**Math 211, Multivariable Calculus, Fall 2011**
**Midterm III Practice Exam 2 Solutions**

1. (a) *What does it mean to say that $(a, b)$ is a* ***saddle point*** *of the function $f(x, y)$?*
   (b) *Find the critical points of the function*

$$f(x, y) = x^3 - xy + y^2.$$

   (c) *For each critical point, decide if it is a local maximum, local minimum or saddle point.*

   (a) It means that $(a, b)$ is a strict local maximum for $f$ when restricted to one cross-section through $(a, b)$ and a strict local minimum for $f$ when restricted to another cross-section through $(a, b)$.
   (b) The critical points occur when $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, that is

$$3x^2 - y = 0, \quad -x + 2y = 0.$$

   Therefore $x = 2y$ and so

$$3(2y)^2 - y = 0$$

   and so

$$y(12y - 1) = 0.$$

   This implies that $y = 0$ or $y = \frac{1}{12}$. Therefore, the two critical points are $(0, 0)$ and $(\frac{1}{6}, \frac{1}{12})$.
   (c) We use the Second Derivative Test. We have

$$D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(2) - (-1)^2 = 12x - 1.$$

   Therefore $D(0, 0) = -1$, so $(0, 0)$ is a saddle point, and $D(\frac{1}{6}, \frac{1}{12}) = 2$, $f_{xx}(\frac{1}{6}, \frac{1}{12}) > 0$, so $(\frac{1}{6}, \frac{1}{12})$ is a local minimum.

2. *Find the absolute maximum of the function*

$$f(x, y) = x - y^2$$

*on the region*

$$x^2 + y^2 \leq 1.$$

*(Make sure you explain how you know that your answer is the absolute maximum.)*

We know there is an absolute maximum by the Extreme Value Theorem because the region $x^2 + y^2 \leq 1$ is closed and bounded. The possible points are:

- points where $\nabla f = \mathbf{0}$ (but there are no such points because $\nabla f = \langle 1, -2y \rangle$)
- points where $f$ is not differentiable (none)

- points on the boundary $x^2 + y^2 = 1$

To find the maximum value of $f$ on the boundary we can use the Lagrange Multiplier Method with $g(x) = x^2 + y^2$. The possibilities are

- points on the boundary where $\nabla f = \lambda \nabla g$ for some $\lambda$
- points on the boundary where $\nabla g = \mathbf{0}$ (there are none of these because $\nabla g = \langle 2x, 2y \rangle$ so is only zero at $(0,0)$ which is not on the boundary)
- points where $g$ is not differentiable (none)

The possible points then are where

$$1 = 2x\lambda, \quad -2y = 2y\lambda.$$

The first equation tells us that $x \neq 0$ so, $\lambda = \frac{1}{2x}$ and so

$$-2y = \frac{y}{x}$$

or

$$y(1 + 2x) = 0.$$

Therefore, either $y = 0$, in which case $x = \pm 1$ (from $x^2 + y^2 = 1$) or $x = -\frac{1}{2}$, in which case $y = \pm\frac{\sqrt{3}}{2}$. So there are four candidates:

$$(1,0),\ (-1,0),\ (-1/2, \sqrt{3}/2),\ (-1/2, -\sqrt{3}/2).$$

It's easy to see that $f(x,y)$ is negative when $x$ is negative, so the absolute maximum must occur at $(1,0)$ with $f(1,0) = 1$.

3. *Let $R$ be the triangular region with vertices $(0,0)$, $(0,1)$, $(2,2)$. Calculate*

$$\iint_R xy\, dA.$$

We can express this region as

$$0 \leq x \leq 2, \quad x \leq y \leq \frac{1}{2}x + 1.$$

The integral is therefore

$$\begin{aligned}
\int_{x=0}^{x=2} \int_{y=x}^{y=\frac{1}{2}x+1} xy\, dy\, dx &= \int_{x=0}^{x=2} x\ \left[y^2/2\right]_{y=x}^{y=\frac{1}{2}x+1}\ dx \\
&= \int_{x=0}^{x=2} \frac{x}{2}((x/2+1)^2 - x^2)\, dx \\
&= \int_{x=0}^{x=2} (x/2 + x^2/2 - 3x^3/8)\, dx \\
&= \left[x^2/4 + x^3/6 - 3x^4/32\right]_{x=0}^{x=2} \\
&= 1 + 8/6 - 3/2 \\
&= 5/6
\end{aligned}$$

