Dimensional analysis, falling bodies, and the fine art of *not* solving differential equations

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Dimensional analysis is a simple, physically transparent and intuitive method for obtaining approximate solutions to physics problems, especially in mechanics. It may—indeed sometimes should—precede or even supplant mathematical analysis. And yet dimensional analysis usually is given short shrift in physics textbooks, presented mostly as a diagnostic tool for finding errors in solutions rather than in finding solutions in the first place. Dimensional analysis is especially well suited to estimating the magnitude of errors associated with the inevitable simplifying assumptions in physics problems. For example, dimensional arguments quickly yield estimates for the errors in the simple expression $\sqrt{2h/g}$ for the descent time of a body dropped from a height h on a spherical, rotating planet with an atmosphere as a consequence of ignoring the variation of the acceleration due to gravity g with height, rotation, relativity, and atmospheric drag.

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I. INTRODUCTION

Dimensional analysis usually gets short shrift in physics textbooks. What one finds is mostly admonitions about checking the dimensional homogeneity of equations as a way of ferreting out errors,[1–4] which falls in the category of telling people to brush their teeth after meals. Good advice but a bit lacking in profundity. Although Serway[2] does note, in a brief section on dimensional analysis, that equations can be derived by dimensional arguments, he then does not follow this prescription. And Giancoli[4] shows (in an appendix) how the period of a simple pendulum can be obtained by dimensional analysis. But by and large textbooks approach problems, especially in mechanics, by solving differential equations. The remarkable power of dimensional analysis to obtain approximate results quickly and easily is not often fully exploited.

My attempts to teach dimensional analysis have been disappointing. By the time I get students they seem to have been thoroughly inculcated in the belief that physics problems must entail the pain and suffering of solving differential equations, and any attempt to side step this is a cheat. No pain, no gain. And it does seem a bit of cheat that so much can be obtained with so little effort. But solving problems by dimensional analysis is quite respectable physics. For example, Lord Rayleigh's inverse fourth power of wavelength law for scattering by objects small compared with the wavelength[5] was first obtained by dimensional arguments no different from those in the following sections.

The aim of physics is physical understanding, not solving differential equations. They are sometimes a means to an end, but not the end itself, and if that end can be reached by simpler means, especially more physically transparent and intuitive ones, all to the good. Even to mathematicians, solving differential equations is not very interesting mathematics: "real" mathematicians[6] don't solve differential equations, they prove theorems.

In the sections that follow the power of dimensional analysis is demonstrated by using it to estimate the errors associated with simplifying the problem of a falling body in the gravitational field of a rotating planet with an atmosphere. Dimensional arguments are particularly well suited to this kind of error analysis.

II. DESCENT TIME OF A FALLING BODY

The time τ_o for an object dropped from rest at a height h to reach the surface of a planet is

$$\tau_o = \sqrt{\frac{2h}{g}}, \tag{1}$$

where g is the acceleration due to gravity near the surface. Several physical factors are ignored in standard derivations of this equation, some explicitly, others implicitly, often with no justification. Let τ be this time taking into account only one of the factors assumed negligible in the derivation of (1). The error resulting from ignoring this single factor is the dimensionless quantity

$$\varepsilon = \frac{\tau - \tau_o}{\tau_o}. \tag{2}$$

Because ε is dimensionless it must depend on some dimensionless parameter ξ (or parameters, but here we assume only one):

$$\varepsilon = f(\xi), \tag{3}$$

where $f(0)=0$. Expand $f(\xi)$ in a Taylor series:

$$f(\xi) = f(0) + \left(\frac{df}{d\xi}\right)_o \xi + \cdots. \tag{4}$$

The leading term in the error is therefore

$$\frac{\tau - \tau_o}{\tau_o} \approx C\xi, \tag{5}$$

where C is a dimensionless constant the value of which cannot be obtained from dimensional analysis. Although a dimensionless quantity raised to any nonintegral power is also dimensionless, we can rule out an error with this functional dependence if we assume that f is expandable in a power series about $\xi=0$.

The implication of (5) is that we can estimate the errors associated with each neglected physical factor, without solving any equations of motion, by physical reasoning about the dimensionless combination of relevant physical parameters. In the following sections we do so successively for the error associated with ignoring the variation with height of the acceleration due to gravity, the rotation of the planet, relativity, and finally, drag.

III. VARIATION OF g WITH HEIGHT

The acceleration due to gravity for a finite (spherical) planet of radius R varies with height above its surface. Equation (1) does not take this variation into account. What is the associated error in the descent time? Because gravity decreases with height, the descent time τ must be greater than τ_o. The greater the height h, the greater this time difference. So we need another relevant physical parameter with the dimensions of length, and the only one that comes to mind is the planetary radius R. Thus the dimensionless parameter for this problem is $\xi = h/R$ and the error is

$$\varepsilon = C\frac{h}{R}, \tag{6}$$

where C is an undetermined dimensionless constant. Does (6) make physical sense? It certainly satisfies at least two necessary conditions: the error must vanish for $h=0$ and as R approaches infinity.

