Amherst College
Economics 58

A simple Lagrange Multiplier example

The problem:

Choose x and y to maximize $z = xy$ subject to the constraint:
$c - ax - by = 0$.

Setup Lagrangian

$$L = xy + \lambda(c - ax - by)$$

First order conditions

$$\frac{\partial L}{\partial x} = y - \lambda a = 0$$
$$\frac{\partial L}{\partial y} = x - \lambda b = 0$$
$$\frac{\partial L}{\partial \lambda} = c - ax - by = 0$$

Solving the problem

$$\frac{y}{a} = \frac{x}{b} \Rightarrow ax = by \Rightarrow x^* = \frac{c}{2a} \quad y^* = \frac{c}{2b} \quad z^* = x^* y^* = \frac{c^2}{4ab} \quad \lambda = \frac{y}{a} = \frac{c}{2ab}$$

Examples

c	*a*	*b*	*x*	*y*	*z*
1	**1**	**1**	**.5**	**.5**	**.25**
1	**2**	**1**	**.25**	**.5**	**.125**
1	**3**	**2**	**1/6**	**1/4**	**1/24**
1	**.5**	**.5**	**1**	**1**	**1**
1.1	**1**	**1**	**.55**	**.55**	**.3025**
0.9	**1**	**1**	**.45**	**.45**	**.2025**
1	**1.1**	**1**	**.4545**	**.5**	**.2273**

How does changing constraint change objective?

$\frac{\partial z^*}{\partial c} = \frac{c}{2ab}$ **Example shows this (approximately)** $\partial z^* = \frac{c}{2ab}\partial c = \frac{1}{2}(\pm .1) = \pm .05$**.**

Also: $\frac{\partial z^*}{\partial a} = \frac{-c^2}{4a^2b}$ **Example shows (approximately)**

$\partial z^* = \frac{-1}{4}\partial a = -0.25(+0.1) = -0.025$.

Can we get these results without actually solving the problem for an explicit solution? Yes, use the envelope theorem:

$$\frac{\partial z^*}{\partial c} = \frac{\partial L}{\partial c} = \lambda = \frac{c}{2ab} \quad \frac{\partial z^*}{\partial a} = \frac{\partial L}{\partial a} = -\lambda x = -\frac{c}{2ab}\cdot\frac{c}{2a} = \frac{-c^2}{4a^2b}$$

Recovering optimal *x* from the original Lagrangian expression using envelope results:

$$x^* = \frac{\frac{-\partial L}{\partial a}}{\frac{\partial L}{\partial c}} = \frac{\frac{c^2}{4a^2b}}{\frac{c}{2ab}} = \frac{c}{2a}$$

Second Order Conditions:

Intuitive: The function $z = xy$ is obviously not concave – doubling x and y would multiply z by four. But the function is quasi-concave as can be shown by looking at a level curve of the form $c = xy$.

Formal: The Hessian here is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $H_1 = 0$, $H_2 = -1$. Hence H is not negative definite.

The bordered Hessian is $\begin{bmatrix} 0 & y & x \\ y & 0 & 1 \\ x & 1 & 0 \end{bmatrix}$, so $H_2 = -y^2$, $H_3 = 2xy$. Hence the leading principal minors of H follow the required pattern for quasi-concavity.