Solutions to the Calculus and Linear Algebra problems on the Comprehensive Examination of January 27, 2012

Solutions to Problems 1-4 are omitted since they involve topics no longer covered on the Comprehensive Examination.

5. [15 points] Evaluate the following integrals:

   (a) $\displaystyle\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx.$

   Solution: We need to switch the order of integration. The region of integration lies between $y = x^2$ and $y = 1$ for $0 \le x \le 1$. This is the same as the region between $x = 0$ and $x = \sqrt{y}$ for $0 \le y \le 1$. Thus

$$\begin{aligned}
\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx &= \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy \\
&= \int_0^1 \tfrac{1}{4} x^4 \sin(y^3)\Big|_0^{\sqrt{y}} dy = \tfrac{1}{4} \int_0^1 y^2 \sin(y^3)\, dy \\
&= \tfrac{1}{12} \int_0^1 \sin(u)\, du = \tfrac{1}{12}\Big(-\cos(u)\Big|_0^1 \Big) \\
&= \tfrac{1}{12}\big(-\cos(1) - (-\cos(0))\big) = \tfrac{1}{12}\big(1 - \cos(1)\big),
\end{aligned}$$

   where we used the substitution $u = y^3$, $du = 3y^2\, dy$.

   (b) $\int_C (y^2 + 6y)dx + \big(\cos(y^2) + 2x(y+1)\big)dy$, where C is some circle of radius 3 in the xy-plane, oriented counterclockwise.

   Solution: Just apply Greens's Theorem, letting D be the domain enclosed by C:

$$\begin{aligned}
\int_C (y^2 + 6y)dx + \big(\cos(y^2) + 2x(y+1)\big)dy &= \iint_D (2(y+1) - (2y+6))\, dA \\
&= \iint_D -4\, dA = -4 \iint_D dA
\end{aligned}$$

   You can do this integration if you like, but the area of a circle of radius 3 is just 9π so the answer is -36π

6. [10 points] Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Note that, while the sphere is not centered at the origin, using spherical coordinates still works nicely for this problem.

   Solution: Above the cone $z^2 = x^2 + y^2$ just means, for example, $z \ge x$ on the xz-plane; in spherical coordinates, clearly $\tan(\phi) \le 1$ so $\phi \le \pi/4$. We can also convert the other constraint to spherical: $x^2 + y^2 + z^2 \le z \Rightarrow \rho^2 \le \rho\cos(\phi) \Rightarrow r \le \cos(\phi)$, so:

$$
\begin{aligned}
V &= \iiint\limits_V dV \\
&= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= 2\pi \int_0^{\pi/4} \frac{1}{3}\rho^3 \sin(\phi)\Big|_{\rho=0}^{\cos(\phi)} d\phi \\
&= \frac{2\pi}{3}\int_0^{\pi/4} \cos^3(\phi)\sin(\phi)\, d\phi \\
&= \frac{2\pi}{3}\Big(-\frac{1}{4}\cos^4(\phi)\Big)\Big|_0^{\pi/4} = -\frac{\pi}{6}\Big(\Big(\frac{\sqrt{2}}{2}\Big)^4 - 1\Big) = \frac{\pi}{8}
\end{aligned}
$$

where we used the substitution $u = \cos(\phi)$, $du = -\sin(\phi)\,d\phi$.

7. [12 points] Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Is f continuous at $(0,0)$? Justify your answer.

Solution: f is not continuous. Take $\lim_{(x,y)\to(0,0)} f(x,y)$ along the line $y = x$:

$$\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = 1/2 \neq 0 = f(0,0)$$

If $\lim_{(x,y)\to(0,0)} f(x,y)$ exists then it must be $1/2$ because that's the result of approaching along $y = x$, but then the limit does not equal the actual value of f at $(0,0)$ so f is not continuous. (Alternatively, it is sufficient to show that $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE by approaching along, for example, the y-axis. That would give a limit of 0, which $\neq 1/2$.)

