Solutions to the Calculus and Linear Algebra problems on the Comprehensive Examination of February 3, 2006

Solutions to Problems 1-4 and 8 are omitted since they involve topics no longer covered on the Comprehensive Examination.

5. Find the volume of the region in 3-dimensional space inside the cylinder $x^2 + y^2 = 1$, above the xy plane, and below the plane $x + z = 1$.

   Solution: We'll be working with cylindrical coordinates here. The inside-the-cylinder constraint is pretty obvious: it just means that $r \leq 1$. The other two constraints bound z: $0 \leq z \leq 1 - x = 1 - r\cos\theta$ (note that $0 \leq 1 - x$ inside the cylinder so this is a legitimate inequality for this problem).

$$\begin{aligned} V &= \iiint_V dV \\ &= \int_0^1 \int_0^{2\pi} \int_0^{1-r\cos\theta} r \, dz \, d\theta \, dr \\ &= \int_0^1 \int_0^{2\pi} r(1 - r\cos\theta) \, d\theta \, dr = \int_0^1 \int_0^{2\pi} r - r^2\cos\theta \, d\theta \, dr \\ &= \int_0^1 (r\theta - r^2\sin\theta)\Big|_{\theta=0}^{2\pi} dr \\ &= \int_0^1 (r(2\pi) - r^2\sin(2\pi)) - (r(0) - r^2\sin(0)) \, dr \\ &= \int_0^1 2\pi r \, dr = \pi r^2\Big|_0^1 = \pi \end{aligned}$$

6. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

   (a) Compute $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$.

   Solution: Use the definition of partial derivative. Note that since f is symmetric with respect to switching x and y, we only need one calculation:

$$\frac{\partial f}{\partial y}(0, 0) = \frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{\frac{h(0)}{h^2 + 0^2}}{h} = \lim_{h \to 0} \frac{0}{h^3} = 0$$

   (b) Prove that $f(x, y)$ is not differentiable at $(0, 0)$.

   Solution: Will not need to prove differentiability in the new comps.

7. Let $f(x,y) = xy + \int_0^y \sin(t^2)\,dt$

   (a) Compute $\nabla f(a,b)$

   Solution: In finding $\frac{\partial f}{\partial y}(a,b)$ remember the Fundamental Theorem of Calculus ($\frac{d}{dy}\int_0^y g(t)\,dt = g(y)$); the rest is straightforward:

$$\nabla f(a,b) = \frac{\partial f}{\partial y}(a,b)\mathbf{i} + \frac{\partial f}{\partial y}(a,b)\mathbf{j} = b\mathbf{i} + (a + \sin(b^2))\mathbf{j}$$

   (b) Show that $(0,0)$ is a saddle point of $f(x,y)$

   Solution: Use the second derivative test:

$$\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & 2y\cos(y^2)\Big|_{(0,0)} \end{pmatrix} = -1 < 0$$

   Thus $(0,0)$ is a saddle point of $f(x,y)$ as desired.✓

9. Let $T : V \to V$ be a linear transformation such that $T \circ T$ is the zero linear transformation. Also let $v \in V$ satisfy $T(v) \neq 0$. Prove that the set $\{v, T(v)\}$ is linearly independent.

   Solution: Given sccalars α and β such that $\alpha v + \beta T(v) = 0$, take T of both sides to get $T(\alpha v + \beta T(v)) = T(0)$. Since T is linear, $T(0) = 0$ and $T(\alpha v + \beta T(v)) = \alpha T(v) + \beta T(T(v))$. But since $T \circ T$ is the zero transformation, $T(T(v)) = 0$, so $\alpha T(v) + 0 = T(0) = 0$. Since $T(v) \neq 0$, this means that $\alpha = 0$. Substituting that into the original equation $\alpha v + \beta T(v) = 0$ gives us $\beta T(v) = 0$. Again, since $T(v) \neq 0$, $\beta = 0$. Thus $\alpha = \beta = 0$, so $\{v, T(v)\}$ is linearly independent as desired. QED

10. Compute the inverse of the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 4 & 1 & 2 \end{pmatrix}.$$

   Check your answer by matrix multiplication.

   Solution: Nothing fancy about this:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & -4 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -1 \\ -1 & -2 & 1 \end{pmatrix}$$

   To check, just note that $AA^{-1} = I$ or $A^{-1}A = I$.

11. Let $M_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real entries. We say that $A, B \in M_n(\mathbb{R})$ commute if $AB = BA$.

   (a) Fix $A \in M_n(\mathbb{R})$. Prove that the set of all matrices in $M_n(\mathbb{R})$ that commute with A is a subspace of $M_n(\mathbb{R})$.

   Solution: Let $S = \{X \in M_n(\mathbb{R}) \big| AX = XA\}$ (the set of all matrices in $M_n(\mathbb{R})$ that commute with A). Since $AA = AA$, $A \in S$ so $S \neq \emptyset$.✓
   Now, given $B, C \in S$, $A(B+C) = AB + AC = BA + CA = (B+C)A$ (using distributive property and the fact that B, C both commute with A), so $B+C \in S$. Thus, S is closed under addition.✓
   Given $\alpha \in \mathbb{R}$ and $B \in S$, $A(\alpha B) = \alpha(AB) = \alpha(BA) = (\alpha B)A$ so $\alpha B \in S$. Thus, S is closed under scalar multiplication.✓
   Thus, S is a subspace of $M_n(\mathbb{R})$ as desired. QED

   Comment: This solution used the version of the subspace criterion that states that the subspace must be nonempty. Another version of the criterion states that the subspace must contain the zero vector. If $\mathbf{O}$ denotes the zero matrix in $M_n(\mathbb{R})$, then $A\mathbf{O} = \mathbf{O} = \mathbf{O}A$, so that $\mathbf{O} \in S$.✓

   (b) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{R})$ and let $W \subseteq M_2(\mathbb{R})$ be the subspace of all matrices of $M_2(\mathbb{R})$ that commute with A. Find a basis of W.

   Solution: A matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ lies in W if and only if $AB = BA$, so we simply do matrix multiplication for AB and BA and set them equal:

$$AB = \begin{pmatrix} a+c & b+d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix} = BA$$

   Setting the entries equal gives us the linear system $a + c = a + b$, $b + d = a + b$, $a + c = c + d$, and $b + d = c + d$. This system is equivalent to the equations $c = b$, $d = a$. Thus

$$W = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\} = \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$
$$= \operatorname{span}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

   Since these two matrices are clearly linearly independent (neither is a multiple of the other), we have a basis for W: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

12. Let $T : V \to V$ and $U : V \to V$ be linear transformations that commute, i.e. $T \circ U = U \circ T$. Let $v \in V$ be an eigenvector of T such that $U(v) \neq 0$. Prove that $U(v)$ is also an eigenvector of T.

Solution: Let λ be the corresponding eigenvalue for v, so $T(v) = \lambda v$. We want to prove something about $T(U(v))$ (namely, that it is equal to $\alpha U(v)$ for some scalar α), so let's go for it:

$$T(U(v)) = (T \circ U)(v) = (U \circ T)(v) = U(T(v)) = U(\lambda v) = \lambda U(v)$$

as desired. Note that the $T \circ U = U \circ T$ equation is given, and the $U(\lambda v) = \lambda U(v)$ equation is a property of linear trandformations. QED