sqrt{3mk_B T}}. \]  
(1.4)

Meanwhile, the interparticle spacing is determined by the temperature of the gas cloud and the tightness of the potential. According to the virial theorem, the average kinetic and potential energies for a particle in a harmonic oscillator potential are equal [18], so we have
\[ \frac{3}{2} k_B T = \frac{1}{2} m \omega_0^2 \langle r^2 \rangle, \]  
(1.5)

where \( \langle r^2 \rangle \) is the expectation value of the square of the radial distance from the center of the trap. The radius \( R \) of the trap is of order the square root of this quantity:
\[ R = \sqrt{\frac{3k_B T}{m \omega_0^2}}. \]  
(1.6)

The volume per atom is given by the volume of the trap divided by the number of atoms in the trap. The interparticle spacing, or the radius allotted to each atom, is roughly the cube root of the volume per atom:
\[ l \sim \left( \frac{4\pi}{3N} \right)^{\frac{1}{3}} R \sim 1.61 N^{-\frac{1}{3}} \sqrt{\frac{3k_B T}{m \omega_0^2}}. \]  
(1.7)

Recall that the phase transition to BEC occurs when the interparticle spacing is roughly equal to the spatial extent of the atomic wavefunction. At the critical temperature \( T_c \), therefore, we have \( l \sim \lambda_B \). To estimate \( T_c \), we set Eq. (1.7) and (1.4) equal and solve for temperature:
\[ T_c \sim 1.3 \frac{\hbar \omega_0 N^{\frac{1}{3}}}{k_B}, \]  
(1.8)
where I have substituted $2\pi\hbar$ for $\hbar$. The actual coefficient determined by a more careful analysis (See, for example, Ref. [16] and [17]) is 0.94 [17], so our order-of-magnitude analysis is within 40% of the correct answer.

The long-range phase coherence of the atoms that compose a condensate [19] implies that a BEC can be described by a single wave function. In the mean field picture, the interatomic interactions provide an extra nonlinear term in the Schrödinger equation [17], which goes as the square of the overall wavefunction:

$$i\hbar \frac{\partial}{\partial t} \Psi(r, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + U_0 |\Psi(r, t)|^2 \right] \Psi(r, t). \quad (1.9)$$

This modified Schrödinger equation is known as the Gross-Pitaevskii equation, and it governs the dynamics of a BEC. The variable $\Psi(r, t)$ is the overall wavefunction describing the condensate, $V(r)$ is the external potential, which is assumed to be static, and $m$ represents the particle mass. The coefficient $U_0$ of the nonlinear term is equal to $4\pi\hbar^2a/m$, where $a$ is the interatomic scattering length, a parameter that determines the sign and strength of the interatomic interactions.

1.2 Condensation at Amherst, Condensed

I will abbreviate this section since excellent discussions of the apparatus and methods used at Amherst can be found in many past student theses in Refs. [2, 20–23]. This section will provide an overview merely of what is necessary to an understanding of our present work.

1.2.1 Properties of $^{87}$Rb

Any discussion of BEC at Amherst must include an encomium on our atom of choice $^{87}$Rb, a bosonic alkali metal with many “forgiving” traits. Its hyperfine groundstate and first excited state structure are diagrammed in Fig. 1.4.
Foremost among the useful qualities of $^{87}$Rb is the availability of a cycling transition from the $F = 2$ to the $F = 3'$ state. This transition can only decay back to the $F = 2$ state, making it useful for laser cooling, in which we repeatedly induce the $F = 2 \rightarrow F = 3'$ transition to slow the atoms down, as discussed in Sec. 1.2.2. We also use this transition for imaging (See Sec. 1.2.5) and magneto-optical trapping (Sec. 1.2.2). Other useful transitions are the “optical pumping” transition $F = 2 \rightarrow F = 2'$, which can be used to change the magnetic quantum number within a hyperfine manifold and to transfer atoms from the upper to the lower hyperfine manifold, and the “repump” transition $F = 1 \rightarrow F = 2'$, which is used to transfer atoms from the lower to the upper hyperfine manifold.

The $^{87}$Rb groundstate hyperfine structure with Zeeman splitting is diagrammed in Fig. 1.5. Only the states whose energy increases in the presence
of a magnetic field are capable of being magnetically trapped. Before the advent of the optical trap, they were the only states on which we could perform experiments.

![Energy Levels Diagram](image)

Figure 1.5: $^{87}$Rb hyperfine ground state with Zeeman splitting. The energy levels in the absence of an external magnetic field are shown in black, and changes caused by the presence of a field are shown in color. The red states are magnetically trappable.

### 1.2.2 Cooling and Trapping

In order to make BECs, we must cool our atoms down to temperatures of order 100 nK—about 10 million times colder than the temperature of outer space. Achieving these temperatures requires an extremely high vacuum and sophisticated cooling and trapping techniques.

The apparatus we use is diagrammed in Fig. 1.6. Room temperature $^{87}$Rb atoms are released from the getters into the “collection cell,” where the first stage of cooling takes place. Laser beams slightly red-detuned from the cycling (See Fig. 1.4) transition are shined from all six directions into the cell. Atoms moving sufficiently quickly toward a beam see it Doppler-shifted onto resonance and are slowed down by the absorption of photons from their direction of travel. Repump light is used to bring atoms initially in the lower hyperfine level to the upper hyperfine level, and then as long as the cycling
transition-frequency laser beams are appropriately polarized, they drive the atoms to and keep them in the $|2, 2\rangle$ state, so that they are attracted to a magnetic field minimum (See Fig. 1.5). A pair of anti-Helmholtz coils creates the field minimum that serves to keep the atoms confined. This configuration is called a magneto-optical trap or MOT, and the cooling technique is referred to as optical molasses.

The atoms are then transferred to a new, higher vacuum cell called the “science cell,” where they undergo the next stage of cooling. This stage, called the compressed MOT, works by further detuning the MOT to reduce the radiation pressure keeping the atoms apart, then turning off the magnetic trap, leaving a small, cold, spatially unconfined blob of atoms.

The final stage of cooling must be done in the absence of laser light because the shot noise sets a lower bound on our achievable temperature. After the compressed MOT, we return as many atoms as possible to a magnetically trappable state and then shut the lasers off completely. The atoms are remain confined by their attraction to a magnetic field minimum produced by a pair of anti-Helmholtz coils and an audio-frequency circularly rotating bias field produced by a pair of Helmholtz coils orthogonal both to each other and to the anti-Helmholtz pair. The rotating bias field is needed to “hide” the zero-field point in the trap from the atoms by swinging it around in a circle more quickly than their thermal energy allows them to follow. If they could catch this so-called “hole of death,” their magnetic quantum number would no longer be well defined, and they could undergo spontaneous transitions, called Majorana transitions, to untrapped states. This trapping scheme is called the TOP trap, short for Time-averaged Orbiting Potential.

Cooling in the TOP trap relies on the Zeeman shifting of the hyperfine levels shown in Fig. 1.5. Atoms with more energy spend most of their time
Figure 1.6: Schematic of optical table showing the vacuum system; MOT, repump, optical pumping, probe, and FORT beams; and magnetic coils. Optics below the science cell, vertical MOT beams, and the magnetic coils and repump beams for the collection cell are omitted for simplicity. From Dan Guest’s thesis [2].
far from the center of the trap in regions of higher magnetic field. They consequently experience stronger Zeeman shifts than lower-energy atoms and require higher radio-frequencies to undergo transitions to untrapped states within the hyperfine manifold. In fact, there is an oblate spheroidal surface in space associated with each transition frequency. In order to cool the atom cloud, we bathe the trap in radio-frequency radiation, starting out at high frequencies and sweeping downward. This way, the radiation acts as a sort of “knife,” cutting out the most energetic atoms first—the ones near the rim of the trap—and gradually slicing inward. The atoms rethermalize with a smaller and smaller average energy, and the temperature drops. At the critical temperature, the atoms fall into the lowest available energy state and condense.

1.2.3 Optical Trap

The optical trap consists of two beams produced by a Nd:YVO$_4$ laser with a wavelength of 1064 nm and a maximum optical power output of 10 W. These beams will henceforth be referred to as FORTs X and Y (for Far Off-Resonant Trap), as labeled in Fig. 1.6 and in more detail, in Fig. 3.3. The beams are piped from the laser to the main optical table by means of optical fibers and emerge through collimating and polarizing optics as round, horizontally polarized, $\sim$1.1 mm-waisted beams. A small fraction of each beam is picked off and sent into a photodiode, which controls the overall beam power through a servoloop connected to an acousto-optic modulator. The beams are presently calibrated to 21.4 mW of optical power per programmed volt for FORT Y and 22.78 mW per volt for FORT X. The beams then pass through a “telescope” consisting of a pair of lenses that expands each beam by a factor of 3. This step is to tighten the focus later on, since as discussed in Sec. 3.2.4, the waist at the focus of a beam is inversely proportional to its collimated diameter. Each
beam is then directed off a pair of mirrors, which give us complete control over the beam pointing. They finally pass through a 300 mm focusing lens, are elevated to the correct height by another two mirror bounces, and come to a focus in the center of the cell at right angles to each other. Although the beams are nearly the same frequency, the horizontal polarization ensures that they do not interfere at the focus. Moreover, the beams are shifted slightly in frequency relative to one another by the AOMs so that any interference that does occur is averaged out in time.

The mechanism by which these beams produce a trapping potential will be discussed extensively in Ch. 2.

1.2.4 Introducing Vortices

Our setup allows us to produce vortices in a magnetically trapped condensate and then transfer the rotating condensate into the optical trap.

Our method for producing vortices is diagrammed in Fig. 1.7. The tip of the magnetic field vector produced by the AF bias coils (See Fig. 1.6) normally traces out a circle to create a cylindrically symmetrical potential. When spinning up the condensate, we change its path to an ellipse, which we then rotate like a propeller to stir the condensate. The resulting vortex lattice for a $|1, -1\rangle$ condensate is shown in Fig. 1.8(a).

To load the rotating condensate into the optical trap, we turn on the beam power to about 0.8 V (17.6 mW) per beam in an 80 ms linear ramp. Sixty ms after the start of this ramp, we begin a 50 ms ramp-off of the magnetic trap. In principle, the magnetic and optical trap axes of symmetry should coincide, so this process should be completely isotropic. In practice, however, only about 20% of vortices survive the transfer, as shown in Fig. 1.8, indicating that some part of the process dissipates angular momentum.
1.2.5 Imaging

Our data consists of a snapshot of each condensate we produce, which can be taken either from the top or from the side. A short burst of on-resonant (cycling transition) light is focused onto the condensate, which absorbs some of the light and projects a shadow onto the beam whose darkness at a given point indicates the density of the condensate at that point. The beam is then allowed to expand, recollimated and sent to a detector, which collects information about the intensity profile of the beam. The condensate appears as a shadow projected onto the beam.
Figure 1.8: Vortex-laden $|1,-1\rangle$ condensate released from a magnetic trap (a) and from an optical trap (b) shortly after the transfer from the magnetic trap. These images are not to scale—the vortical condensate from the magnetic trap is actually about twice the diameter of the one from the optical trap. Note that approximately 20% of vortices survive the transfer.

