Math 13: Project 2
Due Friday 11/20

Introduction

Recall that a sphere of radius r in $\mathbb{R}^3$ is defined by the equation

$$x^2 + y^2 + z^2 = r^2.$$

If we relabel the coordinates as $x = x_1$, $y = x_2$, and $z = x_3$, then the sphere is defined by the equation

$$x_1^2 + x_2^2 + x_3^3 = r^2.$$

With this in mind we can generalize the notion of a sphere to any positive dimension $n \geq 1$ by labeling the coordinates as $x_1, x_2, \ldots, x_n$ and defining an *n-sphere* as the set of points in $\mathbb{R}^n$ which satisfy

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2.$$

We will denote $S^n(r)$ as the sphere in n dimensions of radius r, called the *n-sphere of radius r*. For example, in dimensions 1, 2, and 3 we have

Dimension	Notation	Equation	Object
1	$S^1(r)$	$x_1^2 = r^2$	the interval $[-r, r]$
2	$S^2(r)$	$x_1^2 + x_2^2 = r^2$	the circle of radius r
3	$S^3(r)$	$x_1^2 + x_2^2 + x_3^2 = r^2$	the sphere of radius r

We also need a general notion of volume. We will define volume as the n-fold integral over the region enclosed by $S^n(r)$. So for $n = 1$ we have

$$\text{Volume}(S^1(r)) = V(S^1(r)) = \int_{-r}^{r} dx_1 = 2r$$

a single integral determining the length of the interval. For $n = 2$ we have

$$V(S^2(r)) = \int_{-r}^{r} \int_{-\sqrt{r^2 - x_1^2}}^{\sqrt{r^2 - x_1^2}} dx_2 dx_1 = \pi r^2$$

a double integral determining the area of the circle. For $n = 3$ we have the triple integral determining the volume of the sphere. Your task in this project is to find a formula for the volume of an n-sphere of radius r, for any $n \geq 1$.

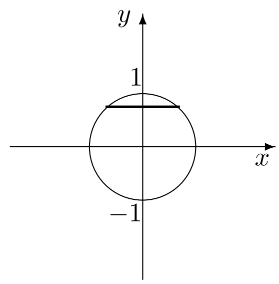
To help you with this task you can use as fact that

$$V(S^n(r)) = r^n V(S^n(1)).$$

So we only need to be concerned with how to compute the volume of the unit n-sphere.

Assignment

1. Write, but do not evaluate an n-tuple integral to compute the volume of $S^n(r)$.
2. Show that
$$V(S^n(r)) = r^n V(S^{n-1}(1)) \int_{-1}^{1} (\sqrt{1-x_n^2})^{n-1} dx_n.$$
Hint: Recall that the volume is computed by summing (integrating) the cross-sectional areas. So for $S^2(r)$ the cross-sections are line segments (which are 1-spheres) as depicted below.



Their radii are the distance from the y-axis to the circle. So we have
$$\begin{aligned} V(S^2(r)) &= r^2 V(S^2(1)) \\ &= r^2 \int_{-1}^{1} V(S^1(\text{distance from } y\text{-axis to the circle}))dy \\ &= r^2 \int_{-1}^{1} V(S^1(1))[\text{distance from } y\text{-axis to the circle}]dy \\ &= r^2 V(S^1(1)) \int_{-1}^{1} [\text{distance from } y\text{-axis to the circle}]dy. \end{aligned}$$
So you need to find the distance from the y-axis to the circle in terms of y.

In dimension 3 you would have a sphere with a circular cross-section (2-spheres). In dimension 4, you would have a hypersphere with spherical cross-sections (3-spheres), etc.
3. For $n \geq 2$, make a trigonometric substitution to show that
$$\int_{-1}^{1} \left(\sqrt{1-x_n^2}\right)^{n-1} dx_n = \int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta.$$
and use integration by parts to show that
$$\int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta = \frac{n-1}{n} \int_{-\pi/2}^{\pi/2} \cos^{n-2} \theta d\theta.$$

4. Let

$$I_n = \int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta$$

compute I_0 and I_1 and show that for all $n \geq 1$ we have

$$I_n I_{n-1} = \frac{1}{n} I_0 I_1 = \frac{2\pi}{n}.$$

5. Determine $V(S^n(r))$ in terms of r^n, I_n, and $V(S^{n-1}(1))$ and use that to determine formulas for $V(S^n(r))$ for even and odd n.

Having now computed the volume of the n-sphere, you will examine some of its properties.

6. What can you say about the sequence $\{V(S^1(1)), V(S^2(1)), V(S^3(1)), \ldots\}$? In other words, when it increasing/decreasing, what are the maximum/minimum, does it converge, if so to what value?

   Hint: Use the ratio $\frac{V(S^n(1))}{V(S^{n-1}(1))} = I_n$ to determine when the sequence is increasing/decreasing.
7. Compare the volume of $S^n(1)$ to the volume of the n-cube with vertices $(x_1, \ldots, x_n)$ with $x_i = \pm 1$ for $1 \leq i \leq n$. What does this say about the amount of space the unit n-sphere takes up inside of the n-cube in the limit as n goes to infinity?
8. How many points do the n-cube and the n-sphere have in common? What is the limit of the number of points as n goes to infinity.

Checklist for Your Writing Projects

Based on checklists by Annalisa Crannell at Franklin & Marshall and Tommy Ratliff at Wheaton College.

Does this paper:

1. clearly (re)state the problem to be solved?
2. provide an explanation as to how the problem will be approached?
3. state the answer in a few complete sentences which stand on their own?
4. give a precise and well-organized explanation of how the answer was found?
5. clearly label diagrams, tables, graphs, or other visual representations of the math?
6. define all variables, terminology, and notation used?
7. clearly state the assumptions which underlie the formulas and theorems, and explain how each formula or theorem is derived, or where it can be found?
8. give acknowledgment where it is due?
9. use correct spelling, grammar, and punctuation?
10. contain correct mathematics?
11. solve the questions that were originally asked?