To check this result we turn to the equation of motion

$$\frac{d^2z}{dt^2} = \frac{-g}{(1+z/R)^2}, \tag{7}$$

where z is the height above the surface and g is the acceleration due to gravity at the surface. The only hope we have of solving this equation (approximately) is to linearize the denominator on the right side, that is, assume $z/R \ll 1$ and expand, which yields the approximate equation of motion

$$\frac{d^2z}{dt^2} = -g(1-2z/R). \tag{8}$$

The solution to (8) subject to the initial conditions $z=h$ and $dz/dt=0$ at $t=0$ is

$$\frac{2z-R}{2h-R} = \cosh t\sqrt{2g/R}, \tag{9}$$

from which it follows that the descent time is the solution to

$$\frac{R}{R-2h} = \cosh \tau\sqrt{2g/R}. \tag{10}$$

Because of the restriction $h \ll R$, cosh here is close to 1, so it can be expanded as a power series in its argument and truncated after the second term. This yields for the error

$$\varepsilon = \frac{h}{R}, \tag{11}$$

which is (6) to within the undetermined (by dimensional analysis) constant C (here equal to 1). Estimating the error by dimensional arguments was certainly much simpler than solving the equation of motion and yet yielded essentially the same result. For Earth R is of order 10^7 m, so for objects dropped from heights of order 10^3 m or less, the error associated with neglecting the variation with height of gravity is less than 0.01%.

IV. ROTATION

Now we turn to the more difficult problem of estimating the consequences of ignoring the rotation of the planet. The error in the descent time must depend on the planetary angular speed ω. This introduces a parameter containing time, so we need another parameter containing time to balance it, which brings to mind g. But now length enters into our analysis, so we need a parameter with the dimensions of length as a counterbalance. The radius R of the planet is a candidate. If these three parameters are combined into a dimensionless group we obtain for the error (to within an undetermined constant)

$$\varepsilon = \frac{\omega^2 R}{g}. \tag{12}$$

The inverse of a dimensionless group is also dimensionless, of course, but in this instance the inverse of the group in (12) would not make physical sense. Equation (12) satisfies the minimal condition that the error vanishes as the rotation vanishes. And when g approaches infinity, the error approaches zero, which also makes sense: when g is infinite, rotation is irrelevant. The only remaining question might be why we chose R instead of h. Both have dimensions of length, and hence both are equally likely candidates at first glance. To choose between them requires a bit of physical reasoning. The correction resulting from rotation is a consequence of centripetal acceleration, which depends on the distance from the origin of the coordinate system. If we limit ourselves to heights $h \ll R$, we can use R in the dimensionless group.

To verify our result obtained from dimensional arguments, we turn to the equations of motion (neglecting the Coriolis term)

$$\ddot{r} + r\dot{\vartheta}^2 = -g + \omega^2 r\cos^2\vartheta, \tag{13}$$

$$2\dot{r}\dot{\vartheta} + r\ddot{\vartheta} = -\omega^2 r\sin\vartheta\cos\vartheta, \tag{14}$$

where r is the distance from the origin of the coordinate system (center of the planet) and ϑ is the latitude. These nonlinear equations are insoluble, but we are not daunted given that the theme of this article is how to extract physics from differential equations without suffering the pain of solving them. If the object is dropped from rest, $\dot{\vartheta}=0$ and $\dot{r}=0$ at $t=0$, and hence from (14) it follows that

$$\ddot{\vartheta}_o = -\omega^2\sin\vartheta_o\cos\vartheta_o, \tag{15}$$

where the subscript o denotes $t=0$. Our aim is to compare the magnitude of the second term on the left side of (13) with that of the second term on the right side. To that end, integrate (15) to obtain

$$\dot{\vartheta} \approx \omega^2 t\sin\vartheta_o\cos\vartheta_o \tag{16}$$

for $t>0$. This yields the approximation

$$\frac{r\dot{\vartheta}^2}{\omega^2 r\cos^2\vartheta} \approx \omega^2 t^2\sin^2\vartheta_o. \tag{17}$$

To approximate the maximum value of t we may use Eq. (1). For Earth ω is around $10^{-4}\ \mathrm{s}^{-1}$ and for h of order 10^3 m, t is around 10 s, so the product ωt is of order 10^{-3} and its square is of order 10^{-6}. We are therefore justified in neglecting the second term on the left side of (13) to obtain the approximate equation of motion

$$\ddot{r} = -g + \omega^2 r \cos^2 \vartheta_o . \tag{18}$$

This equation can be readily solved to obtain for the error in the descent time

$$\varepsilon = \frac{\omega^2 R \cos^2 \vartheta_o}{g}. \tag{19}$$

Except for the factor $\cos^2 \vartheta_o$, this is what we obtained by simple dimensional analysis. But the cosine is dimensionless so it lies beyond dimensional analysis. Nevertheless, we know on physical grounds that (12) must be missing a dimensionless factor that vanishes at the poles. For Earth the error is around 1%, considerably larger than the error associated with neglecting the variation of g with height.

