Math 211, Multivariable Calculus, Fall 2011
Midterm III Practice Exam 1

You will have 50 minutes for the exam and are not allowed to use books, notes or calculators. Each question is worth 10 points.

1. *Find the critical points of the function*

$$f(x,y) = (x-1)e^{xy}$$

and classify each as a local maximum, a local minimum, or a saddle point.

The gradient vector of f is

$$\nabla f(x,y) = \langle (1+y(x-1))e^{xy}, x(x-1)e^{xy} \rangle .$$

The critical points therefore satisfy

$$(1+y(x-1))e^{xy} = 0 \text{ and } x(x-1)e^{xy} = 0.$$

Since e^{xy} cannot be zero, this means

$$1+y(x-1) = 0 \text{ and } x(x-1) = 0.$$

The second equation implies that $x = 0$ or $x = 1$, but substituting $x = 1$ into the first equation gives $1 = 0$ so $x = 1$ is not possible. Therefore we must have $x = 0$. The first equation then gives $y = 1$. There is therefore only one critical point which is $(0, 1)$.

We use the second derivative test to classify the critical point. The second-order partial derivatives of f are

$$f_{xx} = (y+y(1+y(x-1)))e^{xy}, \quad f_{xy} = f_{yx} = (x-1+x(1+y(x-1)))e^{xy}, \quad f_{yy} = x^2(x-1)e^{xy}.$$

Evaluating at the critical point we get

$$f_{xx}(0,1) = 1, \quad f_{xy}(0,1) = f_{yx}(0,1) = -1, \quad f_{yy}(0,1) = 0.$$

[Notice that it is much better to evaluate these at $(0, 1)$ first, rather than combining to form D in terms of xs and y.]

Therefore

$$D(0,1) = (1)(0) - (-1)^2 = -1.$$

Since $D(0, 1) < 0$ it follows that $(0, 1)$ is a saddle point.

2. *Find the absolute minimum of the function*

$$f(x,y) = 3x + y$$

on the region

$$x^2 + y^2 = 10.$$

(Make sure you explain how you know that your answer is the absolute minimum.)

The constrained region is a circle of radius $\sqrt{10}$. This is closed and bounded so the Extreme Value Theorem tells us there is an absolute minimum of f on this domain.

We use the Lagrange multiplier method with $g(x, y) = x^2 + y^2$. We then have

$$\nabla f = \langle 3, 1 \rangle, \quad \nabla g = \langle 2x, 2y \rangle.$$

So the constrained critical points occur when

$$3 = 2x\lambda, \quad 1 = 2y\lambda.$$

These equations tells us that x and y are not zero, so we get

$$\frac{3}{2x} = \frac{1}{2y}$$

and so

$$3y = x.$$

Substituting this into the constraint equation $x^2 + y^2 = 10$ we get

$$10y^2 = 10$$

so $y = \pm 1$. Since $x = 3y$ there are two constrained critical points: $(3, 1)$ and $(-3, -1)$. We should also check when $\nabla g = \mathbf{0}$ which tells us $x = y = 0$. But $(0, 0)$ does not satisfy the constraint.

Therefore, the absolute minimum must occur at either $(3, 1)$ or $(-3, -1)$. The values of f at these points are

$$f(3, 1) = 10, f(-3, -1) = -10$$

and so the absolute minimum occurs at $(-3, -1)$.

3. *Let R be the part of the disc $x^2 + y^2 \leq 4$ that lies in the region where $x, y \geq 0$. Calculate the integral*

$$\iint_R x(x^2 + y^2)\, dA.$$

In polar coordinates this region is given by

$$0 \leq r \leq 2, \quad 0 \leq \frac{\pi}{2}.$$

Since $x = r\cos\theta$ and $x^2 + y^2 = r^2$, the integral becomes

$$\int_{r=0}^{r=2} \int_{\theta=0}^{\theta=\pi/2} (r\cos\theta)(r^2) r\; d\theta\; dr$$

which equals

$$\int_{r=0}^{r=2} r^4 \int_{\theta=0}^{\theta=\pi/2} \cos\theta \, d\theta \, dr = \int_{r=0}^{r=2} r^4 \left[\sin\theta\right]_{\theta=0}^{\theta=\pi/2} \, dr$$
$$= \int_{r=0}^{r=2} r^4 \, dr$$
$$= \left[r^5/5\right]_{r=0}^{r=2}$$
$$= \frac{32}{5}$$

