Physics 400, Fall 2011: Problem Set 1

Due Thursday Sept. 22, 11:59 pm

1. Nordlund 1.6
2. Nordlund 2.2
3. Nordlund 2.7
4. Nordlund 4.4
5. Nordlund 4.6
6. Nordlund 4.9
7. What is the Debye screening length for charge? How close do two charges need to get to feel a mutual electrostatic interaction?
1.6

- From eqns. 1.2 and 1.3, the eqn for linear drag is
  \[ f = b_1 v \]  
  and for quadratic drag is
  \[ f = b_2 v^2 \]

- What are reasonable values for \( b_1 \) and \( b_2 \)?

  From the textbook Classical Mechanics, by John Taylor, we can use
  \[
  b_1 = \beta D \\
  b_2 = \gamma D^2
  \]
  where \( D \) is the diameter of the sphere and
  \[
  \beta = 1.6 \times 10^{-4} \text{ Ns} m^{-1} \quad \text{(eq. 2.5 of Taylor)} \\
  \gamma = 0.25 \text{ Ns} m^{-2} \quad \text{(eq. 2.6 of Taylor)}
  \]

  For \( D = 10^{-5} \text{ m} \) (arbitrary choice), we have
  \[
  b_1 = (1.6 \times 10^{-4} \text{ Ns} m^{-1}) (10^{-5} \text{ m}) = 1.6 \times 10^{-9} \text{ Ns} m^{-1} \\
  b_2 = \gamma D^2 = (0.25 \text{ Ns} m^{-2}) (10^{-5} \text{ m})^2 = 2.5 \times 10^{-11} \text{ Ns}^2 m^{-2}
  \]

- Plot: see attached page

- Crossover:
  \[
  f_1 = f_2 \quad \Rightarrow \quad b_1 v = b_2 v^2 \\
  \frac{b_1}{b_2} = v \\
  v = \frac{1.6 \times 10^{-9} \text{ Ns} m^{-1}}{2.5 \times 10^{-11} \text{ Ns}^2 m^{-2}} = 64 \text{ m/s}
  \]
\[b_1 = 1.6 \times 10^{-9}\]
\[b_2 = 2.5 \times 10^{-11}\]

Solve for \(v\):
\[f_1(v) = b_1 v\]
\[f_2(v) = b_2 v^2\]

Plot \([f_1(v), f_2(v)], (v, 0, 100)\]
(1) Calculate the average distance between molecules in liquid water

- Density of water is \( \rho = \frac{1}{2} \text{ g/cm}^3 \)
- 1 mole of water = 18 g = 6.02 x \(10^{23}\) particles

so:

\[
\frac{1}{2} \text{ g/cm}^3 \cdot \frac{6.02 \times 10^{23} \text{ molecules}}{18 \text{ g}} = \frac{3.34 \times 10^{22} \text{ molecules}}{\text{cm}^2}
\]

or:

\[
3 \times 10^{-23} \text{ cm}^3 \text{ molecule}
\]

since 1 cm = \(10^{-8}\) A, this is

\[
\left(3 \times 10^{-23} \text{ cm}^3 / \text{ molecule}\right) \left(10^{-8} \text{ cm} / \text{ molecule}\right)^3 = 30 \text{ A}^3
\]

So if we imagine each molecule has a little box \(30 \text{ A}^3\) in volume

\[
\frac{30 \text{ A}^3}{2}\]

[Diagram of 3 molecules]

From the distance box between water molecules is \(r \approx \frac{30 \text{ A}^3}{2}\) center-to-center

(2) Ave distance between Na ions in 1 M aqueous soln of NaCl

\[
\left(\frac{1 \text{ mole NaCl}}{6.02 \times 10^{23} \text{ particles}}\right) \left(\frac{1 \text{ L}}{1000 \text{ cm}^3}\right) = 6.02 \times 10^{20} \text{ particles/cm}^3
\]

or take \(\frac{1}{\text{cm}^3}\) to get:

\[
\left(1.6 \times 10^{-21} \text{ cm}^3 / \text{ particle}\right)
\]

Take the cube root to get distance between particles:

\[
\left(1.2 \times 10^{-7} \text{ cm}\right) \left(\frac{10^{-10} \text{ cm}}{\text{ particle}}\right) = 1.2 \text{ nm}
\]

(3) From above, if the concentration is \(C \text{ mol NaCl}\), the distance between particles is

\[
\frac{C}{1} \left(1.2 \text{ nm}\right) = \frac{C^{1/3}}{(1.2 \times 10^{-7})^{1/3}} (10^{-9} \text{ m})
\]

(approximate)

This is by \(10^{-9} \text{ m}\)
So: \( \log_{10}(e^{\frac{1}{3}} \cdot 10^{-9}) = \log_{10}(10^{-9}) + \log_{10} e \)
\[= -9 + \frac{1}{3} \log_{10} e \]

If the x-axis is \( \log_{10} C \), the curve of the log-log plot is a line with slope \( \frac{1}{3} \) and y-intercept \( -9 \).