The downside to this process is that the pulse of resonant light completely obliterates the condensate, preventing real-time observation of condensate dynamics. Non-destructive imaging techniques do exist for observing many real-time characteristics; however, no one has come up with a method for non-destructively imaging a vortex-laden condensate. This is because vortex lines are so narrow in the trap—the theory, described in Ch. 2, only requires them to be one-dimensional—that the only way to make them visible is to release the condensate and allow its angular momentum to expand them. Once a vortex-bearing condensate is released and imaged, however non-destructively, there is no way of putting it back into its earlier state. Hence, vortex lattice evolution has never been observed in real time.
Chapter 2

Atoms in the Presence of Far Off-Resonant Light

There are two regimes of optical trapping, which can loosely be designated “far off-resonant,” or the dipole force regime, and “way the heck off-resonant,” or the quasi-electrostatic trapping (QUEST) [26] regime. Steve Chu’s group first demonstrated dipole force trapping in 1986 [27], while QUEST, which requires greater laser intensities and tighter foci, was only achieved a decade later by Knize’s group [28]. Our trap lies in the no-man’s land between these two regimes, so, for completeness, I will discuss both; however, it is common practice to use the dipole force approximation to model traps like ours [1], and this is the model we have used for purposes of numerical simulation and trap characterization.

In the presence of an external electric field, a neutral atom acquires an induced electric dipole moment.\(^1\) As long as the external field is quasistatic (i.e., changing at frequencies much lower than any allowed atomic transition) and small compared to the field binding the electron to the nucleus, the induced dipole moment will be proportional to the external field (See Fig. 2.1) [29].

---

\(^1\)For the following heuristic discussion of the dipole force trap, I rely heavily upon Dan Guest’s presentation of the same in his senior thesis [2].
That is,

\[ d = \alpha_s E \tag{2.1} \]

where \( d \) is the induced dipole moment, \( \alpha_s \) is the atomic polarizability,\(^2\) and \( E \) is the external electric field. This equation is analogous to Hook’s law for springs, \( F = -kx \). \( E \) corresponds to \( F \), \( d \) to \( x \), and \( \alpha_s \) to \(-1/k\). Using this analogy, the energy of the dipole-field interaction will be the total amount of energy required to “stretch” the dipole to its final value in the presence of the field. The interaction energy is therefore:

\[ \Delta E = -\int d \mathbf{d} \cdot \mathbf{E}. \tag{2.2} \]

Since Eq. (2.1) tells us that \( d \mathbf{d} = \alpha_s d \mathbf{E} \), this equation simplifies to:

\[ \Delta E = -\frac{1}{2} \alpha_s E^2, \tag{2.3} \]

where \( E \) is the magnitude of the electric field.

Eq. (2.3) suggests that the polarized atom seeks the region of strongest electric field, as illustrated in Fig. 2.1. Earnshaw’s theorem \([29]\) forbids electrostatic field maxima in free space, but rms maxima in electromagnetic waves detuned far enough to the red of any atomic transitions that the atom just “sees” a slowly varying field adequately approximate the electrostatic condition. This is the QUEST regime, in which the trapping potential is approximated by the “lowest order perturbation theory expression for the Stark shift of a ground state \( |g\rangle \) due to excited states” \( |e_n\rangle \) \([26]\). The same effects are in play in the dipole trapping regime, except that the electric fields are varying rapidly enough that we can no longer use the dc Stark shift approximation and

\(^2\)The subscript \( s \) is to indicate that the polarizability is a scalar quantity for the ground state alkali atoms we are considering. In general, the polarizability may be a tensor, but the S subshell is rotationally symmetric, so the polarizability is completely defined by a single quantity \([17]\).
must take into account the dependence of the polarizability on the radiation frequency. In our harmonic oscillator analogy, pursued in Fig. 2.1, this means that the driving force is near enough to resonance that we can no longer assume that the oscillator is perfectly in phase with it. A dipole slightly out of phase with an oscillating field will experience a smaller energy shift from that of an in-phase dipole.

Figure 2.1: In the presence of an electric field, the valence electron of $^{87}\text{Rb}$ behaves roughly like a mass on a spring subject to a driving force proportional to the electric field. As long as the field is red-detuned from the atomic resonance, as shown in (a), the dipole oscillates in phase with the electric field. The atom then experiences a force along the gradient of the rms field, drawing it towards the region of greatest intensity—in our case, the beam focus. Just like a classical harmonic oscillator, the dipole will oscillate out of phase with a driving force whose frequency is greater than the natural frequency. Thus, as illustrated in (b), a blue-detuned electromagnetic field induces a dipole antiparallel to the electric field, driving the atom towards the region of lowest field intensity.

Consider a two-state atom, for instance, an alkali atom with a single excited state available to its valence electron. This turns out to be a reasonable model for our $^{87}\text{Rb}$ atoms: The coupling between the ground and excited states will be treated as the “spring,” and in the case of a far red-detuned laser beam, only
the lowest couple of excited states will have stiff enough “spring constants” to contribute significantly to the trapping potential. Moreover, coupling between the excited states would require two-photon transitions, which are extremely rare. For our purposes, therefore, we can treat each relevant ground-to-excited state transition as a separate two-state system.

Since we have stipulated that the external electric field is much smaller than the nuclear field at the radius of the valence electron, we can treat the interaction Hamiltonian between our two-state dipole and the electric field as a perturbation to the overall Hamiltonian. Therefore, as intimated above, the trapping potential seen by an atom in the ground state is the first nonzero term in the perturbation theory correction to the ground state energy due to the atom-field interaction Hamiltonian given in the dipole approximation by [29]:

$$H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}. \quad (2.4)$$

In order to determine this lowest-order term, let us continue with the spring analogy and rewrite Eq. (2.4) in terms of the quantum mechanical harmonic oscillator ladder operators, following the method of C. S. Adams et al. [30]. The electric dipole is simply charge times displacement, so if we choose the dipole to be in the $\hat{z}$ direction with the nucleus at the origin, we can write it as

$$\mathbf{d} = e \mathbf{z} \quad (2.5)$$

where $e$ is the electron charge. If we label the two states $|g\rangle$ and $|e\rangle$, the atomic raising and lowering operators are given by$^3$ [10, 30, 31]:

$$a^\dagger \equiv |e\rangle \langle g| \equiv (2\hbar m\omega)^{-\frac{3}{2}}(m\omega z - ip) \quad \text{and} \quad (2.6)$$

$$a \equiv |g\rangle \langle e| \equiv (2\hbar m\omega)^{-\frac{3}{2}}(m\omega z + ip) \quad (2.7)$$

$^3$These are actually the standard harmonic oscillator ladder operators (See Appendix B).
respectively, where $m$ is the atomic mass, $\omega$ is the angular frequency separation between the two atomic levels, $z$ is the one-dimensional displacement, and $p$ is the momentum. We now solve for $z$ in terms of $a^\dagger$ and $a$:

$$z = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a), \quad (2.8)$$

and substitute this result into Eq. (2.5):

$$\mathbf{d} = e\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)\hat{z}. \quad (2.9)$$

We can now achieve a simplification by taking the inner product of $\mathbf{d}$ in Eq. (2.9) between $\langle g|$ and $|e \rangle$. The term involving the raising operator vanishes, and the term involving the lowering operator becomes 1, yielding

$$d_{ge} = e\sqrt{\frac{\hbar}{2m\omega}}. \quad (2.10)$$

Substituting this result into Eq. (2.9) gives the tidy form

$$\mathbf{d} = d_{ge}(a^\dagger + a)\hat{z}. \quad (2.11)$$

It remains to write the electric field in terms of quantum mechanical operators. The derivation of this quantization is somewhat beyond the scope of the undergraduate curriculum and is therefore relegated to Appendix B, but we may proceed using the result for a single radiation frequency $\omega_L$ polarized in the $\hat{e}$ direction and propagating in the $\hat{k}$ direction [30]:

$$\mathbf{E}(r, t) = \frac{1}{2}\hat{e} \left[ \alpha \mathcal{E} e^{i(\omega_L t - k \cdot r)} + \alpha^\dagger \mathcal{E} e^{-i(\omega_L t - k \cdot r)} \right] \quad (2.12)$$

where $\alpha = (\sqrt{n + 1})|n\rangle\langle n + 1|$ and $\alpha^\dagger = (\sqrt{n + 1})|n + 1\rangle\langle n|$ are the photon annihilation and creation operators, respectively, $\mathcal{E}$ is a constant, and $k$ is the wavenumber, whose magnitude is $\omega_L/c$. Now, inserting Eq. (2.11) and Eq.
(2.12) into the dipole-field interaction Hamiltonian, Eq. (2.4), gives a four-term sum:

$$\hat{H}_{\text{int}} = -\frac{1}{2} d_{ge} (\hat{z} \cdot \hat{e}) \left( a \alpha e^{i(\omega_L t - \mathbf{k} \cdot \mathbf{r})} + \alpha \dagger e^{-i(\omega_L t - \mathbf{k} \cdot \mathbf{r})} \right). \quad (2.13)$$

The terms in $a \alpha \dagger$ and $a \dagger \alpha$ are naturally interpreted as the emission of a photon into the field accompanied by a decay to the ground state and an absorption of a photon from the field accompanied by an atomic excitation, respectively. The other two terms, in $a \dagger \alpha \dagger$ and $a \alpha$, correspond to an emission accompanied by an excitation and an absorption accompanied by a decay, respectively. These terms do not contribute to the interaction Hamiltonian and may be neglected [17, 30], yielding the simplified formula:

$$\hat{H}_{\text{int}} = -\frac{1}{2} d_{ge} (\hat{z} \cdot \hat{e}) \left( a \dagger \alpha e^{i(\omega_L t - \mathbf{k} \cdot \mathbf{r})} + a \alpha \dagger e^{-i(\omega_L t - \mathbf{k} \cdot \mathbf{r})} \right). \quad (2.14)$$

The perturbation theory expression for the first order correction to the ground state energy is [10]

$$E_{g}^{1} = \langle g | \hat{H}_{\text{int}} | g \rangle, \quad (2.15)$$

where I have adopted Griffiths’s [10] convention of writing the order of the correction as a superscript and the state as a subscript. However, inspection of Eq. (2.14) for the interaction Hamiltonian reveals that this inner product vanishes, since $a \dagger = |e \rangle \langle g |$, $a = |g \rangle \langle e |$, and $|g \rangle$ and $|e \rangle$ are orthogonal. As Pethick and Smith point out, we could have anticipated this vanishing from Eq. (2.4) [17]: The zeroth and first order terms in the Taylor series expansion of $\hat{H}_{\text{int}} = -1/2 \alpha_s E^2$ about $E = 0$ vanish. The second order correction is given by [10]

$$E_{g}^{2} = \left| \frac{\langle e | \hat{H}_{\text{int}} | g \rangle}{E_{g}^{0} - E_{e}^{0}} \right|^2, \quad (2.16)$$

These terms involve adding the atomic transition and the electromagnetic field frequencies in the exponent; consequently, they are cycling so quickly that they average out over time. This is known as the rotating wave approximation [30].
where $E_g^0$ and $E_e^0$ are the unperturbed ground and excited state energies, respectively. Setting this perturbation energy equal to the energy $\Delta E$ given in Eq. (2.3) and substituting Eq. (2.4) in for the interaction Hamiltonian on the right gives:

$$-rac{1}{2} \alpha_s E^2 = \left| \langle e| \mathbf{d} \cdot \hat{\mathbf{e}} |g \rangle \right|^2 \frac{E^2}{E_g^0 - E_e^0}.$$  

(2.17)

We can now solve this equation for the atomic polarizability to obtain [17, 30]:

$$\alpha_s = (\hat{\mathbf{z}} \cdot \hat{\mathbf{e}})^2 \frac{2d_{eg}^2}{E_g^0 - E_e^0}.$$  

(2.18)

Note that by using Eq. (2.3), we have assumed that the dipole is always perfectly in phase with the electric field. This polarizability is therefore only applicable in the QUEST regime; we will have to modify it to account for the dependence on the radiation frequency when we turn to the dipole regime.

