Math 13 Fall 2009: Exam 1

Name:

Instructions: Each problem is scored out of 8 points for a total of 32 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1.

(a) Find the equations of the planes given by the following information.

(1) Perpendicular to the line $\langle 2+t, 2t, 1+t\rangle$ and containing the point $(1, 2, 0)$.

(2) Containing the vectors $\langle 0, 1, 1\rangle$ and $\langle 1, 0, 2\rangle$ and containing the point $(1, 0, 1)$.

(b) Find an equation for their line of intersection.

Proof.

(a) (1) We have $\vec{n_1} = \langle 1, 2, 1\rangle$ and so the plane is

$$1(x-1) + 2(y-2) + 1(z-0) = 0$$

which is

$$x + 2y + z - 5 = 0.$$

(2) To find the normal we take a cross product to get

$$\vec{n_2} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = 2\hat{i} + \hat{j} - \hat{k} = \langle 2, 1, -1\rangle .$$

So the equation is

$$2(x-1) + 1(y-0) - 1(z-1) = 0$$

which is

$$2x + y - z - 1 = 0.$$

(b) The direction of the line of intersection is the cross product of the normals of the two planes, so we compute

$$\det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{pmatrix} = -2\hat{i} + 2\hat{j} + \hat{k} - 4\hat{k} - \hat{i} + \hat{j} = \langle -3, 3, -3\rangle$$

Since we only need the direction we might as well take $\langle -1, 1, -1\rangle$. Now we need a point on the line, which must be on both planes. Adding the two equations together we get

$$3x + 3y - 6 = 0$$

so we can choose $x = 1, y = 1$, and then we have $z = 2$. So we have the line in parametric equations as

$$\langle 1-t, 1+t, 2-t\rangle$$

and in symmetric equations as

$$1 - x = y - 1 = 2 - z.$$

□

Problem 2. Let C be the curve of intersection of the two surfaces $y = \frac{2}{x}$ and $z = \frac{4x^2}{3y}$.

(a) Find parametric equations for C.

(b) Set-up the integral for the length of the curve C from $(1, 2, \frac{2}{3})$ to $(2, 1, \frac{16}{3})$.

Proof.

(a) We can parameterize as $x = t$ to get $y = \frac{2}{t}$ and then $z = \frac{2t^3}{3}$ to get

$$\vec{r}(t) = \left\langle t, \frac{2}{t}, \frac{2t^3}{3} \right\rangle.$$

(b) Using the parametrization

$$\vec{r}(t) = \left\langle t, \frac{2}{t}, \frac{2t^3}{3} \right\rangle$$

we get

$$\vec{r'}(t) = \left\langle 1, -\frac{2}{t^2}, 2t^2 \right\rangle$$

and hence

$$\left|\vec{r'}\right| = \sqrt{1 + \frac{4}{t^4} + 4t^4}.$$

So we have

$$\int_1^2 \sqrt{1 + \frac{4}{t^4} + 4t^4}dt.$$

□

Problem 3. A particle starts ($t = 0$) at the origin with initial velocity $\vec{v}(0) = \hat{\imath} + \hat{\jmath} - \hat{k}$. Its acceleration is given by $\vec{a}(t) = t\hat{\imath} + \hat{\jmath} + t\hat{k}$.

(a) Find $\vec{r}(t)$.

(b) Find the curvature at $(\frac{10}{3}, 4, -\frac{2}{3})$.

(c) Find a_T and a_N at $(\frac{10}{3}, 4, -\frac{2}{3})$.

Proof.