The slope of the line is \( -\frac{1}{3} \). If the concentration was expressed in different units, the y-intercept would change (no longer the \(-9 \) I have above), but the slope would be the same.
\texttt{In[24]:= (* Nordlund 2.2 *)}

\texttt{In[24]:= Plot[-1/3 x - 9, \{x, -12, 2\}]}

\texttt{Out[24]:=}

![Graph showing a line with equation $y = -\frac{1}{3}x - 9$.]
What are dairy cow's ave mass, daily consumption of food, ave discharge of methane (or other gases) in cubic meters (at 1 atm)?

I find:

- Wikipedia entry on Holstein: mass = 570 kg
- http://cpi.arsusda.gov/ke/dkindex.html:
  - Food intake/day = 40 lb = 18 kg
- Ave methane discharge: 480 liters/cm/day = 0.48 m³/cm/day (Google search under "methane discharge of cow" brings to MIT website with this)
- Other sites say 200 /day, 280 /day.
  - Let's say 300 /day, as a rough average.

(1) Determine average available energy, in J, per kg of food.

Take on trial:

- \( \frac{300 \text{ kcal}}{} \) for a 35 g swing
- 1 kcal = 4184 J

so: \( \frac{300 \text{ kcal}}{35 \text{ g}} \times \frac{1000 \text{ g}}{\text{kg}} \times \frac{4184 \text{ J}}{\text{1 kcal}} = 15500 \text{ J/kg} \)

available energy per kg of food.

(2) Cow's intake in food energy:

\( \frac{(18 \text{ kg})}{\text{day}} \times \frac{15500 \text{ J}}{\text{kg}} = 279,000 \text{ J/day} = 2.8 \times 10^5 \text{ J/day} \)

or \( = 2.8 \times 10^5 \text{ J/day} \)

- Average production of volatile gas energy
  - Combustion of methane gives an energy output of ~ 890 kJ/mole
  - Density of methane at 20°C = 0.67 kg/m³
  - Molecular wt. of methane = 16 g/mole

so: \( \frac{(0.3 \text{ m³})}{\text{day}} \times \frac{890 \text{ kJ}}{1 \text{ m³}} \times \frac{1 \text{ kg}}{1000 \text{ g}} \times \frac{1 \text{ mole}}{16 \text{ g}} \times \frac{1 \text{ day}}{\text{day}} = 12.5 \text{ moles methane/day} \)
Compare to car's efficiency

1. **Toyota Camry**
   - wt = 3240 lb
   - mass = \( \frac{3240 \text{ lb}}{2.2 \text{ lb}} = 1473 \text{ kg} \)
   - mileage = 25 miles \( \text{ gallon}^{-1} \)

   \[
   \text{convert (miles) to (meters)}
   \]

   - to do this, we need the fact that gasoline has an energy content of \( \frac{35 \text{ MJ}}{2} = 3.5 \times 10^7 \text{ J m}^{-1} \)

   - For the Camry
     \[
     \left( \frac{25 \text{ miles}}{\text{ gallon}} \right) \left( \frac{1 \text{ gallon}}{3.585 \text{ L}} \right) \left( \frac{1609 \text{ m}}{1 \text{ mile}} \right) \left( \frac{1 \text{ J}}{3.5 \times 10^7 \text{ J m}^{-1}} \right) = 3.0 \times 10^{-4} \text{ m J}^{-1}
     \]

   - Normalizing to 1000 kg, as Newton instructs:
     \[
     (3.0 \times 10^{-4} \text{ m J}^{-1}) \left( \frac{1000 \text{ kg}}{1473 \text{ kg}} \right) = 2.1 \times 10^{-4} \text{ m J}^{-1}
     \]

2. **Cow**: estimate a cow's normal 20 p.m. (meters) per day.