In the presence of a radiation field, Eq. (2.16) for the energy shift of the ground state takes a slightly different form because there are two overall intermediate states that involve an excited atomic state. If the ground state is $|g, n\rangle$, then the excited state can either be $|e, n - 1\rangle$ or $|e, n + 1\rangle$ [17]. That the change in photon number is one is enforced by the radiation raising and lowering operators in each term of the interaction Hamiltonian. Rather than just $E_e$, the energies for these intermediate states are $E_e - \hbar \omega_L$ for the state $|e, n - 1\rangle$, or $E_e + \hbar \omega_L$ for the state $|e, n + 1\rangle$. The energy shift is therefore given by:

$$E_g'^2 = \left| \langle e| \hat{H}_{\text{int}} |g \rangle \right|^2 \left( \frac{1}{E_g^0 - E_e^0 + \hbar \omega_L} + \frac{1}{E_g^0 - E_e^0 - \hbar \omega_L} \right).$$  

(2.19)

Substituting Eq. (2.14) into this equation for $\hat{H}_{\text{int}}$ gives [17]:

$$E_g'^2 = (\hat{\mathbf{z}} \cdot \hat{\mathbf{e}})^2 \frac{1}{4} d_{eg}^2 \sqrt{nE}^2 \left( \frac{1}{E_g^0 - E_e^0 + \hbar \omega_L} + \frac{1}{E_g^0 - E_e^0 - \hbar \omega_L} \right).$$  

(2.20)

where $\sqrt{nE}$ is the amplitude of the radiation field.
Now we can finally invoke the QUEST approximation by substituting the dc polarizability from Eq. (2.18) into Eq. (2.20) [26]:

$$E_g^2 = \frac{1}{4} \alpha_s |\sqrt{nE}|^2 \left( \frac{1}{1 - (\omega/\omega_L)^2} \right),$$  \hspace{1cm} (2.21)

where $\omega$ is the atomic transition frequency given by $1/\hbar(E_0^g - E_0^e)$. This approximation is accurate within 0.1% as long as $\omega_L < \omega/2$ [26]. In our case, where $\omega_L \approx 0.73\omega$, the approximation is less ideal.

A slightly better model for our purposes may be derived from a different approximation starting from Eq. (2.20). Instead of substituting the dc polarizability, we rewrite the equation in the evocative form [17],

$$E_g^2 = -\frac{1}{2} \left[ (\hat{z} \cdot \hat{e})^2 d_{eg}^2 \left( \frac{1}{E_e^0 - E_g^0 + \hbar\omega_L} + \frac{1}{E_e^0 - E_g^0 - \hbar\omega_L} \right) \right] \langle E(r, t)^2 \rangle,$$  \hspace{1cm} (2.22)

where the sign change in the denominators in parentheses have been cancelled by the overall sign change, and we have used the fact that the expectation value of the square of a sine wave is equal to half the amplitude of the sine wave. Note that this equation has the same form as Eq. (2.3), suggesting that we define the quantity in brackets as the ac polarizability:

$$\alpha_s(\omega_L) = (\hat{z} \cdot \hat{e})^2 d_{eg}^2 \left( \frac{1}{E_e^0 - E_g^0 + \hbar\omega_L} + \frac{1}{E_e^0 - E_g^0 - \hbar\omega_L} \right).$$  \hspace{1cm} (2.23)

Now, in the dipole force approximation, we assume that the laser frequency is close enough to the atomic resonance that the second term in parentheses, which is proportional to the reciprocal of the detuning between the laser and transition frequencies, will dominate. We therefore neglect the first term and rewrite Eq. (2.23) as [17]

$$\alpha_s(\omega_L) \approx (\hat{z} \cdot \hat{e} d_{eg})^2 \left( \frac{1}{E_e^0 - E_g^0 - \hbar\omega_L} \right).$$  \hspace{1cm} (2.24)

Since we are still far off resonance, this approximation is also imperfect. The term we have neglected in fact contributes about 13% of the ac polarizability of the dominant transition.
Up to this point, we have been assuming that the excited state never undergoes spontaneous emission [17]. In the QUEST regime, this assumption makes sense because the radiation is so far detuned that the population of the excited state is negligible anyway. In the dipole regime, however, we must take into account the finite lifetime of the excited state.

Pethick and Smith [17] proceed by noting that a finite lifetime implies a decay envelope on the wavefunction. We may therefore assume that the excited state wavefunction takes the separable form:

$$|e⟩ = |ψ(r)⟩ e^{(-iω − Γ_e/2)t}$$

so that the probability amplitude varies as $$⟨e|e⟩ = e^{-Γ_e t}$$. This implies that $$Γ_e$$ is the lifetime of the state. We also know [10] that separable solutions take the form $$|ψ(r)⟩ e^{-iEt/\hbar}$$, where $$E$$ is the energy of the state. Thus, we can interpret the state lifetime as an imaginary term in the energy of the excited state, leading us to the substitution $$E^0_e → E^0_e − iℏΓ_e/2$$ in Eq. (2.24). Plugging this new term into Eq. (2.22) gives the perturbation in the ground state energy due to the ac Stark effect:

$$E'^2_g ≈ \frac{1}{2} \left( \frac{(e · \hat{z} \cdot d_{eg})^2 ⟨E(r,t)^2⟩}{(E^0_e − E^0_g − ℏω_L)^2 + (ℏΓ_e/2)^2} \right) \left( (E^0_e − E^0_g − ℏω_L) + iℏΓ_e/2 \right) .$$

Thus, like the excited state energy, this ground state energy shift consists of a real part, which may be interpreted as the effective potential, and an imaginary part, which is related to the lifetime of the ground state. The ground state lifetime is finite because even a far off-resonant radiation field will eventually excite the atoms.

To simplify the form of the effective potential $$ℜ(E'^2_g)$$, we introduce the
quantities:

\[ \Omega_R = \langle e | d \cdot \mathbf{E}(r, t) | g \rangle / (2\hbar) = \hat{e} \cdot \hat{z} \, d_{eg} \sqrt{\langle \mathbf{E}(r, t)^2 \rangle / 2} / \hbar, \quad \text{and} \quad (2.27) \]

\[ \Delta = \omega - \omega_L = (E_e^0 - E_g^0) / \hbar - \omega_L, \quad (2.28) \]

where \( \Omega_R \) is the Rabi frequency, and \( \Delta \) is the detuning. With these substitutions, the pseudopotential \( U = \Re(E_g'^2) \) becomes [17]:

\[ U \approx -\frac{\hbar \Delta \Omega_R^2}{\Delta^2 + \Gamma_e^2 / 4}. \quad (2.29) \]

The lifetime of atoms in this pseudopotential is given by \( \Gamma_g = -\frac{2}{\hbar} \Im(E_g'^2) \), or:

\[ \Gamma_g \approx \frac{\Gamma \Omega_R^2}{\Delta^2 + \Gamma_e^2 / 4}. \quad (2.30) \]

All of the quantities in these formulae are physically determinable. The Rabi frequency is just [2]:

\[ \Omega_R = \frac{\Gamma_e}{2} \sqrt{\frac{I}{2I_{sat}}}, \quad (2.31) \]

where \( I \) is the radiation intensity determined by our beam power and waist size, and \( I_{sat} \) is the saturation intensity, which is given in Appendix A for the relevant transitions. The linewidth \( \Gamma_e \) is likewise given in Appendix A, as are the atomic transitions from which we can determine the detuning \( \Delta \).

For our experimental setup, the relevant transitions are the \( D_2 \) (5\(^2\)S\(_{1/2} \rightarrow \)5\(^2\)P\(_{1/2} \)) and \( D_1 \) (5\(^2\)S\(_{1/2} \rightarrow \)5\(^2\)P\(_{1/2} \)) lines. These are the lowest-frequency single-photon transitions available to the atom in the infrared range, at 780 nm and 795 nm, respectively; all other transitions are sufficiently far detuned from our 1064 nm laser beam to be negligible. I argued above that we can treat each of these transitions separately as a two-state system. This assertion is justified by a glance at the quality factors \( \omega / \Gamma_e \) for \( D_2 \) and \( D_1 \): The transition frequencies are THz apart, while the linewidths are mere MHz. We can therefore treat these transitions as “uncoupled springs,” meaning that the
overall pseudopotential is a linear combination of the pseudopotential for each transition.

So far, we have examined the general behavior of atoms in far off-resonant light and the relevant features of our atom $^{87}$Rb. The final step to modeling our trapping potential is calculating the beam intensity, which appears implicitly in the pseudopotential in Eq. (2.29) as one of the parameters of the Rabi frequency. Dan Guest [2] discusses a derivation of this result from Gaussian optics, and similar discussions can be found in any optics text, such as Ref. [32]. I will simply cite the result that a laser beam with focal waist $w_0$ traveling in free space has a two-dimensional Gaussian profile given by:

$$I(\rho, z) = \left( \frac{2P}{\pi w(z)^2} \right) e^{-2\rho^2/w(z)^2},$$  \hspace{1cm} (2.32)

where $z$ is the axial coordinate along the direction of propagation measured from the focus, $\rho$ is the radial coordinate, $P$ is the total beam power, and $w(z)$ is the $1/e^2$ intensity falloff radius at position $z$, given by the formula [2]:

$$w(z) = w_0 \sqrt{1 + \frac{z^2\lambda^2}{\pi^2w_0^4}},$$ \hspace{1cm} (2.33)

where $\lambda$ is the radiation frequency.

Thus, we can write down the beam intensity as a function of position as long as we know the total beam power, the wavelength, and the beam waist at the focus. The first quantity is easy to measure with an optical power meter. In our present setup, the beam powers are approximately 22.8(1) mW per programmed volt for FORT X and 21.4(1) mW per volt for FORT Y. The wavelength is also easy: We use a Nd:YVO$_4$ laser specified to have a 1064 nm beam.