(a) We first need to find $\vec{v}(t)$ to get

$$\vec{v}(t) = \left\langle \frac{t^2}{2} + 1, t + 1, \frac{t^2}{2} - 1 \right\rangle$$

Then we find

$$\vec{r}(t) = \left\langle \frac{t^3}{6} + t, \frac{t^2}{2} + t, \frac{t^3}{6} - t \right\rangle.$$

(b) From the y coordinate we have $t^2/2 + t = 4$ and hence $(t + 4)(t - 2) = 0$. So we have $t = 2$ by examining either the x or z coordinate. Computing

$$\vec{r'}(2) = \vec{v}(2) = \langle 3, 3, 1 \rangle \quad \text{and} \quad \vec{r''}(2) = \vec{a}(2) = \langle 2, 1, 2 \rangle.$$

So we compute curvature as

$$\kappa = \frac{|r' \times r''|}{|r'|^3}$$

to get

$$\det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} = 6\hat{\imath} + 2\hat{\jmath} + 3\hat{k} - 6\hat{k} - \hat{\imath} - 6\hat{\jmath} = \langle 5, -4, -3 \rangle$$

so we have

$$\kappa = \frac{\sqrt{25 + 16 + 9}}{(\sqrt{19})^{3/2}} = \frac{5\sqrt{2}}{19\sqrt{19}}.$$

(c) We know

$$a_T = \frac{r' \cdot r''}{|r'|} \quad \text{and} \quad a_N = \frac{|r' \times r''|}{|r'|}.$$

So we have

$$a_N = \sqrt{\frac{50}{19}}$$

and

$$a_T = \frac{6 + 3 + 2}{\sqrt{19}} = \frac{11}{\sqrt{19}}$$

□

Problem 4. Given vectors $\vec{a}$ and $\vec{b}$ show that

$$\vec{c} = |b|\,\vec{a} + |a|\,\vec{b}$$

bisects the angle between $\vec{a}$ and $\vec{b}$.

Proof. Bisection means that the angle between $\vec{a}$ and $\vec{c}$ is equal to the angle between $\vec{b}$ and $\vec{c}$. Since we know the angle θ between two vectors satisfies

$$\cos\theta = \frac{\vec{a}\cdot\vec{c}}{|a|\,|c|}$$

we must have

$$\frac{\vec{a}\cdot\vec{c}}{|a|\,|c|} = \frac{\vec{b}\cdot\vec{c}}{|b|\,|c|}.$$

So we compute both sides. We first get

$$\frac{\vec{a}\cdot\vec{c}}{|a|\,|c|} = \frac{|b|\,(\vec{a}\cdot\vec{a}) + |a|\,(\vec{a}\cdot\vec{b})}{|a|\,|c|} = \frac{|a|\,(|a|\,|b| + (\vec{a}\cdot\vec{b}))}{|a|\,|c|} = \frac{|a|\,|b| + (\vec{a}\cdot\vec{b})}{|c|}.$$

Computing the other side we have

$$\frac{\vec{b}\cdot\vec{c}}{|b|\,|c|} = \frac{|b|\,(\vec{b}\cdot\vec{a}) + |a|\,(\vec{b}\cdot\vec{b})}{|b|\,|c|} = \frac{|b|\,((\vec{b}\cdot\vec{a}) + |b|\,|a|)}{|b|\,|c|} = \frac{(\vec{b}\cdot\vec{a}) + |b|\,|a|}{|c|}$$

Since the dot product is commutative these two quantities are equal.

Alternate Solution: Notice that

$$\frac{\vec{c}}{|a|\,|b|} = \frac{\vec{a}}{|a|} + \frac{\vec{b}}{|b|}$$

is the sum of two unit vectors. Note also that the magnitude of a vector has no effect on its direction, so if $\frac{\vec{c}}{|a||b|}$ bisects the angle, then so does $\vec{c}$. So we may assume that $\vec{c} = \vec{a} + \vec{b}$ for two unit vectors $\vec{a}$ and $\vec{b}$.

The sum of two unit vectors forms an equilateral parallelogram whose diagonal is $\vec{c}$. The two halves of the parallelogram form isosceles triangles which are in fact identical. (draw a nice picture here). So we have that $\vec{c}$ bisects the angle between $\vec{a}$ and $\vec{b}$. □