Math 111, Introduction to the Calculus, Fall 2011
Midterm III Practice Exam 1

You will have 50 minutes for the exam and are not allowed to use books, notes or calculators. Each question is worth 10 points.

1. *Sketch a graph of the function*

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

to show the horizontal asymptote, as well as the intervals on which f is increasing or decreasing. You should explain how you worked out each part of your answer.

To find the horizontal asymptote we have to look at the limit of $f(x)$ as $x \to \pm\infty$. We have

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{\lim_{x\to\infty} 1-\frac{1}{x^2}}{\lim_{x\to\infty} 1+\frac{1}{x^2}} = \frac{1}{1} = 1.$$

Similarly

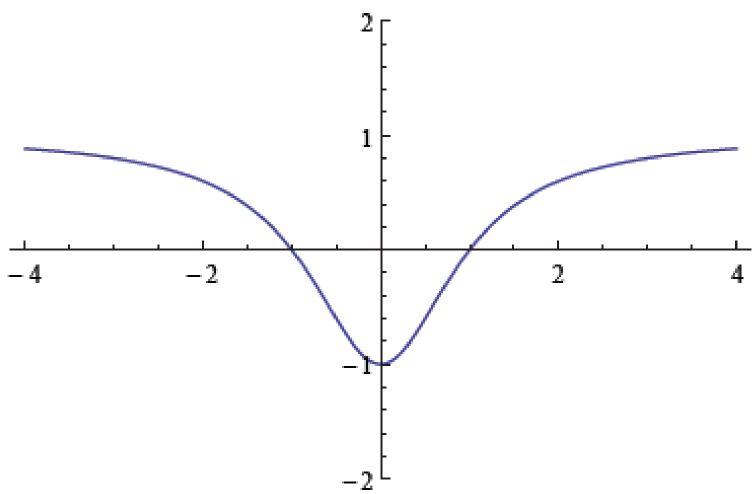
$$\lim_{x\to-\infty} = 1.$$

Therefore, there is a horizontal asymptote at $y = 1$.

To decide where f is increasing or decreasing, we have to look at f'. We have

$$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

Since the denominator is always positive, we see that $f'(x)$ is negative when $x < 0$, zero when $x = 0$ and positive when $x > 0$. Therefore, $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$ with a local minimum at $x = 0$. At $x = 0$ the value of f is -1 and (although the question did not ask for it) the graph crosses the x-axis at $x = \pm 1$. Therefore the graph looks as follows:



2. *Find the critical points of the function*

$$f(x) = x^2\sqrt{x+1}$$

and classify each critical point is a local maximum, local minimum, or neither.

The derivative is

$$f'(x) = 2x\sqrt{x+1} + \frac{x^2}{2\sqrt{x+1}} = \frac{4x(x+1)+x^2}{2\sqrt{x+1}} = \frac{x(5x+4)}{2\sqrt{x+1}}.$$

The critical points are when this is zero, that is

$$x(5x+4) = 0$$

so $x = 0$ or $x = -\frac{4}{5}$.

Rather than using the Second Derivative Test, we can just see where $f'(x)$ is positive or negative. For $x < -4/5$, we have $x < 0$ and $5x + 4 < 0$, so $f'(x) > 0$. Therefore, f is increasing on this region. For $-4/5 < x < 0$, we have $x < 0$ but $5x + 4 > 0$, so $f'(x) < 0$. Therefore f is decreasing on this region. For $x > 0$, we have $x > 0$ and $5x + 4 > 0$, so $f'(x) > 0$ and f is increasing again on this region. This means that $x = -4/5$ is a local maximum and $x = 0$ is a local minimum.

3. *Find the point on the curve* $y = \sqrt{2x+9}$ *that is closest to the origin.*

We are trying to minimize the quantity

$$d = \sqrt{x^2+y^2}$$

where x, y are related by the equation

$$y = \sqrt{2x+9}.$$

Therefore

$$d(x) = \sqrt{x^2+2x+9}.$$

The range of possible x values is $x \geq -9/2$ (because we cannot take the square-root of a negative number). To find the minimum we therefore have to compare the value of $d(x)$ at any critical points and at the endpoint $x = -9/2$.