The only quantity we are unable to obtain directly is the beam waist; however, we can calculate it indirectly from the trap itself using a program
written by Mike Goldman [33]: We simply find the minimum beam power at which the trap holds atoms and then use the pseudopotential in Eq. (2.29) to calculate the beam waist required for the potential barrier to vanish at that power. At the time of this writing, we believe that the beam waist for FORT X is approximately 30 µm and FORT Y is approximately 31 µm. The waist measurement process is unfortunately open to many systematic errors. For example, a tilted beam that lets atoms slide downward away from the focus or a violent transfer of atoms from the magnetic to the optical trap could both result in atoms being ejected from the trap for reasons that have nothing to do with beam waist or power. If anything, therefore, we are likely to underestimate our waist size, since other loss mechanisms would tend to simulate a weak focus.
Chapter 3

Refining the Optical Trap

After constructing the optical dipole trap, Dan Guest was able to load rotating condensates into it and obtain some preliminary results [2]. When we attempted to refine these results, however, we found that FORT Y drifted quickly out of alignment, introducing asymmetries into the trap that drastically reduced vortex and condensate lifetimes. The time scale of this drifting was on the order of minutes—too rapid to take a complete data set with a well-behaved trap. Moreover, the drift was often discrete and unpredictable, so that we could work for five or ten minutes without diminution in the quality of our results, and then in the next shot be unable to load any vortices at all into the trap. Figure 3.1 illustrates one such abrupt deterioration in data quality.

After many refinements to the FORT beam paths, which are discussed below, we saw some improvement in the stability of the trap. Figure 3.2 shows a series of data from November 20, 2007, in which the number of vortices in the trap remained stable for a period of about forty minutes, with some fluctuations in the number and variations in vortex quality.
Figure 3.1: Data taken May 21, 2007. Number of vortices (diamonds) and number of atoms (squares) plotted against the time the free evolution time of the condensate in a (0.8 V, 0.8 V) optical trap. The error bars are due to unclear vortices and vortices on the edge of a condensate. The smooth curves are offset exponential decays whose respective time constants are shown on the plot. The data points plotted in red were taken at the end of this set. They lie significantly below the trendlines, indicating a sudden deterioration in trap quality. We believe that this was caused by a change in the FORT Y beam alignment that made the trap rotationally asymmetrical, spinning down the condensate and making it lose atoms. For some data points, condensate images are shown for comparison. Note that the lattice structure does not stabilize until about 500ms, indicating some excitation introduced in the loading process. This excitation may be exacerbated when FORT Y drifts out of alignment, reducing trap lifetimes still further.

3.1 Improvements to the Beam Path

Figure 3.3 shows many of the modifications we have made to the optical trap to improve its stability as well as our ability to align, control, and monitor it.

The broad slotted mount foot on the 300 mm focusing lens for FORT Y provides stability as well as alignment advantages that will be discussed in Sec. 3.2.2. The stabilizing mount foot supports the final dichroic plate for FORT
Figure 3.2: Time series from November 20, 2007 showing the stability of the optical trap after six months of improvements. Whereas in May (See Fig. 3.1), vortex number used to drop precipitously on the five to ten minute scale, this chart shows its relative stability, with some fluctuations in number and quality, on a forty- to fifty-minute time scale. These data were taken in a (0.8V, 0.84V) trap after 500ms of free evolution.

Y, which we discovered rests on an uneven part of the table. The width of the foot helps to average out this unevenness, keeping the plate from rocking on its mount.

The 14 mm-diameter shutter replaces a smaller shutter, ensuring that our ∼2.2 mm-diameter beam experiences no clipping. A standard rule of thumb is never to send a beam through an aperture whose diameter is less than three times the beam diameter [2].

The “Apertured Plug” label refers to a rubber beam stop with a hole in the center placed in front of the photodetector. The reason for this is that the beam pick is a rather thick plate of glass that reflects two beams into the photodetector—a main one from the front of the plate and a weaker one from the back. The two beams are far enough apart that the weaker beam is near the
edge of the sensitive region of the photodiode, meaning that small fluctuations in its alignment could lead to fluctuating power readings. Since this photodiode signal is used to stabilize the beam power, this false signal could result in an unreliable overall beam power. The plug prevents this problem by blocking the subsidiary beam while allowing the main beam through to the center of the photodetector.

The cage mount shown in Fig. 3.4 has proved to be a particularly useful modification. It replaced a less versatile ThorLabs LM1-XY mount, which held the fiber at a fixed distance from the collimating lens but allowed it to move transversely with respect to it. These turned out to be precisely the reverse of the degrees of freedom we wished the fiber launch to have. The XY translation freedom merely introduced instability into the system in exchange for two degrees of freedom which were unnecessary once the fiber
and collimating lens were properly aligned. The ability to translate the fiber along the z-axis, by contrast, has profound advantages that will be discussed in Secs. 3.2.2 and 3.2.4.

Figure 3.4: Cage mount for the fiber launch. The fiber launch is secured to a plate that can be translated along the cage bars to adjust the distance between it and the collimating lens. This arrangement allows us to manipulate the position and size of the beam focus. The quarter- and half-wave plates fix the beam polarization.

In addition to these improvements in the beam path proper, we have also taken steps to monitor the beam drift. Toward this end, we have installed three mounts for quadrant photodetectors along the FORT Y beam path and one on the FORT X path. Quadrant photodetectors are devices for tracking beam position by separately measuring the light incident on each quarter of their surfaces and then reporting the voltage differences between their top and bottom halves and their left and right halves. These photodetectors can easily be moved in and out of the beam path to make alignment measurements and then allow the experiment to run.
So far, we have had little luck making use of them since they seem to be subject to at least as much drift as the beams themselves; however, if we can get them to remain stable, they could provide a helpful feedback signal that would simplify the alignment process considerably.

3.2 Alignment Techniques

In order to trap rotating condensates without introducing asymmetries that rapidly spin the condensates down, the foci of the two FORT beams, with waists of approximately 30 $\mu$m, must intersect orthogonally with each other at the location of the condensate, also approximately 30 $\mu$m in radius. The last available alignment optic is about 17 cm from the atoms, meaning that an angular deviation of just 20 arcsecs in this final mirror is enough to cause us to miss the atoms entirely. For optics upstream, matters are even worse, though they may be compensated somewhat by the presence of the 300 mm focusing lens just before the upward-facing mirror (See Fig. 3.5). The upshot is that beam alignment is an extremely delicate process. At first, we were losing our ability to trap atoms in FORT Y on an almost daily basis, which made correcting for the beam drift a highly time-consuming process—recovering signal alone could take up to four or five hours.

Our first step was therefore to streamline our alignment techniques. The basic procedures are clearly and carefully discussed in Dan Guest’s thesis [2], and I refer anyone looking for a beginning-to-end guide to the alignment process to his “Beam Alignment” section. However, some improvements have been made since the time of his writing. I hope that this section may serve as a reference to future students learning to optimize the optical trap. The bulk of alignment is done using the optics indicated in Fig. 3.5, and I will refer to optics by the labels in this diagram throughout the following sections.
3.2.1 Finding the Trap

Once the FORT beams have been carefully aligned with the horizontal MOT beams [2], the atoms will be well within range of the coarse actuators on the final dichroic plate (henceforth M2, following Dan’s nomenclature) placed about 17 cm from the center of the cell. These actuators have an angular range of $\sim 20^\circ$ and can therefore cover a roughly 60 mm $\times$ 60 mm square in the plane of the atoms. Unfortunately, since the atoms are only sensitive when M2 is within $\sim 20$ arcseconds of the correct position, hunting for signal at this point is impractical—it would take roughly 3000 shots to sweep out a region subtending a solid angle of $1^\circ \times 1^\circ$ ($\sim 3$ mm$^2$).

To overcome this difficulty, we use a “pilot beam” [2] that is on resonance with the atoms (wavelength=780 nm) and carefully aligned with the FORT
beam as a rough alignment tool (See Fig. 3.5). The atoms are much more sensitive to the presence of resonant light than to the far-detuned FORT beams: even a fraction of the 1.3 μW pilot beam is sufficient to excite condensate atoms, resulting in a visible depletion of the trapped atoms. We proceed by aligning the pilot to maximize the damage to a cloud of thermal atoms, which provides a broad target. We then switch to a condensate for increased sensitivity, using briefer and briefer pulses of pilot light to maximize signal (i.e., minimize the number of trapped atoms) as the alignment improves. Once we have optimized alignment using this method, we must turn to the FORT.

Unfortunately, the pilot beam rarely coincides so perfectly with the FORT that we can trap atoms immediately. Even if the two beams are perfectly aligned before the focusing lens, the foci will suffer a longitudinal displacement caused by chromatic aberration when passing through the lens. Since the FORT is in the infrared while the pilot is in the near infrared, the pilot beam will suffer more deflection when passing through a lens and therefore have a shorter focal length than the FORT, assuming both beams are collimated before the focusing lens. Moreover, if one of the beams is slightly off-center on the focusing lens, the bounces off the final two optics M1 and M2 can amplify the angular deviation of the two beams, resulting in a transverse displacement of the foci as well. The upshot is that we often have to hunt for signal with the FORT even after we have fully exploited the pilot beam.

Hunting for signal with the FORT is often the most time-consuming part of alignment. Fortunately, we made several discoveries and modifications to the beam path that expedited this process considerably. One of the major problems we noticed was that even slight alterations in the beam path could change the FORT’s collimation or the optical path length after the focusing lens, causing the beam to get badly out of focus. The depth of a single 100
mW FORT (approximately the strongest power we ever use) with a 30 µm focus decreases by roughly a factor of two if the beam is 1.1 mm out of focus (See Fig. 3.6).

Figure 3.6: Plot of the trap depth in µK of a 100 mW, 30 µm-waisted single-beam trap as a function of y, the distance along the beam axis, and z, the vertical distance. Both distances are measured in µm from the point of greatest trap depth, slightly below the beam focus. An atom 1100 µm along the beam from the focus sees a trap depth of about half the maximum. Our ability to trap atoms in a single beam is therefore fairly sensitive to the placement of the focus.

I will discuss our method for finding the focus in the next section; however, we can obtain signal on even a badly focused FORT as long as the other FORT is trapping. Our method is to load a condensate into the working FORT with a beam strength of about 44 mW (2V) and look for distortions in the condensate. We have found this method to be more sensitive than searching for signal using a thermal cloud, probably because a condensate is less energetic and therefore
settles more readily into local potential minima. We have also found the alignment method using the other FORT more effective than using atoms in the magnetic trap, possibly because the functioning single-beam trap provides a broad orthogonal target in the horizontal plane, so we only need to get the other FORT roughly aligned on the vertical axis in order to see signal. Fig. 3.7 shows a common distortion in a single-beam trap that indicates the proximity of the other beam.

Figure 3.7: Crashing helicopter-shaped distortion of a condensate trapped in FORT X caused by the proximity of the misaligned FORT Y. The green streak of spilling condensate on the right is caused by the presence of FORT Y; we corrected this misalignment by adjusting the beam leftward. We hypothesize that the spilling occurs because FORT Y exerts a force on the trapped atoms, imparting enough momentum that they eventually escape over the potential barrier. Though it is beyond the scope of this thesis, it may be fruitful to study and characterize such distortions further. See, for example, the discussion of gap-like distortions in Ref. [2].