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

Solution: Use the definition of partial derivative. Note that since f is symmetric with respect to x and y, $f_x = f_y$ so only one calculation is needed:

$$f_y(0,0) = f_x(0,0) = \lim_{h\to 0} \frac{\frac{h(0)}{h^2+0^2}}{h} = \lim_{h\to 0} \frac{0}{h^3} = 0$$

(c) Is f differentiable at (0,0)? Justify your answer.

Solution: Will not need to prove differentiability in the new comps. At any rate, though, f is obviously not differentiable at $(0,0)$ because it isn't even continuous.

8. [10 points] Assume that the temperature in degrees Celsius at a point (x, y) on the circle $x^2 + y^2 = 4$ is given by $T(x, y) = x^2 + 4x - y^2 + 12$. Find the points on the circle at which the temperature is highest and lowest, and state the temperature at each of these points.

   Solution: We use Lagrange multipliers here to find the max and min of T with the constraint $g(x, y) = x^2 + y^2 = 4$. To find the points to test, we solve the system $\nabla T(x, y) = \lambda \nabla g(x, y)$ (so $2x + 4 = \lambda(2x)$ and $-2y = \lambda(2y)$) and $g(x, y) = 4$. We have two cases here. If $y = 0$, then clearly $x = \pm 2$. If $y \neq 0$, then $\lambda = -1$ so $x = -1$, which in turn gives us $y = \pm\sqrt{3}$. So we just plug in the points $(\pm 2, 0)$ and $(-1, \pm\sqrt{3})$ to get the max and min. We find that $(2, 0)$ gives us the max T of 24, and $(-1, \pm\sqrt{3})$ gives us the min of 6.

9. [10 points] Let C be a 3×5 real-valued matrix. Answer the following questions about C and briefly justify your answers:

   (a) Can the columns of C be linearly independent?

   Solution: No. The columns are $\in \mathbb{R}^3$ so no more than 3 columns can be linearly independent. (If 3 columns of C are linearly independent, then they form a basis for $\mathbb{R}^3$ so they span, among other things, the fourth and fifth columns of C.)

   (b) Does the equation $C\mathbf{x} = \mathbf{0}$ have a unique solution with $\mathbf{x} \in \mathbb{R}^5$?

   Solution: No. There are 3 equations in 5 variables so there are at least 2 free variables, which means infinitely many solutions.

   (c) Assume that the span of the columns of C is all of $\mathbb{R}^3$. Can you determine the nullity (= dimension of the null space or kernel) of C?

   Solution: Let n = nullity and r = rank of C. Since the columns of C span all $\mathbb{R}^3$, $r = 3$. Thus by the rank-nullity theorem, $n = 5 - r = 2$.

10. [8 points] Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}.$$

   (a) Determine the eigenvalues and eigenvectors of A.

   Solution: We want to find all possible nontrivial values $\mathbf{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$, which means that the system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has a nontrivial solution. Thus, $A - \lambda I$ is non-invertible, or $\det(A - \lambda I) = 0$. This gives us the characteristic polynomial $(1 - \lambda)(2 - \lambda) - 12 = 0$. The solutions to this quadratic are $\lambda = 5$ and $\lambda = -2$. Now we solve for the nullspace of $A - \lambda I$ for each λ; these are our eigenvectors (plus zero):

$$\lambda = 5 : \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}.$$

The nullspace of this matrix is clearly $c\begin{bmatrix}3\\4\end{bmatrix}$, $c \in \mathbb{R}$ (that is, $\text{span}\left(\begin{bmatrix}3\\4\end{bmatrix}\right)$).

$$\lambda = -2 : \begin{bmatrix}3 & 3\\4 & 4\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}.$$

The nullspace of this matrix is clearly $c\begin{bmatrix}-1\\1\end{bmatrix}$, $c \in \mathbb{R}$ (that is, $\text{span}\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$).