4. *Let $D$ be the solid cylinder whose ends are given by the planes $z = -1$ and $z = 2$, and whose curved surface is given by $x^2 + y^2 = 1$. Calculate*

$$\iiint_D z \, dV.$$

The region $D$ can be expressed in cylindrical coordinates as

$$-1 \leq z \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1.$$

The integral is therefore

$$\begin{aligned}
\int_{z=-1}^{z=2} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} zr \, dr \, d\theta \, dz &= \int_{z=-1}^{z=2} z \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} \, d\theta \, dz \\
&= \int_{z=-1}^{z=2} \pi z \, dz \\
&= \pi [z^2/2]_{z=-1}^{z=2} \\
&= \frac{3\pi}{2}
\end{aligned}$$

5. The *hyperbolic coordinates* of a point in the $xy$-plane are the variables $s, t$ given (when $x > 0$) by

$$s = xy, \quad t = \frac{y}{x}.$$

   (a) Find the Jacobian $\frac{\partial(x,y)}{\partial(s,t)}$ for the change of variables from $x, y$ to $s, t$.

   (b) Let $R$ be region in the $xy$-plane bounded by the lines $y = x$ and $y = 2x$, and the curves $y = \frac{1}{x}$ and $y = \frac{2}{x}$. Sketch a diagram of the region $R$.

   (c) Use hyperbolic coordinates to calculate the integral

$$\iint_R xy \, dA.$$

   (a) (This question should have specified to assume that $x, y > 0$ to avoid problems when $y = 0$.) We first need to find expressions for $x$ and $y$ in terms of $s$ and $t$. The second equation gives

$$y = xt$$

   so then the first gives

$$s = x^2 t$$

   and so

$$x = \sqrt{\frac{s}{t}}.$$

   (We take the positive square-root because we are told in the question that these coordinates only work for the region where $x > 0$.)

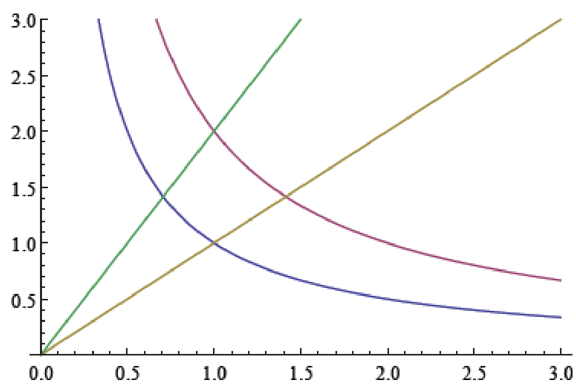
We then get

$$y = xt = \sqrt{st}.$$

We then have

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(s,t)} &= (\frac{\partial x}{\partial s})(\frac{\partial y}{\partial t}) - (\frac{\partial x}{\partial t})(\frac{\partial y}{\partial s}) \\
&= \frac{1}{2\sqrt{st}}\frac{\sqrt{s}}{2\sqrt{t}} - \frac{-\sqrt{s}}{2t^{3/2}}\frac{\sqrt{t}}{2\sqrt{s}} \\
&= \frac{1}{4t} + \frac{1}{4t} \\
&= \frac{1}{2t}
\end{aligned}$$

(b) The picture looks like:




(c) The four boundary curves become, respectively, $t = 1$, $t = 2$, $s = 1$, $s = 2$, so the integral is

$$\begin{aligned}
\int_{s=1}^{s=2}\int_{t=1}^{t=2} \frac{s}{2t}\,dt\,ds &= \int_{s=1}^{s=2} \frac{s}{2}\ [\ln t]_{t=1}^{t=2}\ ds \\
&= \int_{s=1}^{s=2} \frac{s}{2} \ln 2\, ds \\
&= \ln 2\left[s^2/4\right]_{s=1}^{s=2} \\
&= \frac{3\ln 2}{4}
\end{aligned}$$