   - If we guess that a cow ambles along at a mile per hour over maybe six hours per day, we guess it moves
     \[
     \left( \frac{1 \text{ mi}}{\text{ hr}} \times 6 \text{ hr} \right) \left( \frac{1609 \text{ m}}{1 \text{ mi}} \right) = 9654 \text{ m day}^{-1}
     \]

   - It takes in \( 2.8 \times 10^{-6} \text{ J day}^{-1} \)

   - so:
     \[
     (9654 \text{ m day}^{-1}) \left( \frac{1 \text{ day}}{2.8 \times 10^{-6} \text{ J}} \right) = 3.4 \times 10^5 \text{ m J}^{-1}
     \]
Convert the cow's energy use to miles/gallon.

\[
(3.4 \times 10^{-5} \text{ m}^3) \times \left( \frac{1 \text{ mi}}{1609 \text{ m}} \right) \times \left( \frac{3.5 \times 10^7 \text{ J}}{3.7854 \text{ gal}} \right) = \left( \frac{2.8 \text{ mi}}{\text{ gal}} \right)
\]

<table>
<thead>
<tr>
<th>Cows</th>
<th>Car</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.8</td>
<td>25</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{mpg} & = 2.8 \\
\text{meters/Joule} & = 3.4 \times 10^{-5} \\
\text{meters/Joule} & = 3.0 \times 10^{-4}
\end{align*}
\]

\(*\text{Number of dairy cows in the US: } 9 \times 10^6 \text{ (wikipedia)}\)
- Each cow produces methane with energy content \(1.1 \times 10^{14} \text{ J} \text{ day}\) each day
- \( (9 \times 10^6 \text{ cows}) \times \left( \frac{1.1 \times 10^{14} \text{ J}}{\text{ cow day}} \right) \approx 10 \times 10^{17} \text{ J} \text{ day} \)
- \(10^{14} \frac{\text{ J}}{\text{ day}} \text{ from methane}\)
- Let's say a typical car drives 40 miles/day
  \[
  (40 \text{ mi}) \left( \frac{1609 \text{ m}}{\text{ mi}} \right) = 6.4 \times 10^4 \frac{\text{ m}}{\text{ day}}
  \]
  If it has an efficiency \(3.0 \times 10^{-4} \frac{\text{ m}}{\text{ Joule}}\), it uses
  \[
  \frac{6.4 \times 10^4 \frac{\text{ m}}{\text{ day}}}{3.0 \times 10^{-4} \frac{\text{ m}}{\text{ J}}} = 2.1 \times 10^7 \frac{\text{ J}}{\text{ day}}
  \]
- So, cow methane exhaust provides power equivalent to
  \[
  (10^{14} \frac{\text{ J}}{\text{ day}}) \left( \frac{1 \text{ cow day}}{2.1 \times 10^8 \text{ J}} \right) = 480,000 \text{ cars}
  \]
If ice didn't form on the surface and water became increasingly dense as it went from 40°C to 0°C until it froze, then the surface water would cool the surface water, as the surface water would sink and be replaced by warmer water, which would in turn be cooled. The convection would very efficiently cool the water in the pool until it finally began to freeze from the bottom up. Eventually, the entire pool would freeze. Some heat is from the Earth might keep a layer of liquid water at the bottom.

Smaller bodies of water would freeze solid relatively rapidly. They'd also thaw more slowly — warmer surface water wouldn't sink to melt the bottom ice, so in melting energy transfer would take place by conduction rather than convection and would take longer.

Smaller bodies of water would freeze quickly in winter as they slowly in summer. Larger bodies might not freeze over in summer and after thousands of years large bodies of water would freeze solid.
The electrostatic potential energy of a charge in a dielectric medium with a constant dielectric constant $\varepsilon = \varepsilon_0$ is

$$U = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{r} = \frac{1}{4\pi \varepsilon_0} \frac{e^2}{r}$$

The change in energy from $r_o = 5 \times 10^{-10}$ m to $r = x$ is

$$\Delta U = \frac{e^2}{4\pi \varepsilon_0} \left( \frac{1}{x} - \frac{1}{r_o} \right)$$

So, in cases (1) $\varepsilon = 4$ and (2) $\varepsilon = 78.5$

we take

$$20k_B T = \frac{e^2}{4\pi \varepsilon_0} \left( \frac{1}{x} - \frac{1}{r_o} \right)$$

From back of book

$$k_B T = 4.1 \times 10^{-21} J$$

We also know:

$$e = 1.6 \times 10^{-19} C$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{J \cdot m}$$

$$\frac{e^2}{4\pi \varepsilon_0} = 2.3 \times 10^{-20} J \cdot m$$

$$\frac{20k_B T}{\left( \frac{e^2}{4\pi \varepsilon_0} \right)} = (20 \times 4.1 \times 10^{-21} J) = 3.6 \times 3.6 \times 10^8 m^{-1}$$