4. *Use a triple integral to calculate the volume of the sphere of radius a.*

If we put the center at the origin, the sphere is given in spherical coordinates by

$$0 \le \rho \le a, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

The volume is found by integrating the constant function 1 over the sphere. This integral becomes

$$\int_{\rho=0}^{\rho=a} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\phi=\pi} (1)\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

which equals

$$\int_{\rho=0}^{\rho=a} \rho^2 \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=2\pi} \sin\phi \, d\phi \, d\theta \, d\rho = \int_{\rho=0}^{\rho=a} \rho^2 \int_{\theta=0}^{\theta=2\pi} \left[-\cos\phi\right]_{\phi=0}^{\phi=\pi} \, d\theta \, d\rho$$
$$= \int_{\rho=0}^{\rho=a} \rho^2 \int_{\theta=0}^{\theta=2\pi} 2 \, d\theta \, d\rho$$
$$= \int_{\rho=0}^{\rho=a} 4\pi\rho^2 \, d\rho$$
$$= 4\pi \left[\rho^3/3\right]_{\rho=0}^{\rho=a}$$
$$= \frac{4\pi a^3}{3}$$

5. *Use the change of variables*

$$x = u + 2v, \quad y = u$$

to calculate

$$\iint_R xy \, dA$$

over the triangular region R with vertices $(0,0)$, $(1,1)$, $(2,0)$.

First we figure out what the region R corresponds to in the uv-plane. The line connecting $(0,0)$ and $(1,1)$ is the line $y = x$, which becomes the line

$$u = u + 2v$$

and so $v = 0$. The line connecting $(1,1)$ and $(2,0)$ is the line $y = 2 - x$ which becomes

$$u = 2 - u - 2v.$$

This simplifies to

$$v = 1 - u.$$

Finally, the line connecting $(2, 0)$ and $(0, 0)$ is $x = 0$ which gives $u = 0$. The region in the uv-plane is therefore the the triangle consisting of the points $(0, 0)$, $(0, 1)$ and $(1, 0)$. In terms of the variables u and v we can describe this as

$$0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u.$$

We also have

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = (1)(0) - (2)(1) = -2.$$

The integral therefore becomes

$$\int_{u=0}^{u=1} \int_{v=0}^{v=1-u} (u + 2v)u \ |-2| \ dv \ du$$

which is

$$\begin{aligned} 2\int_{u=0}^{u=1} \left[u^2 v + v^2 u\right]_{v=0}^{v=1-u} \ du &= 2\int_{u=0}^{u=1} u^2(1-u) + (1-u)^2 u \ du \\ &= 2\int_{u=0}^{u=1} u(1-u) \ du \\ &= 2\int_{u=0}^{u=1} u - u^2 \ du \\ &= 1 - \frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

The original question 5 which required integration by parts is below:

Use the change of variables

$$x = u + 2v, \quad y = u$$

to calculate

$$\iint_R (x - y)e^{2y}$$

over the triangular region R with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$.

First we figure out what the region R corresponds to in the uv-plane. The line connecting $(0, 0)$ and $(1, 1)$ is the line $y = x$, which becomes the line

$$u = u + 2v$$

and so $v = 0$. The line connecting $(1, 1)$ and $(2, 0)$ is the line $y = 2 - x$ which becomes

$$u = 2 - u - 2v.$$

This simplifies to

$$v = 1 - u.$$

Finally, the line connecting $(2, 0)$ and $(0, 0)$ is $x = 0$ which gives $u = 0$. The region in the uv-plane is therefore the the triangle consisting of the points $(0, 0)$, $(0, 1)$ and $(1, 0)$. In terms of the variables u and v we can describe this as

$$0 \leq u \leq 1, \quad 0 \leq v \leq 1 - u.$$

We also have

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = (1)(0) - (2)(1) = -2.$$

The integral therefore becomes

$$\int_{u=0}^{u=1} \int_{v=0}^{v=1-u} (2v)e^{2u} |-2| \, dv \, du$$

which is

$$\int_{u=0}^{u=1} 2e^{2u} \left[v^2\right]_{v=0}^{v=1-u} \, du = \int_{u=0}^{u=1} 2e^{2u}(1-u)^2 \, du$$

Unfortunately, this requires integration by parts (twice!) which I didn't intend and do not require you to know for the exam. One integration by parts gives

$$\left[e^{2u}(1-u)^2\right]_{u=0}^{u=1} - \int_{u=0}^{u=1} 2e^{2u}(u-1) \, du = (-1) - \int_{u=0}^{u=1} 2e^{2u}(u-1) \, du.$$

We now use integration by parts again on this integral which gives overall

$$(-1) - \left[e^{2u}(u-1)\right]_{u=0}^{u=1} + \int_{u=0}^{u=1} e^{2u} \, du$$

which equals

$$(-2) + \left[e^{2u}/2\right]_{u=0}^{u=1}$$

which is

$$\frac{e^2 - 5}{2}.$$