V. RELATIVITY

Up to this point analysis has been based on classical mechanics. What are the consequences of special relativity? The dimensionless group determining the error in the descent time surely must depend on c, the free-space speed of light. This introduces the dimension of time, so we need another parameter with time in its dimensions, which immediately suggests g. But no combination of c and g can be made dimensionless, so we need either a parameter with the dimensions of time or of length. The height h is the only plausible candidate given that we again assume a flat planet. The only time that comes to mind is the (classical) descent time (1). But this depends on h and g, and so is not an independent parameter. Our dimensionless group for this problem is therefore

$$\xi = \frac{hg}{c^2}, \tag{20}$$

which gives the error (to within a constant). To check this result consider the relativistic equation of motion for the (vertical) velocity v,

$$\frac{d}{dt} \frac{v}{\sqrt{1 - v^2/c^2}} = -g. \tag{21}$$

For a body dropped from rest at $t = 0$, the first integral of the motion is

$$y = \frac{dz}{dt} = \frac{-gt}{\sqrt{1 + g^2 t^2 / c^2}}. \tag{22}$$

This equation is readily integrated to obtain z at any time $t > 0$, from which the descent time (time at which $z = 0$) follows:

$$\tau = \tau_o \sqrt{1 + hg/2c^2}. \tag{23}$$

With the assumption that $hg/2c^2 \ll 1$, the error in the descent time is to good approximation

$$\varepsilon = \frac{hg}{4c^2}, \tag{24}$$

which is what we obtained (within a factor of 4) by dimensional arguments. As expected, the relativistic correction is a tiny fraction of a percent. Of course, we could have obtained this same result by noting that the (classical) speed squared at $z = 0$ is $2hg$, and we would expect the relativistic correction to be of order a characteristic speed squared divided by c^2. For this problem we had three possible paths to follow: dimensional analysis, solving a not especially difficult differential equation, or an argument based on the ratio of speeds squared (which is also a dimensional argument).

VI. DRAG

Last but not least are the consequences of ignoring drag, the existence of which often is at least acknowledged in textbooks even if it is whisked under the carpet, out of sight and therefore out of mind. But one cannot simply ignore drag because to account for it would be uncomfortable. One has to at least estimate the error in the descent time (1) as a consequence of ignoring drag.

For this problem it is best to begin with the equation of motion for a body of mass m falling in a uniform gravitational field and subject to atmospheric drag[7] even though it is not necessary to solve it:

$$m \frac{dv}{dt} = -mg + \frac{1}{2} \rho_a A C_D v^2, \tag{25}$$

where ρ_a is the density of the atmosphere through which the body falls, A is the cross-sectional area of the body, and all the complicated fluid mechanics of drag are wrapped up in the drag coefficient C_D. Equation (25) can be re-written as

$$\frac{dv}{dt} = -g + bv^2, \tag{26}$$

where $b = \rho_a A C_D / 2m$. The error in the descent time must vanish with b, which has the dimensions of inverse length. We therefore need another parameter with the dimensions of length as a counterbalance, and the only one that comes to mind is h, the height at which the body is dropped from rest. So we are led to postulate an error in the descent time (again to within a constant)

$$\varepsilon = hb. \tag{27}$$

This is general, but to check its correctness we have to make some specific assumptions. One is that we can ignore the variation of density ρ_a with height.[8] The drag coefficient poses more of a problem because it depends on Reynolds number $\mathrm{Re} = \rho_a v d / \mu$ (at speeds less than about a third that of sound), where d is a characteristic linear dimension of the object and μ the (dynamic) viscosity of the fluid through which it moves. Measurements of the drag coefficient for bodies of simple shape and of the sizes of balls and similar everyday objects show that over a large range of Reynolds numbers, the drag coefficient is approximately constant.[9] With the assumption of constant b, the solution to (27) is

$$v = \frac{dz}{dt} = -\sqrt{\frac{g}{b}} \tanh(t\sqrt{gb}). \tag{28}$$

Note that as $t\sqrt{gb}$ approaches infinity, v approaches a limit $v_\infty = \sqrt{g/b}$ called the terminal velocity. For times t such that $t\sqrt{gb} \ll 1$, $v \approx -gt$, which is the usual expression for the endless increase in velocity of a falling object with time in the absence of drag.