Then

$$3.6 \times 10^8 m^{-1} = \frac{1}{\varepsilon} \left( \frac{1}{x} - \frac{1}{r_o} \right)$$

$$\frac{1}{r_o} = \frac{2 \times 10^{-9} m^{-1}}{2 \times 10^{-9} m^{-1}}$$
For $k = 4$

\[ 1.4 \times 10^8 \text{ m}^{-1} = \frac{1}{x} - 2 \times 10^9 \text{ m}^{-1} \]

\[ 3.4 \times 10^9 \text{ m}^{-1} = \frac{1}{x} \]

\[ x = 2.9 \times 10^{-10} \text{ m} \approx 0.3 \text{ nm} \]

For $k = 78.5$

\[ 283 \times 10^8 \text{ m}^{-1} = \frac{1}{x} - 2 \times 10^9 \text{ m}^{-1} \]

\[ 26.3 \times 10^9 \text{ m}^{-1} = \frac{1}{x} \]

\[ x = 3.8 \times 10^{-11} \text{ m} = 0.038 \text{ nm} \]

For $k_{\text{variable}}$, as in Fig 4.10, we could use the data from the table to create an interpolating function in Mathematica (using Interpolation), then integrate the force

\[ \Delta U = - \int_{r_0}^{x} F \, dr = \frac{1}{4 \pi \varepsilon_0} \frac{\Phi^2}{x} \int_{r_0}^{x} \frac{1}{r^2} \, dr \]

\[ 20 k_B T = - \frac{\Phi^2}{4 \pi \varepsilon_0} \int_{r_0}^{x} \frac{1}{r^2} \, dr \]

\[ \frac{20 k_B T \text{ } \Phi^2}{(4 \pi \varepsilon_0)} = - \int_{r_0}^{x} \frac{1}{r^2} \, dr \]

\[ 3.6 \times 10^8 \text{ } \Phi^2 = - \int_{r_0}^{x} \frac{1}{r^2} \, dr \]

I can play with the value of the upper limit, $x$, until I find a solution. I get about

\[ x = 0.135 \times 10^{-9} \text{ m} = 0.135 \text{ nm} \]
In[1]:= data = {{0.0, 4}, {0.1 \times 10^{-9}, 5}, {0.15 \times 10^{-9}, 8}, {0.2 \times 10^{-9}, 14}, {0.25 \times 10^{-9}, 35}, {0.3 \times 10^{-9}, 50}, {0.35 \times 10^{-9}, 57}, {0.4 \times 10^{-9}, 59}, {0.5 \times 10^{-9}, 64}, {0.6 \times 10^{-9}, 65}}

Out[1]=

In[2]:= wow = Interpolation[data]

Out[2]= InterpolatingFunction[{{0.6 \times 10^{-10}}, <>}, <>]

In[3]:= Plot[wow[x], {x, 0, 6 \times 10^{-10}}]

Out[3]=

In[4]:= NIntegrate[-(x^2) / wow[x], {x, 0.5 \times 10^{-9}, 0.135 \times 10^{-9}}]

Out[4]= 3.56891 \times 10^{6}

In[5]:= wow[0.4 \times 10^{-9}]

(1) Use Debye, Onsager, and Kirkwood-Freedlich approximations to calculate dielectric constant of water.

Constants:
\[
\begin{align*}
\kappa_0 & = 1.36 \times 10^{-23} \frac{1}{\text{m}} \\
T & = 298 \text{K} \\
N_A & = 6.02 \times 10^{23} \\
\rho & = \text{density} = 1 \frac{\text{g}}{\text{cm}^3} = 1000 \frac{\text{kg}}{\text{m}^3} \\
M & = \text{molecular mass} = 18.015 \text{g} = 18.01 \times 10^{-3} \text{kg} \\
\end{align*}
\]

\[\kappa \int \frac{\alpha'}{E_0} \rightarrow \text{what Nordlund means is the polarizability} \]
\[
\alpha' = \frac{\alpha}{4 \pi \kappa_0} \text{ in Atkins Physical Chemistry (8th ed, 06/12/2017)}
\]