We have

$$d'(x) = \frac{2x+2}{2\sqrt{x^2+2x+9}} = \frac{x+1}{\sqrt{x^2+2x+9}}.$$

The critical points are when $x + 1 = 0$, that is $x = -1$. Notice that as long as $2x + 9 > 0$ then $x^2 + 2x + 9 > 0$ so the function $d(x)$ is differentiable at all points we are considering.

Our two candidates for the closest point are therefore $x = -9/2$ where $d = 9/2$ and $x = -1$ where $d = \sqrt{8} < 3$. Therefore, the closest point is when $x = -1$, that is, at the point $(-1, \sqrt{7})$.

4. *Calculate the integral*

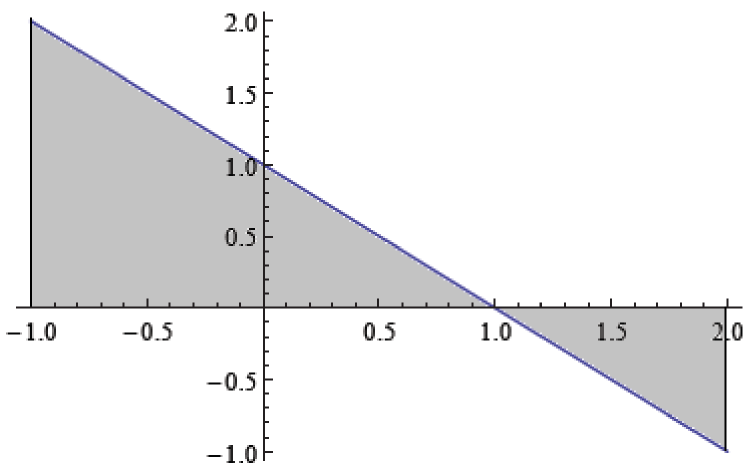
$$\int_{-1}^{2} (1-x)\,dx$$

in two ways:

(a) by drawing a graph and finding the appropriate area or areas;

(b) using the Fundamental Theorem of Calculus.

(a) The graph looks at follows



Areas of regions below the x-axis count negative so the integral is equal to

$$\frac{1}{2}(2)(2) - \frac{1}{2}(1)(1) = \frac{3}{2}.$$

(b) The integral is equal to

$$\left[x - x^2/2\right]_{x=-1}^{x=2} = (2 - 2^2/2) - ((-1) - (-1)^2/2) = 0 + 1 + 1/2 = \frac{3}{2}.$$

5. *(a) Find the derivative of the function*

$$f(t) = \int_1^{t^2} \frac{1}{x}\,dx.$$

(b) Find an antiderivative $G(x)$ for the function

$$g(x) = 3\sin(x+\pi)$$

that has the property that $G(0) = 5$.

(a) We can write

$$f(t) = F(t^2)$$

where

$$F(u) = \int_1^u \frac{1}{x}\, dx.$$

By the Chain Rule we have

$$f'(t) = F'(t^2)(2t)$$

and by the Fundamental Theorem of Calculus (Part I) $F'(u) = \frac{1}{u}$, so

$$f'(t) = \frac{1}{t^2}(2t) = \frac{2}{t}.$$

(b) We might guess that an antiderivative involves $\cos(x+\pi)$. The derivative of $\cos(x+\pi)$ is $-\sin(x+\pi)$, so the derivative of $-3\cos(x+\pi)$ is $3\sin(x+\pi)$. This means that our antiderivative of $g(x)$ must be of the form

$$G(x) = -3\cos(x+\pi) + c$$

for some constant c. We want $G(0) = 5$ which means that

$$-3\cos(0+\pi) + c = 5$$

which tells us that

$$3 + c = 5$$

or $c = 2$. Therefore, the antiderivative that we require is

$$G(x) = -3\cos(x+\pi) + 2.$$