If this method fails, the likely reasons are either that the pilot beam is out
of alignment with the FORT or that the focus of the FORT is too far off. In the former case, we must realign the pilot and start over. In the latter, we proceed to the methods of the next section.

3.2.2 Focus

Focusing a FORT beam is conceptually simple enough: we adjust the focusing lens (Fig. 3.5) along the beam until the focus intersects with the atoms. In practice, however, this process is fraught with challenges. When the beam is badly out of focus, the gradient in the signal is often insufficient to determine in which direction the lens should be moved unless a large range of lens positions is tested at the risk of worsening the misalignment.

Our trick for figuring this out is to adjust the fiber launch slightly (∼5 µm) towards or away from the collimating lens using the actuator on the cage mount and observe whether the signal worsens or improves. If we imagine the fiber as a point source of light and the collimating lens as a thin lens,\(^1\) we can understand how this technique works using the lens equation [34]:

\[
\frac{1}{s} + \frac{1}{s'} = \frac{1}{f},
\]

where \(s\) is the distance of the object in front of the lens, \(s'\) is the distance of the image behind the lens, and \(f\) is the lens’s focal length. If the fiber is at the focus, the image will be at \(\infty\), meaning that the beam is collimated. If the fiber is closer to the collimating lens than the focus, then the image distance must be negative, meaning that it is a virtual image on the same side of the lens as the fiber. Consequently, the beam must diverge as it passes through the

\(^1\)Our ray optics approach should be taken with a grain of salt. We must ignore dispersive effects and non-pointlike beam waists as well as the fact that the collimating lens is not thin and the fiber is an extended source. Gaussian optics provides a better description of the behavior of the beam; however, since we are not here concerned with quantitative accuracy, we use the simpler ray-optics method to get a qualitative understanding of the technique—correct within a sign, you might say.
lens. This causes the focus to move farther away from the final focusing lens. If, on the other hand, the fiber is farther from the collimating lens than the focus, the image distance will be positive, and the beam will converge. This causes the focus to move closer to the final focusing lens. These scenarios are illustrated in Fig. 3.8. Thus, if moving the fiber launch towards the collimating lens (clockwise rotation of the actuator) improves the signal, then we know that the focusing lens is too far from the cell. Conversely, if moving the fiber launch away from the collimating lens improves the signal, then we know that the focusing lens is too close to the cell.

Once the misaligned beam is trapping atoms, we optimize the focus and alignment by minimizing the potential at which it traps after the magnetic trap has been gradually ramped off over a period of 50 ms. The trap should also be symmetrical: At just below the minimum trapping potential, atoms should trickle out from the center of the trap. Spilling from either side is an indication either of poor focus or poor alignment. For these tests to be accurate indicators of the location of the focus, however, it is important to ensure that the beam is level. I will discuss the procedure for leveling in the next section.

The actual adjustment of the focusing lens is done with an actuator whose total travel is 12.7 mm at 0.35 mm per revolution. We often found that we ran out of travel after major beam overhauls and had to move the lens mount by hand on FORT Y—a painstaking process that often caused us to lose signal entirely. To alleviate this problem, we designed a mounting foot the FORT Y focusing lens that is bolted to the table through parallel slots (See Fig. 3.5). Since the beam path is designed to run parallel to the holes in the optical table, constraining the mount to move along these slots ensures that once the lens is aligned, we can change the focus without changing the beam pointing.
3.2.3 Leveling the FORT

The process of leveling the beam is bound to be headache-inducing since it requires changing the height of the final optic M2. If M2 is too high, the beam will slope downward when the focus intersects with the atoms. Likewise, if M2
is too low, the beam will slope upward when it is properly aligned. Even after careful placement of the mirror to align the FORT with the MOT beam, as described in Ref. [2], it is common for the mirror to be displaced by 1–2 mm, resulting in a tilt of about 0.5°. Figure 3.9 shows the consequences of a tilt of this magnitude for the trapping potential; as one might expect, the atoms tend to slide down the slope, resulting in a lower trapping potential barrier in the z-direction and a potential minimum offset from the focus. As Fig. 3.9 shows, this effect is surprisingly pronounced at a tilt of only 0.5°.

To determine which way the beam is tilted, we trap a condensate in the weakest single-beam trap that holds atoms—usually about 44 mW or 2V. Then we allow it to evolve for several hundred milliseconds and image it using TopView to see which way it has drifted. For FORT Y, an upward drift on the image corresponds to an upward-tilted beam, and a downward drift on the image corresponds to a downward-tilted beam. For FORT X, a leftward drift means an upward tilt, and a rightward drift means a downward tilt. We then move the final plate M2 up or down its post—up if the beam is tilted upward and down if downward—by about 1 mm at a time.

For realignment purposes, it helps to observe certain protocols: Before loosening the plate mount’s clamp, we measure its distance from the top of the post and mark its position in case we need to restore it. We also turn on the beam to a low power and center it on a target on the far side of the cell beneath the beam-dumping dichroic plate, or, in the case of FORT Y, we record the voltages of the quadrant photodetector in that position. Then, after moving M2 to the desired height, we adjust its pitch and yaw to restore the beam’s position on the target or quadrant photodetector. We then resume the alignment process, returning to the pilot beam if we fail to get signal using the FORT. Note that after a change in the height of M2 the beam is likely to be
Figure 3.9: Plots (a) and (b) show the turning point and trap depth, respectively, of a 66 mW beam (33 µm waist) as a function of tilt angle. The turning points meet and the trap depth goes to zero at \( \sim 0.45^\circ \), implying that even a modest beam slope can destroy our ability to trap stably. Plots (c) and (d) show the trapping potential of an 88 mW trap with and without a 0.5° tilt, respectively, as a function of the along-beam distance from the focus \( x \) and the cross-beam distance \( y \). Plots (e) and (f) show a 44 mW trap under similar conditions. The tilted 88 mW trap barely confines atoms while the tilted 44 mW trap does not confine at all. Calculations by Mike Goldman [33].

out of focus since the optical path length from the focusing lens has changed.

According to our numerical model [33], a 44 mW trap with a 33 µm waist will trap until the tilt is just under 0.3°, so this is approximately our uncer-
tainty in our beam tilt. The trapping potential minimum is then at most about 2.5 mm from the beam’s actual focus. However, the Rayleigh range, which characterizes the length of a beam’s focus [2] is given by:

\[ z_0 = \frac{\pi w_0^2}{\lambda}, \]  

(3.2)

where \( w_0 \) is the beam waist and \( \lambda \) is the wavelength. For the beam described above, this quantity is about 3 mm, so our procedure brings us within range of the tightest part of the focus.

### 3.2.4 Waist Size

One of our major improvements in the beam this year is the addition of the cage mount for the fiber launch and collimating lens system. A photograph of this setup is shown in Fig. 3.4. Prior to this installment, the fiber was set at a fixed distance from the collimating lens, and the two moved together. The cage mount with the z-axis actuator gives us the freedom to adjust the distance between the fiber launch and the collimating lens. One advantage of this option has been discussed in Sec. 3.2.2 of this thesis, but the more important advantage is the ability to adjust the waist size of the beam at its focus.

For a collimated laser beam, the diffraction-limited beam waist at the focus is given to a good approximation by [35]:

\[ w_0 = \frac{0.61 f \lambda}{w}, \]  

(3.3)

where \( f \) is the focal length of the lens, \( \lambda \) is the wavelength of the light, and \( w \) is the beam waist before the lens. The noteworthy aspect of this relation for our purposes is that the focal spot size is inversely related to the beam diameter. Figure 3.8 therefore suggests a method for adjusting the beam waist using the collimating lens: Moving the fiber closer to the lens causes the beam to
diverge slightly, resulting in a larger beam and consequently a tighter focus; moving the fiber farther away, on the other hand, causes the beam to converge, resulting in a smaller beam and a weaker focus. Of course, we would also like to preserve the beams’ collimation as much as possible, but the optical path length between the fiber launch and the focusing lens is roughly 2.5 m, so the angular divergence required to make significant changes in the spot size is negligible. For example, if we wanted to halve the waist at the focus, we would double the waist at the lens; and for our usual prelens waist size of about 3 mm, this would require an average divergence of only about 0.4°.

Since the waist size in part determines the tightness of the trap (See Eq. (2.29)), our ability to change the distance between the fiber and the collimating lens is an important adjustment knob both for optimization of the trap and for studying its properties. For example, we found that the cross-beam trap is still able to support vortex-laden condensates despite a profound (∼2:1 ratio) asymmetry in beam waists, as long as the beam powers are adjusted to compensate. This suggests that these two parameters, beam power and beam waist, are fairly interchangeable in terms of securing the rotational symmetry of the trap.

### 3.2.5 Fine Alignment Tricks

Once the beams are separately focused, leveled, and aligned using the techniques described above, they may still not be well enough aligned with each other and with the atoms to support a vortex lattice. At this stage, all the alignment is done using the fine actuators on M2 (See Fig. 3.5), and we have developed several techniques to expedite the process.

First [2], we turn on the two-beam trap to a potential just low enough that a small stream of atoms is able to escape from the bottom of the trap, as shown
in Fig. 3.10. Then we tweak the actuators on one of the beams, usually FORT Y, about 1 µm at a time until the stream is shut off. We then reduce the beam powers to restore the stream and iterate this process until optimization. At our current waist size and beam power calibration, the minimum two-beam trapping potential is usually approximately (0.8 V, 0.8 V), or (18 mW, 17 mW).

Figure 3.10: Trap spilling used for alignment. This image shows a condensate spilling slightly from a (0.75 V, 0.75V) or (17 mW, 16 mW) trap after the magnetic trap has been ramped off.

If we still fail to sustain a condensate with vortex lines, vertical misalignment of the beams is the usual culprit. The trap is much less sensitive to horizontal deviations because of the long Rayleigh range discussed earlier: As long as the beams intersect each other within ∼ 3 mm of each other’s foci, and the intersection is near the center of the magnetic trap, we will not notice a significant difference in the trapping potential. If the beams are vertically misaligned, on the other hand, they may not intersect at all.

We have learned to interpret the distortions in the condensate imaged from the top rather than from the side in terms of vertical misalignment. A condensate stretched out in the vertical direction means that FORT Y is above FORT X, while a condensate stretched out in the horizontal direction means the reverse. We correct these deviations as explained in Fig. 3.11; usually an...
adjustment of a micron or two on the vertical actuator is sufficient.

Vertically elongated condensate:
- FORT Y vertical actuator CCW
- FORT X vertical actuator CW

Horizontally elongated condensate:
- FORT Y vertical actuator CW
- FORT X vertical actuator CCW

Figure 3.11: Vertical fine alignment signal.