Atkins Table 18.1 has for water:
\[\alpha' = 1.48 \times 10^{-20} \text{m}^3, \text{ so I'll take that as my value for } \alpha \]
\[\alpha = \alpha' + \alpha'' \]

\[P = \text{electric dipole moment} \]
\[= (1.85 \text{ Debye})(3.34 \times 10^{-30} \text{ C\cdotm}) = 6.2 \times 10^{-30} \text{ C\cdotm} \]
\[\text{ (Atkins, Table 18.1)} \]

\[N = \text{number of molecules per unit vol} = 3.34 \times 10^{25} \text{ molecules/m}^3 = 3.34 \times 10^{25} \text{ molecules/m}^3 \]

\[g = \text{molecular correlation function (between 1.5 and 1.6)} \]

\[V = \text{molar volume} = 18 \text{ml} = 18 \times 10^{-6} \text{ m}^3 \]
\[e_0 = 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2} \]
\[n = \text{index of refraction} = 1.337 \]
Debye:

\[
\frac{e^{-1}}{e^{+2}} = \frac{P}{M} R_n = \frac{P}{M} \frac{N_a}{3e_0} \left( \alpha + \frac{p^2}{3kT} \right)
\]

= \frac{P}{M} \frac{4\pi N_a}{3e_0} \left( \alpha + \frac{p^2}{3kT} \frac{1}{4\pi e_0} \right)

= \frac{P}{M} \frac{4\pi N_a}{3e_0} \left( \alpha + \frac{p^2}{3kT} \frac{1}{4\pi e_0} \right)

= \text{call it } C

\text{Eqn: } C = 4.1

Onsager:

\[
\frac{(e^{-n^2})(2e+n^2)}{e(n^2+2)} = \frac{4\pi N_a p^2}{9k_b T} \frac{1}{e_0} \rightarrow \text{formula yields } \frac{e_0}{e_0} \text{ not shown in text}
\]

Solving using Mathematica gives \( e_0 = 94 \)

Kirkwood-Fredlich:

\[
\frac{(e^{-n^2})(2e+n^2)}{e(n^2+2)} = \frac{4\pi N_a p^2}{9k_b T} \frac{1}{e_0} \rightarrow I'll \text{ take } e_0 = 1.5
\]

\[
\frac{1}{74}
\]

Solving: \( e_0 = 140 \)
\( n_a = 6.02 \times 10^{-23} \), \( \rho = 1000 \), \( m = 18 \times 10^{-3} \), \( \alpha = 1.48 \times 10^{-30} \),
\( p = 6.2 \times 10^{-30} \), \( n_{vol} = 3.344 \times 10^{-28} \), \( q = 1.33 \), \( \gamma = 1.5 \),
\( v = 18 \times 10^{-6} \), \( \varepsilon_0 = 8.85 \times 10^{-12} \), \( t = 298 \), \( k_B = 1.38 \times 10^{-23} \).

\[
I_{Debye} = \frac{4\pi n_a}{3} \times \frac{\rho}{m} \times (\alpha + p^2) (3k_B t \varepsilon_0 4\pi)
\]

\[
\varepsilon_{Debye}(t) = \frac{1 + 2I_{Debye}}{1 - I_{Debye}}
\]

(* here's the plot for the debye model, which doesn't make sense *)

\[
\text{Plot}[\varepsilon_{Debye}(t), \{t, 273, 373\}]
\]

\[
I_{Onsager} = 4\pi n_{vol} p^2 / (9k_B t \varepsilon_0)
\]

\[
\text{FindRoot}[(x - n^2) (2x + n^2) / (x (n^2 + 2)) = \text{conons}[t], \{x, 10\}]
\]

(* here's the plot for the onsager model *)

\[
\text{Plot}[\text{conons}[t], \{t, 273, 373\}]
\]

\[
\text{conons}[t] := 4\pi n_{vol} p^2 / (9k_B t \varepsilon_0)
\]

\[
\text{FindRoot}[(x - n^2) (2x + n^2) / (x (n^2 + 2)) = \text{conons}[t], \{x, 10\}]
\]

(* here's the plot for the kirkwood-frohlich model *)

\[
\text{Plot}[[x - n^2] (2x + n^2) / (x (n^2 + 2)) = \text{conons}[t], \{x, 10\}]
\]

\[
\text{conons}[t] := 4\pi n_{vol} p^2 / (9k_B t \varepsilon_0)
\]

\[
\text{FindRoot}[(x - n^2) (2x + n^2) / (x (n^2 + 2)) = \text{conons}[t], \{x, 10\}]
\]

(* here's the plot for the kirkwood-frohlich model *)

\[
\text{Plot}[(x - n^2) (2x + n^2) / (x (n^2 + 2)) = \text{conons}[t], \{x, 10\}]
\]
Plot[z2[tt], {tt, 273, 373}]
What is the Debye screening length for forge?

- The Debye length is the distance over which mobile charge carriers, such as those in an ionic solution, screen out electric fields.

How close do two charges need to get to feel an electrostatic repulsion?

- In an electrolyte or ionic solution, they need to be within a cluster Debye length or less.