So to conclude, A has an eigenvalue of 5 with corresponding eigenvectors $c\begin{bmatrix}3\\4\end{bmatrix}$, $c \neq 0$, and an eigenvalue of -2 with corresponding eigenvectors $c\begin{bmatrix}-1\\1\end{bmatrix}$, $c \neq 0$.

(b) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: Following the diagonalization procedure, we let D be the matrix with A's eigenvalues in the diagonal entries: $D = \begin{bmatrix}-2 & 0\\0 & 5\end{bmatrix}$. P is the matrix consisting of each eigenvalue's eigenvectors, taking care to keep the order the same as in D: $P = \begin{bmatrix}-1 & 3\\1 & 4\end{bmatrix}$. This is easily checkable; we find that $P^{-1} = \begin{bmatrix}-4/7 & 3/7\\1/7 & 1/7\end{bmatrix}$, and a simple (if boring) exercise in matrix multiplication shows that $A = PDP^{-1}$.

11. [10 points] Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V. Suppose T is one-to-one (injective). Prove that if $\{v_1, \ldots, v_n\}$ is a basis for V, then $\{T(v_1), \ldots, T(v_n)\}$ is also a basis for V.

Solution: First, we show that $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent.
Given scalars $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = 0$:
Since T is linear, $T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = 0$. Note that T is linear, so $T(0)$ also equals 0. Since T is one-to-one, this means that $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$. We know that $\{v_1, \ldots, v_n\}$ is a basis, so it is linearly independent, which means $\alpha_i = 0$ $\forall i \in \{1, \ldots, n\}$. But that's what we were trying to prove! So $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent , as desired.

Now note that if $\{v_1, \ldots, v_n\}$ is a basis for V, then V has dimension n. Thus since $\{T(v_1), \ldots, T(v_n)\}$ is a set of n linearly independent vectors, it is a basis for V (in an n-dimensional vector space, n vectors span $\Leftrightarrow$ they are linearly independent $\Leftrightarrow$ they form a basis). QED

12. [15 points] Let $T : P_2 \to \mathbb{R}^2$, where $P_2 = \{a + bt + ct^2 : a, b, c \in \mathbb{R}\}$, be defined by

$$T(p) = \begin{bmatrix}p(1)\\p(2)\end{bmatrix}.$$

You may assume that T is linear.

(a) Find bases of the null space (kernel) and range of T.

Solution: One way to find the kernel is to note that $T(p) = 0 \Rightarrow p(1) = 0$, $p(2) = 0$. The only quadratics with zeros at 1 and 2 are of the form $c(x-1)(x-2) = c(t^2 - 3t + 2)$, $c \in \mathbb{R}$. Thus $t^2 - 3t + 2$ is a basis of the kernel of T.

Alternatively, we could rewrite T using the basis $\{1, t, t^2\}$:

$$T(a + bt + ct^2) = \begin{bmatrix} a+b+c \\ a+2b+4c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Then row reduce:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

Hence the nullspace of the matrix is span $\left(\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right)$, so that $2 \cdot 1 - 3 \cdot t + 1 \cdot t^2$ spans the kernel of T.

Since T maps to $\mathbb{R}^2$, the range of T is the column space of its matrix representation. From the matrix, we can readily see that the range is all of $\mathbb{R}^2$, since the first two columns span $\mathbb{R}^2$.

(b) Find the matrix representation of this transformation with respect to the bases $\{1, t, t^2\}$ and $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

Solution: The procedure here is to find what T does to each member of the basis for P_2, and put those results in the given basis of $\mathbb{R}^2$. So:

$$T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T(t^2) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The representations of $T(1)$, $T(t)$, and $T(t^2)$ were found by inspection. The more systematic method would be (for $T(1)$, to pick an example) to solve the system of equations:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

At any rate, the matrix representation is $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$.