The last stage of alignment is done by trial and error on the vortices themselves. We attempt to load vortices into the trap and then adjust the final dichroic plates to maximize their number and visual sharpness. We have discovered that a lack of vortex clarity, exhibited in Fig. 3.12, is often the result of beam misalignment, though it may also be an artifact of the imaging process. For example, it can be caused by an incorrect probe beam detuning or by the condensate’s being out of the focus of the probe beam. If realignment does not seem to improve vortex quality, we test for these imaging problems by changing the detuning and the time the condensate is allowed to fall before the image is taken.
Figure 3.12: Clarity comparison for vortex lines. Image (a) shows a condensate with ideal vortex appearance sharp, narrow, and deep blue. The vortex lines in image (b), by contrast, are partly obscured by thermal atoms, which are also apparent in the blue halo around the condensate. Their presence indicates either that the trap is too tight and is consequently capturing thermal atoms or producing them by causing the condensate to become too dense, or that poor alignment is spinning down and heating the atoms. Image (c) likewise exhibits unclear vortex lines, but no thermal cloud is in evidence. This may be due to either a problem with the imaging, as discussed in Sec. 3.2.5, or to the condensate’s being rotated at an angle from the vertical due to misalignment, causing some of the vortex lines to appear as stripes.
Chapter 4

Vortices in an Optically Trapped BEC

4.1 Quantized Vorticity

We might test whether an ordinary fluid is rotating about a given point by placing a small paddle wheel at that point and observing whether it rotates. This is exactly the same as the test for whether a vector field has curl, or equivalently, whether the integral of the field about some sufficiently small closed loop is nonzero. We may conclude that an ordinary fluid is rotating about any given point when the velocity field of the fluid elements has curl about that point. By analogy, we define the condition for a rotating flow in a superfluid to be:

\[ \nabla \times \mathbf{v} \neq 0 \quad \text{or} \quad \kappa \equiv \oint_{L} \mathbf{v} \cdot d\mathbf{l} \neq 0, \]  

(4.1)

where \(\mathbf{v}\) is the fluid velocity, \(L\) is some closed path, and \(\kappa\) is called the “circulation.”

The trouble is that while an ordinary fluid is described by real spatial and momentum distributions, a superfluid is a quantum mechanical object whose observable properties are encoded in a complex wavefunction. Writing down the superfluid velocity therefore takes some work. Following the method of
Pethick and Smith [17] and Tilley and Tilley [36], we begin by deriving the continuity equation from the Gross-Pitaevskii equation, Eq. (1.9), reproduced here:

\[
i\hbar \frac{\partial}{\partial t} \Psi(r, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + U_0|\Psi(r, t)|^2 \right] \Psi(r, t),
\]

(4.2)

where \(\Psi(r, t)\) is the overall condensate wavefunction, \(m\) is the atomic mass, \(V(r)\) is the external, time-independent potential, and \(U_0\) is the mean-field term. Multiplying this equation by \(\Psi^*(r, t)\) and subtracting the complex conjugate of the resulting equation eliminates the potential energy term \(V(r)|\Psi(r, t)|^2\) and the mean-field term \(U_0|\Psi(r, t)|^4\), leaving:

\[
i\hbar \left[ \Psi^*(r, t) \frac{\partial}{\partial t} \Psi(r, t) + \text{c.c.} \right] = -\frac{\hbar^2}{2m} \left[ \Psi^*(r, t) \nabla^2 \Psi(r, t) - \text{c.c.} \right],
\]

(4.3)

where c.c. stands for the complex conjugate of the previous term. We can hammer this equation into a recognizable form by applying the product rule to the left hand side and factoring out one of the \(\nabla\)'s from the right hand side.

Dividing both sides of the new equation by \(i\hbar\) yields:

\[
\frac{\partial}{\partial t} |\Psi(r, t)|^2 = -\frac{i\hbar}{2m} \nabla \cdot [\Psi(r, t) \nabla \Psi^*(r, t) - \text{c.c.}].
\]

(4.4)

This has the form of a continuity equation:

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J},
\]

(4.5)

where \(\rho\) is the density of some quantity and \(\mathbf{J}\) is the volume current density of that quantity, equal to \(\rho \mathbf{v}\), where \(\mathbf{v}\) is the quantity’s velocity. Continuity equations are statements of conservation: the rate of change of a quantity’s density in a region is minus the rate of outflow of that quantity from the region.

In the case of Eq. (4.4), \(|\Psi(r, t)|^2\) may be interpreted as the condensate’s particle density,\(^1\) from which we may conclude that the particle current density

---

\(^1\)Ordinarily, the \(\Psi\) in the Schrödinger equation represents probability amplitude and \(|\Psi|^2\) probability density, but for a many-particle object like BEC, the probability density for a single particle to be in a region corresponds to the actual density of particles in that region [36].
is given by:
\[ J_m = \frac{i\hbar}{2m} [\Psi(r, t) \nabla \Psi^*(r, t) - c.c.] \tag{4.6} \]
and the particle velocity by:
\[ \mathbf{v} = \frac{i\hbar}{2m} \left( \frac{\Psi(r, t) \nabla \Psi^*(r, t) - c.c.}{|\Psi|^2} \right) . \tag{4.7} \]

At this point, it is convenient [17, 36] to write the wavefunction \( \Psi(r, t) \) in the general form:
\[ \Psi(r, t) \equiv \psi_0(r, t) e^{i\phi(r, t)}, \tag{4.8} \]
where \( \psi_0(r, t) \) is the real amplitude of the wavefunction and \( \phi(r, t) \) is the phase. If we substitute this into Eq. (4.7) and observe that \( z - z^* = 2i\Im(z) \), we obtain:
\[ \mathbf{v} = \frac{-\hbar}{m} \Im \left( \psi_0(r, t) e^{i\phi(r, t)} \nabla \left[ \frac{\psi_0(r, t) e^{-i\phi(r, t)}}{\psi_0^2} \right] \right) . \tag{4.9} \]
The gradient in the numerator of Eq. (4.9) simplifies to \( \nabla \psi_0 - i\psi_0 \nabla \phi \), so all the exponential terms in the numerator cancel out. Taking the imaginary part leaves:
\[ \mathbf{v} = \frac{\hbar}{m} \nabla \phi \tag{4.10} \]
as the final simplification of the condensate velocity.

Substituting this result into our condition for rotation from Eq. (4.1) gives \( \frac{\hbar}{m} (\nabla \times \nabla) \phi \neq 0 \); however, the curl of a gradient is always zero in a simply connected region. Thus, a superfluid can only be rotating if its phase has a singularity [17]. This curious result implies that a superfluid with angular momentum has at least one forbidden region. The vortex cores in Fig. 4.1(b) are visible manifestations of these predicted topological defects.

The quantization of angular momentum in a superfluid arises from the usual requirement that the wavefunction be single-valued. To see this, consider the circulation \( \kappa \), defined in Eq. (4.1). If we substitute Eq. (4.10) for velocity...
into this equation, we obtain:

$$\kappa = \frac{\hbar}{m} \oint_L \nabla \phi \cdot dl.$$  \hspace{1cm} (4.11)

By the fundamental theorem of calculus, this integral evaluates to [36]:

$$\kappa = \frac{\hbar}{m} (\Delta \phi)_L$$  \hspace{1cm} (4.12)

where \((\Delta \phi)_L\) is the change in phase about the closed loop \(L\). Since the wavefunction in Eq. (4.8) can take on only one value at any given point, \((\Delta \phi)_L\) must be a multiple of \(2\pi\). From this requirement we derive the quantization condition of the circulation:

$$\kappa = \frac{n\hbar}{m}$$  \hspace{1cm} (4.13)

where \(n = 0, 1, 2, \text{etc.}\)

Figure 4.1: Two images of rotating BECs. Image (a) exhibits distinct vortex lines which are the visible manifestation of the theoretically predicted forbidden region. Image (b) shows an instance of the simplest case of a single vortex line along the axis of a rotationally symmetric condensate.

The fact that the atoms in our condensates repel one another means that it is energetically favorable for each vortex to carry one quantum of circulation. The vortices also mutually “repel” one another, forming a regular triangle lattice to minimize the total energy.
4.2 Results

With the improved stability in the optical trap, we were able to perform some preliminary experiments on optically trapped vortex-laden condensates. In particular, we examined the lifetimes of vortex lattices, observed vortex-laden spinor condensates, and examined the phenomenon of rotational atom lasers. I hope that these results may prove a jumping-off point for future studies.

4.2.1 Vortex Lifetimes

We found that both our ability to trap vortex-laden condensates and the lifetime of the vortex number were strongly dependent on the quality of the optical trap from day to day. Nevertheless, our vortex lifetime data suggest a correlation between the loss rates of vortices and condensate atoms in addition to raising some interesting questions about other loss mechanisms and the possibility of configuring the trap to optimize vortex lifetimes.

Figure 4.2 shows two time series of vortex and atom number decays, one from March and one from October of 2007. The trapping potentials are similar, but the time series from October was taken after a significant number of improvements in the beam path (See Sec. 3.1). The atom and vortex number lifetimes from October are within uncertainty of those from March. The more striking feature of these plots is that the time constants for vortex and atom number are nearly equal for both data series. This strongly suggests that the vortex lifetime is limited by the departure of angular momentum-bearing atoms from the trap. This is a pleasing discovery since it implies that the asymmetry of the optical trap is not the dominant loss mechanism. A previous worry had been that thermal cloud could become pinned by asymmetries in the trap, and collisions with the condensate atoms could deprive the condensate of angular momentum. Much of our data, however, suggest that the
trap is round enough that the rotating condensate is not dragged to a stop more rapidly than the natural lifetime of the condensate in the trap. Since the trap is formed by two beams in the shape of a plus sign, it is not at all obvious that this should be the case.

Our hypothesis that the loss of atoms from the condensate dominates the vortex decay rate is mostly borne out by the data—though one of the exceptions is fairly damning. As shown in Fig. 4.3, on four of the six dates on which lifetime data were taken, the vortex lifetime was within uncertainty of the condensate lifetime. On June 6, 2007, the vortices decayed faster, possibly due to beam misalignment or a difference in beam waists. Both of these defects could cause asymmetries that would tend to spin down the rotating condensate.

The result of September 22, however, in which the atom number decayed significantly faster than the vortex number, is more difficult to explain. Even if the depletion of atoms from the condensate is not the primary mechanism for vortex decay, one might expect it to set an upper limit on the lifetime. One possible explanation for the September 22 result might be that the lifetime of the vortices is not well modeled by an exponential decay with a single time constant. It might instead be governed by two or more mechanisms with decays of roughly equal weight. If this were the case, we would not necessarily expect the lifetime of the vortices always to appear less than or equal to the atom number lifetime since it might be artificially inflated by the other loss rates. In any case, the possibility of other loss mechanisms merits further exploration.

Another interesting feature of Fig. 4.3 and 4.2 is that despite the improvements in beam stability and control over the beam power and waist size made between March and October, the lifetimes show no significant overall increase. This is disappointing, but it also poses a new challenge. We expected that a better-controlled trap would give us stabler condensates and vortex lattices,
Figure 4.2: Time series comparison between March and October.