Equation (29) can be solved[10] for z:

$$z = h - \frac{\ln \cosh(t\sqrt{gb})}{b}, \tag{29}$$

and hence the descent time is the solution to

$$\cosh(\tau\sqrt{gb}) = \exp(hb). \tag{30}$$

To solve this equation we expand both functions in power series and truncate after the third term (if we truncate after the second term we get only τ_o). The result for the error, after tedious but straightforward algebra, is

$$\varepsilon = \frac{hb}{6}, \tag{31}$$

which is essentially what we obtained by dimensional arguments.

Before proceeding it is instructive to give a physical interpretation of b or, rather, its inverse, a length. For large values of the argument of the cosh in (29) the distance the object falls from rest in time t is

$$h - z \approx v_\infty t - \frac{\ln 2}{b}. \tag{32}$$

The first term on the right side of (32) is the distance the object would have fallen if it had its (constant) terminal velocity from the outset ($t=0$). We may therefore interpret $1/b$ as the distance the object falls from rest before it reaches an appreciable fraction of its terminal velocity. Thus the quantity hb is the ratio of the height from which the body is dropped to the distance it falls before reaching (almost) its terminal velocity. When the body is near its terminal velocity, drag certainly cannot be neglected, so the ratio of lengths hb as the key quantity determining whether drag is or is not negligible makes good physical sense.

We can write b as

$$b = \frac{1}{2}\frac{\rho_a}{\rho}\frac{A}{V}C_D, \tag{33}$$

where V is the volume of the object and ρ its density. The quantity A/V is K/d, where K is a constant and d is a characteristic linear dimension of the object. For a sphere $K = 3/2$ if d is its diameter. For Earth's atmosphere (within the troposphere) ρ_a/ρ lies in the approximate range 10^{-3}–10^{-4} (a tennis ball corresponds to the lower end of this range, a shot put to the upper end). The drag coefficient for a sphere[9] is about 0.4 at large Reynolds numbers (say, 10^3–10^5). Thus for more or less spherical objects the error in the descent time is approximately

$$\varepsilon \approx 5 \times 10^{-2}\frac{\rho_a}{\rho}\frac{h}{d}. \tag{34}$$

Balls of many kinds have characteristic lengths d of order 10 cm. At what drop height is the error 10%, say? For a tennis ball, about 200 m, for a shot put about 2000 m. The height of the Leaning Tower of Pisa is about 50 m. Suppose that Galileo really did drop from this tower objects of the same size but of different density (about which there is doubt[11]) to show that they reach the ground at the same time. From (1) the approximate drop time is 3 s. From (34) the fractional error in the drop time as a consequence of neglecting drag is about 0.02 for a tennis ball, 0.002 for a shot put. The difference in times to reach the ground is therefore about 0.05 s. At impact both balls would have speeds of around 30 m/s, which yields a separation of about 15 cm between them at the instant one of them first reaches the ground. This is approximately equal to the diameter of a ball, and hence would be difficult but not impossible to detect, although for the observation to be valid the two balls would have to be dropped at the same time to within 0.01 s or less.

VII. CONCLUDING REMARKS

Have we exhausted all the possible errors lurking behind the simple expression (1) for the descent time of a falling body? Probably not, but I don't want to spoil the pleasure readers might have looking for other sources of error and estimating their magnitude by dimensional arguments.

Is dimensional analysis an infallible method for solving physical problems? Of course not, but the same can be said of *any* method. There is no royal road to physics. The advantage of dimensional analysis is that physical analysis precedes or even supplants mathematical analysis. Solving differential equations teaches students to solve differential equations. But it is the interpretation of solutions that is the essence of good physics. Unfortunately, because of the sheer drudgery of solving equations physical interpretation often is an afterthought instead of occupying pride of place, as it does in dimensional analysis.

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[6]For a delightful exposition of the outlook of a "real" mathematician see G. H. Hardy, *A Mathematician's Apology* (Cambridge U.P., Cambridge, 1969), pp. 121–124.

[7]It is common to find in textbooks a linear drag law of the form kv, where k is independent of speed. This yields an equation of motion that is readily solved at the expense of the solution being largely irrelevant. Such a law is valid only for Reynolds numbers less than 1, which except for exceedingly short time intervals is not satisfied by objects with the dimensions of tennis balls (or even lead shot). The ratio μ/ρ (called the kinematic viscosity) for air at 15 °C is about 0.15 cm^2/s. A ball or other object of comparable size dropped from rest reaches 1 cm/s in about 10^{-3} s. For $d=10$ cm and $v=1$ cm/s, the Reynolds number is of order 10^2, so the linear drag law is invalid for such objects during all but a tiny fraction of their trajectories. This law is valid, however, for cloud droplets, which have diameters of order 10 μm. A rule of thumb is that if you can readily see a falling body, the linear drag law is not applicable to it.

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