Mar. 9: Vortex and Atom Number Decay in a (0.75 V, 0.75V) Trap

Oct. 5: Vortex and Atom Number Decay in a (0.8V, 0.8V) Trap

Exponential decay:
\[ \tau_{\text{atoms}} = 1.8(3) \text{s} \]
\[ \tau_{\text{vortices}} = 2.3(5) \text{s} \]

Exponential Decay
\[ \tau_{\text{atoms}} = 3.1(3) \text{s} \]
\[ \tau_{\text{vortices}} = 3.0(9) \text{s} \]
Figure 4.3: Time series comparison between March and October, 2007.

and its failure to do so raises the question of how else we might approach optimizing the trap for vortex-laden condensates. One avenue we might explore is testing the effect of the beam waists on the lifetimes since it is possible that the roundness of the trap depends on the strength of the focus.

4.2.2 Vortex-Laden Spinor Condensate

A true spinor condensate is one in which all atomic spin states are accessible. Spinor condensates are somewhat difficult to come by because they must be confined non-magnetically. As discussed above, only certain spin states are
magnetically trappable: A magnetic field minimum repels the \( |2, -2\rangle, |2, -1\rangle, \) and \( |1, 1\rangle \) hyperfine ground states of \(^{87}\text{Rb}\) and has no effect on the \( |2, 0\rangle \) and \( |1, 0\rangle \) states. A pure optical trap is therefore required for the study of spinor condensates.

Figure 4.4: Vortices in a spinor condensate in the lower hyperfine level. This condensate was created and stirred in the magnetic trap in the \( |1, -1\rangle \) state, then loaded into the optical trap and subjected to an evenly distributing Landau-Zener sweep \[33\]. The condensate was then subjected to a magnetic field gradient (pointing diagonally down and to the right, in this image) that acted as a Stern-Gerlach discriminator, spatially separating the spin states. Note that the vortex lines appear to have overlapped before the spin states were separated, indicating that the components of the rotating condensate do not immediately separate.

Spinor condensates have been observed in optical traps in the past \[37, 38\], and there is an extensive body of theoretical work on vortex states in a spinor condensate \[39–42\]. We believe that we are the first group to have actually observed vortices in a spinor condensate (See Fig. 4.4) using our cross-beam dipole trap. We achieve this by loading vortices into a condensate in either the \( |2, 2\rangle \) or \( |1, -1\rangle \) state while it is in the magnetic trap, then loading the rotating condensate into the optical trap. Once in the optical trap, the condensate is distributed across the spin states, either by switching off the magnetic field.
to allow the spin states to evolve freely (See Fig. 4.5) or by subjecting it to a Landau-Zener sweep as described in Mike Goldman’s thesis [33]. Our preliminary results indicate that the stray magnetic field gradients in the trap are sufficiently small that the spatial superposition of spin states is preserved. Moreover, in the absence of magnetic field manipulations, the vortices of the different spin states seem to coincide and exhibit the usual evolution dynamics on short timescales, indicating that the macroscopic rotation is only weakly coupled to the atomic spin.

Figure 4.5: Spinor condensate in the upper hyperfine level. This even distribution of states across the upper hyperfine manifold was created by loading the optical trap with atoms in the $|2, 2\rangle$ state and allowing it to evolve freely in the absence of magnetic fields for 75 ms. The condensate was then subjected to a magnetic field gradient that acted as a Stern-Gerlach discriminator. Some spatial separation appears to have occurred due to the relative buoyancies of the spin states even prior to the Stern-Gerlach discrimination: Note the bottom-heavy and top-heavy appearance of the $|2, -2\rangle$ and $|2, -1\rangle$ states, respectively, as well as the horizontal elongation of the $|2, 1\rangle$ and $|2, 2\rangle$ states.

Our ability to create vortex-laden spinor condensates opens up new possibilities for the study of vortex lattice dynamics and the exchange of angular momentum between the macroscopic rotation about the center of mass and the atomic spin. These experiments will require a way to control the coupling between the vortex and spin degrees of freedom, for example, by using Feshbach resonances to manipulate the interspecies interactions, which in turn change the fluid dynamics of the condensate. Figure 4.5 suggests that some of these spinor-induced fluid dynamical evolutions may also occur spontaneously.
due to the relative buoyancies of the spin states. One possible experiment is simply to observe a vortex-laden spinor condensate over longer time scales to see whether these dynamics affect the structure of the vortex lattice.

### 4.2.3 Vortex-Laden Atom Laser

BEC in motion is often referred to as a “matter laser” because like an actual laser, it is a beam of coherent particles that can travel long distances without spreading, be focused down to a point [43], be reflected, be stored in a resonant cavity [44], and even undergo stimulated emission [13]. The first BEC atom laser was produced in 1996, and atom lasers have been outcoupled from optical traps since 2003 [45, 46]; however, we believe that we are the first to observe rotational modes in an atom laser.

After the hard work of alignment is done, the process for producing a rotational atom laser is remarkably simple: We load a vortex-laden condensate into the optical trap and then ramp down the beam powers until the optical potential becomes too weak to support the condensate against gravity. Atoms then begin to drizzle out of the bottom—the same phenomenon we use to optimize the FORT alignment in Sec. 3.2.5. Figure 4.6 shows a sampling of the atom lasers we produce by this method.

Our rotating atom lasers display several intriguing features that are consistent with an intuitive understanding of atom laser behavior. First, note that the atom laser with a single vortex line is wider than the non-rotating atom laser and that the multiple-vortex laser is wider than either of these. This is because the condensate atoms in the rotational mode have a transverse velocity component and thus tend to spread outward when released from the trap. Condensates with more vortices have more angular momentum and so spread out more. Second, Fig. 4.8 shows the precession of the condensate in
Figure 4.6: Images of non-rotating (a), single vortex-bearing (b), and vortex lattice-bearing (c) condensates with atom laser beams outcoupled by gravity. The beams in (b) and (c) are broadened compared to (a) because of centrifugal force. Also note the apparent evolution of the denser part of the condensate in (b) from the left at the bottom of the image to the right at the top. This precession is further explored in Fig. 4.8. In a typical atom laser experiment, the optical trap is ramped down to a potential low enough to spill a stream of atoms and held at that level for about 15 ms before the condensate is released and imaged in the usual way.
Fig. 4.6(b) about the center of mass, as predicted by theory [47, 48]. All condensates in vortex states precess; an atom beam outcoupled from a rotating condensate is interesting in that it translates this temporal precession into a spatial precession about the beam axis, as shown conceptually in Fig. 4.7.

Figure 4.7: Conceptual diagram of the spatial precession of a single-vortex atom laser about its vortex core. The gray helix represents a surface of the same phase winding along the atom beam about the vortex core. From Ref. [47].
Figure 4.8: Density profiles for the atom laser in Fig. 4.6(b) at vertical intervals of 192 µm. The density minimum in the middle of each profile is due to the single vortex core. At the bottom of the laser, which corresponds to an earlier time, the density profile appears fairly symmetrical. As we move up the laser column, however, the density to the left of the vortex core increases to a peak at 576 µm, becomes nearly symmetrical again at 384 µm, and then develops a peak on the right at 0 µm. This is indicative of the precession of the condensate about the vortex core. The precession is more obvious near the top of the laser because the beam is accelerating due to gravity. Thus, although the profiles are at even increments in the spatial domain, the ones near the bottom are closer together in the time domain.
Chapter 5

Conclusion

The past year has been both fruitful and frustrating. At the beginning of the year, we hoped that the instability of the optical trap would be quickly resolved and that we would obtain extensive results on the experiments I outlined in the previous sections. Instead, the optical trap showed it had more tricks up its cladding than we expected, and stabilizing it proved to be a project unto itself. Despite two semesters of improvements, we still occasionally observe drifts that we cannot explain with reference to thermal fluctuations, loose optics, beam power instabilities, or anything else we can think of.

Nevertheless, we were able to make significant advances in understanding and stabilizing the beam. We streamlined the alignment procedure, installed new fiber launches that increased our control over the beam foci, stabilized many potentially loose optics, and examined the possibility of using quadrant photodetectors to monitor and correct for beam drift. The beams are altogether better controlled and understood than they were two semesters ago. I hope that these steps give future students a boost toward performing some of the fascinating experiments on optically trapped vortex-laden condensates treated so fleetingly here.
Appendix A

Properties of $^{87}\text{Rb}$

A.1 Atomic Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atomic Number</td>
<td>$Z$</td>
<td>37</td>
</tr>
<tr>
<td>Total Nucleons</td>
<td>$Z+N$</td>
<td>87</td>
</tr>
<tr>
<td>Atomic Mass</td>
<td>$m$</td>
<td>$86.909180,520(15)$ u</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1.443,160,60(11) \times 10^{-25}$ kg</td>
</tr>
<tr>
<td>Nuclear Spin</td>
<td>$I$</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>Nuclear Lifetime</td>
<td>$\tau_n$</td>
<td>$4.88 \times 10^{10}$ yr</td>
</tr>
<tr>
<td>Relative Natural Abundance</td>
<td>$\eta(^{87}\text{Rb})$</td>
<td>27.83(2)%</td>
</tr>
</tbody>
</table>

Table A.1: Data from Steck [25].
A.2 D₂ Transition Properties

\[ 5^2S_{\frac{1}{2}} \rightarrow 5^2P_{\frac{1}{2}} \]

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>(\omega_0)</td>
<td>(2\pi \times 384.230, 484.4685(62)) THz</td>
</tr>
<tr>
<td>Transition Energy</td>
<td>(\hbar\omega_0)</td>
<td>1.5589049439(58) eV</td>
</tr>
<tr>
<td>Wavelength (Vacuum)</td>
<td>(\lambda)</td>
<td>780.241209686(13) nm</td>
</tr>
<tr>
<td>Wavelength (Air)</td>
<td>(\lambda_{air})</td>
<td>780.241209686(13) nm</td>
</tr>
<tr>
<td>Wave Number (Vacuum)</td>
<td>(k/2\pi)</td>
<td>12816.54938993(21) cm(^{-1})</td>
</tr>
<tr>
<td>Lifetime</td>
<td>(\tau)</td>
<td>26.24(4) ns</td>
</tr>
<tr>
<td>Natural Line Width (FWHM)</td>
<td>(\Gamma)</td>
<td>38.11(6) \times 10^6 s(^{-1})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2\pi \times 6.065(9) MHz</td>
</tr>
<tr>
<td>Far-Detuned Saturation Intensity</td>
<td>(I_{sat})</td>
<td>2.503(3) mW/cm(^2)</td>
</tr>
</tbody>
</table>

Table A.2: The D₂ line is the most important transition used in our experiment since it has a cycling transition [25] that tends to drive the atoms into the \(|2, 2\rangle\) state. We lock our lasers to 780nm and manipulate the polarization to use D₂ for optical pumping, optical molasses, and imaging. This transition also provides an important term in the optical trapping pseudopotential. Data from Steck [25].
### A.3 D\(_1\) Transition Properties

\[ 5^2 S_{1/2} \rightarrow 5^2 P_{1/2} \]

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>( \omega_0 )</td>
<td>( 2\pi \times 377.10746354(4) ) THz</td>
</tr>
<tr>
<td>Transition Energy</td>
<td>( \hbar \omega_0 )</td>
<td>1.55959099(6) eV</td>
</tr>
<tr>
<td>Wavelength (Vacuum)</td>
<td>( \lambda )</td>
<td>794.9788509(8) nm</td>
</tr>
<tr>
<td>Wavelength (Air)</td>
<td>( \lambda_{air} )</td>
<td>794.76569 nm</td>
</tr>
<tr>
<td>Wave Number (Vacuum)</td>
<td>( k/2\pi )</td>
<td>12578.950985(13) cm(^{-1})</td>
</tr>
<tr>
<td>Lifetime</td>
<td>( \tau )</td>
<td>27.70(4) ns</td>
</tr>
<tr>
<td>Natural Line Width (FWHM)</td>
<td>( \Gamma )</td>
<td>( 36.10(5) \times 10^6 ) s(^{-1})</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 2\pi \times 5.746(8) ) MHz</td>
</tr>
<tr>
<td>Far-Detuned Saturation Intensity</td>
<td>( I_{sat} )</td>
<td>4.484(5) mW/cm(^2)</td>
</tr>
</tbody>
</table>

Table A.3: The D\(_1\) line lacks a cycling transition and is therefore less important to the main experiment, but as the lowest-frequency ground-to-excited state transition, it provides the dominant term to the optical trap pseudopotential. Data from Steck [25].
Appendix B

Quantizing the Radiation Field

This appendix shows how the analogy between a laser mode and a harmonic oscillator can be used to write down the electric field in terms of the raising and lowering operators $\alpha^\dagger$ and $\alpha$ for the harmonic oscillator potential [31]:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_L^2 \hat{q}^2$$

(B.1)

Let us first motivate this quantization by exploring the analogy: A laser mode is a single frequency $\omega_L = ck_L$ of radiation polarized in a single direction $\mathbf{e}_{\omega_L}$. The energy per photon in this mode is $\hbar\omega_L$, so the total energy in the mode is $n\hbar\omega_L$, where $n$ is the total number of photons in the mode. This suggests [31] that the laser mode can be treated as a quantum mechanical harmonic oscillator with adjacent levels spaced by one photon energy. In this picture, the absorption of a photon from the mode or the emission of a photon into the mode is equivalent to a transition down or up one energy level, respectively, and the electric field is analogous to the position of a mechanical oscillator.

Recall the atomic raising and lowering operators given in Eq. (2.6) and (2.7). While there were only two available atomic states, $|g\rangle$ and $|e\rangle$, the laser mode is defined by the number of photons it contains, so we write the orthogonal eigenstates as $|n\rangle$, where $n = 0, 1, 2, \text{ etc.}$ The laser mode raising
and lowering operators are therefore:

\[
\alpha^\dagger \equiv (\sqrt{n+1}|n+1\rangle\langle n|) \equiv (2\hbar m\omega_L)^{-\frac{1}{2}}(m\omega_L\hat{q} - i\hat{p}) \quad \text{and} \quad (B.2)
\]

\[
\alpha \equiv (\sqrt{n+1}|n\rangle\langle n+1|) \equiv (2\hbar m\omega_L)^{-\frac{1}{2}}(m\omega_L\hat{q} + i\hat{p}), \quad (B.3)
\]

where the Cartesian coordinate \( z \) from the atomic oscillator has been replaced with the generalized coordinate operator \( \hat{q} \), whose physical interpretation has nothing to do with an actual displacement.\(^1\) Rather, it is a variable with dimensions of length that is related to the amplitude of the electric field. The variable \( m \) also serves as a placeholder in this equation; it is a constant with dimensions of mass rather than a physical quantity.

Note the identities [31]:

\[
\alpha^\dagger\alpha = \frac{1}{\hbar\omega_L} \left[ \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_L^2\hat{q}^2 + \frac{i}{2}\omega_L (\hat{q}\hat{p} - \hat{p}\hat{q}) \right] \quad \text{and} \quad (B.4)
\]

\[
\alpha\alpha^\dagger = \frac{1}{\hbar\omega_L} \left[ \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_L^2\hat{q}^2 + \frac{i}{2}\omega_L (\hat{p}\hat{q} - \hat{q}\hat{p}) \right]. \quad (B.5)
\]

The first two terms in brackets in each equation are just the Hamiltonian for the harmonic oscillator potential given in Eq. (B.1). The expressions in parentheses can be simplified using the canonical commutation relation [10, 31]:

\[
[\hat{p}, \hat{q}] = -i\hbar. \quad (B.6)
\]

With these substitutions, Eq. (B.4) and (B.5) become:

\[
\alpha^\dagger\alpha = \frac{1}{\hbar\omega_L} \left( \hat{H} - \frac{\hbar}{2}\omega_L \right) \quad \text{and} \quad (B.7)
\]

\[
\alpha\alpha^\dagger = \frac{1}{\hbar\omega_L} \left( \hat{H} + \frac{\hbar}{2}\omega_L \right). \quad (B.8)
\]

\(^1\)I suppose \( \hat{q} \) could be interpreted as the displacement of the oscillating charge that emits or absorbs the photon, but since my aim is to write down the quantized electric field for the case of free propagation, I prefer the source-free picture.
Solving for $\hat{H}$ in terms of $\alpha^\dagger \alpha$ and $\alpha \alpha^\dagger$ gives:

$$\hat{H} = \frac{1}{2} \hbar \omega_L \left( \alpha^\dagger \alpha + \alpha \alpha^\dagger \right). \quad (B.9)$$

From these results and some observations from classical electrodynamics, we now proceed to motivate the quantization of the electric field given in Eq. (2.12). Consider a cubic region of space of volume $v$ and length $L$—a cavity without walls, as it were. Let the laser mode in question be the fundamental mode of this cavity, so that $\omega_L = \pi c / L$, and for simplicity, assume that the wave is traveling along the positive $z$-axis and is polarized in the $\hat{x}$ direction. Recall that in the Coulomb gauge, in which we choose the nonphysical quantity $\nabla \cdot A$ to be zero, the vector potential $A$ and scalar potential $V$ obey the field equation [29],

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) A + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla V = \mu_0 J, \quad (B.10)$$

where $J$ is the current density. The electric field is derived from the potentials by [29]:

$$E = -\nabla V - \frac{\partial A}{\partial t}. \quad (B.11)$$

By the Coulomb gauge assumption, the vector potential $A$ has no divergence, and the gradient of the scalar potential $V$ naturally has no curl; therefore, we can divide Eq. (B.10) into solenoidal and irrotational equations [31] to separate the behavior of the vector and scalar potentials:

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) A = \mu_0 J_S, \quad (B.12)$$

where $J_S$ represents the solenoidal part of the current density, and

$$\frac{1}{c^2} \frac{\partial}{\partial t} \nabla V = \mu_0 J_I, \quad (B.13)$$

where $J_I$ is the irrotational part of the current density. Turning back to the laser mode problem, we know since we are in current-free space that $J_S$ and
$\mathbf{J}_I$ are both zero. Eq. (B.13) then implies that the gradient of the scalar potential is a constant, meaning that it can contribute at most a constant overall vector field to $\mathbf{E}$. However, we will only be concerned with the oscillating part of $\mathbf{E}$ since a laser mode is a plane wave. Furthermore, since $\mathbf{J}_S$ vanishes in Eq. (B.12), the vector potential obeys the ordinary wave equation of electrodynamics. For a single laser mode,\(^2\) then, the general solution for the vector potential will the amplitude of a sinusoidal plane wave with the frequency, polarization, and direction of propagation specified above. We may write it in the following form:

$$
\mathbf{A} = \hat{x} \left( A e^{i(\omega_L t - k_L z)} + A^* e^{-i(\omega_L t - k_L z)} \right)
$$

(B.14)

where $k_L = \omega_L / c$ is the wavenumber and $A$ is a constant related to the amplitude. The requirement that the second order wave equation have two linearly independent solutions is satisfied by the real and imaginary parts of $A$, and the coefficients $A$ and $A^*$ are complex conjugates of each other to satisfy the requirement that the electric field be real. The electric field given by Eq. (B.11) is then:

$$
\mathbf{E} = -\hat{x} i\omega_L \left( A e^{i(\omega_L t - k_z)} - A^* e^{-i(\omega_L t - k_z)} \right).
$$

(B.15)

We also know from classical electrodynamics that the energy density in the laser mode is [29]:

$$
u = \epsilon_0 E^2.
$$

(B.16)

Using Eq. (B.11), the total energy $H = \int_v \nu \, dv'$ in the cavity can be written in terms of $A$ and $A^*$ [31]:

$$
H = -\epsilon_0 \omega_L^2 \int_v \left( A e^{i(\omega_L t - k_z)} - A^* e^{-i(\omega_L t - k_z)} \right) \, dv'
$$

(B.17)

$$
= -\epsilon_0 \omega_L^2 \left[ A^2 e^{2i\omega_L t} \int_v e^{-2ikz} \, dv' + c.c. \right] - (AA^* + A^* A) \int_v dv'
$$

(B.18)

\(^2\)We must specify a single laser mode because the most general solution would require summing over all possible frequency modes of the cavity.
Recall that we were examining the fundamental mode of the cavity, so that \( k = \frac{\pi}{L} \). The integrals of the form \( \int_{z_0}^{z_0+L} e^{\pm 2i k z} d z' \) are therefore integrals over a full period of the exponential function, so the first two terms of Eq. (B.18) vanish. The energy then simplifies to:

\[
H = \epsilon_0 \omega^2 v (A^* A + A A^*)
\]

(B.19)

Or, solving for \( v \) in terms of \( \omega_L \) using \( L = \pi c / \omega_L \):

\[
H = \frac{\epsilon_0 \pi^3 c^3}{\omega_L} (A^* A + A^* A)
\]

(B.20)

Note how reminiscent the classical formula for the field energy in Eq. (B.20) is of the quantum field energy in Eq. (B.9). This suggests that the classical amplitudes \( A \) and \( A^* \) are analogous to the quantum mechanical ladder operators \( \alpha \) and \( \alpha^\dagger \). In fact, if we take as the canonical substitutions [31] for a single laser mode:

\[
A \rightarrow \frac{\hbar \omega^2}{2 \epsilon_0 \pi^3 c^3} \alpha \quad \text{and} \quad A^* \rightarrow \frac{\hbar \omega^2}{2 \epsilon_0 \pi^3 c^3} \alpha^\dagger,
\]

(B.21)

(B.22)

the two Hamiltonians become equivalent. In the quantum picture, therefore, the electric field in Eq. (B.11) becomes:

\[
\hat{E} = \frac{i \hbar}{\epsilon_0} \left( \frac{\omega_L}{\pi c} \right)^3 \left( \alpha^\dagger e^{-i(\omega_L t - k z)} - \alpha e^{i(\omega_L t - k z)} \right).
\]

(B.23)

This result is the same as Eq. (2.12) used in the discussion of the optical dipole force, with the substitution \( \mathcal{E} = \frac{2 \pi \hbar}{\epsilon_0} \left( \frac{\omega L}{\pi c} \right)^3 \). This shows that is is appropriate to treat the radiation field in the quantum mechanical fashion required for an understanding of the optical dipole force.
Bibliography


