Math 13 Fall 2008: Exam 3

Name:

Instructions: There are 5 questions on this exam of which you must do 4. Each problem is scored out of 8 points for a total of 32 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Circle below the 4 problems you wish to be graded. Otherwise, I will grade the first 4 completed problems

1 2 3 4 5

Problem 1. Evaluate

$$\int_0^1 \int_{y^{1/3}}^1 \frac{1}{\sqrt{1+x^4}} dxdy$$

Proof. Reversing the order of integration we have

$$\begin{aligned}\int_0^1 \int_{y^{1/3}}^1 \frac{1}{\sqrt{1+x^4}} dxdy &= \int_0^1 \int_0^{x^3} \frac{1}{\sqrt{1+x^4}} dydx \\ &= \int_0^1 \frac{x^3}{\sqrt{1+x^4}} dx \\ &= \left[\frac{1}{2}(1+x^4)^{1/2}\right]_0^1 \\ &= \frac{\sqrt{2}-1}{2}.\end{aligned}$$

□

Problem 2. Find the mass of the ice cream cone bounded by $x^2 + y^2 + z^2 = 18, z = \sqrt{x^2 + y^2}, z = 0$ if the density is given by $\delta = \rho$.

1. Set up the integral in both spherical and cylindrical coordinates.

2. Evaluate one of the two integrals.

Proof. The cone is given by $\tan(\phi) = 1$ and so $\phi = \frac{\pi}{4}$. So we have the integral

$$\begin{aligned} M &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sqrt{18}} \rho^3 \sin\phi d\rho d\theta d\phi \\ &= 2\pi \int_0^{\pi/4} 81 \sin\phi d\phi \\ &= 162\pi[-\cos\phi]_0^{\pi/4} \\ &= 162\pi \left(\frac{1}{\sqrt{2}} - 1 \right) = 81\pi(2 - \sqrt{2}) \end{aligned}$$

We have intersection at $2z^2 = 18$ and hence $z^2 = 9$ and so $z = 3$. At $z = 3$ we have the circle of radius 3.

$$\int_0^{2\pi} \int_0^3 \int_r^{\sqrt{18-r^2}} r\sqrt{r^2 + z^2} dz dr d\theta$$

□

Problem 3. Find the volume of the part of the ellipsoid $4x^2+9y^2+z^2=6$ inside the elliptic paraboloid $z=4x^2+9y^2$.

Proof. We first make the change of coordinates $\frac{u}{2}=x, \frac{v}{3}=y, w=z$. The Jacobian of this transformation is

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{6}.$$

So we have the region contained in the sphere $u^2+v^2+w^2=6$ and paraboloid $w=u^2+v^2$. The intersection is $w^2+w-6=0$. So we have $w=2$. So the uv region is a circle of radius $\sqrt{2}$. So we have the volume is given by

$$\begin{aligned} V &= \frac{1}{6}\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{r^2}^{\sqrt{6-r^2}} r\,dw\,dr\,d\theta \\ &= \frac{1}{3}\pi\int_0^{\sqrt{2}} r\sqrt{6-r^2}-r^3\,dr\,d\theta \\ &= \frac{1}{3}\pi\left[-\frac{1}{3}(6-r^2)^{3/2}-\frac{r^4}{4}\right]_0^{\sqrt{2}} \\ &= \frac{1}{3}\pi\left(-\frac{11}{3}+\frac{6\sqrt{6}}{3}\right) \\ &= \frac{\pi(6\sqrt{6}-11)}{9}. \end{aligned}$$

□

Problem 4. Find the volume above the upper sheet of the hyperboloid $x^2 + y^2 = z^2 - 1$ and below the plane $z = 3$.

Proof. We have that the vertex of the upper sheet is given by $z = 1$ at $(0, 0, 1)$ and at $z = 3$ we have $x^2 + y^2 = 8$.

In cylindrical coordinates we have

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\sqrt{8}} \int_{\sqrt{1+r^2}}^{3} r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^{\sqrt{8}} 3r - r\sqrt{1+r^2} dr d\theta \\ &= \int_0^{2\pi} \left[\frac{3r^2}{2} - \frac{(1+r^2)^{3/2}}{3} \right]_0^{\sqrt{8}} d\theta \\ &= \int_0^{2\pi} \frac{10}{3} d\theta \\ &= \frac{20\pi}{3}. \end{aligned}$$

You could also have

$$\begin{aligned} V &= \int_1^3 \int_0^{2\pi} \int_0^{\sqrt{z^2-1}} r dr d\theta dz \\ &= \int_1^3 \int_0^{2\pi} \frac{1}{2}(z^2 - 1) d\theta dz \\ &= \int_1^3 \pi(z^2 - 1) dz \\ &= \frac{20\pi}{3}. \end{aligned}$$

□

Problem 5. Find the moment of inertia about the z-axis for the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant if the density is given by $\delta = 15$.

Proof. We know that

$$I_z = \iiint_R (x^2 + y^2)\delta dV.$$

In cylindrical coordinates we have

$$\begin{aligned} I_z &= \int_0^2 \int_0^{\pi/2} \int_0^{\sqrt{4-r^2}} 15r^3 dz d\theta dr \\ &= \int_0^2 \int_0^{\pi/2} 15r^3(\sqrt{4-r^2}) d\theta dr \\ &= \int_0^2 \frac{15\pi}{2} r^3(\sqrt{4-r^2}) dr \end{aligned}$$

we set $u = 4 - r^2$ to get

$$\begin{aligned} &= -\frac{15\pi}{4} \int_4^0 (4-u)u^{1/2} du \\ &= -\frac{15\pi}{4} \int_4^0 4u^{1/2} - u^{3/2} du \\ &= \frac{15\pi}{4} \int_0^4 4u^{1/2} - u^{3/2} du \\ &= \frac{15\pi}{4} \left[\frac{8u^{3/2}}{3} - \frac{2u^{5/2}}{5} \right]_0^4 \\ &= 15\pi \left(\frac{16}{3} - \frac{16}{5} \right) \\ &= 32\pi. \end{aligned}$$

In spherical coordinates we have

$$\begin{aligned} I_z &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 15(\rho \sin\phi)^2 \rho^2 \sin\phi d\rho d\phi d\theta \\ &= 15 \int_0^{\pi/2} \int_0^{\pi/2} \frac{32}{5} \sin^3\phi d\phi d\theta \\ &= 96 \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi(1 - \cos^2\phi) d\phi d\theta \\ &= 96 \int_0^{\pi/2} \left[-\cos\phi + \frac{\cos^3\phi}{3} \right]_0^{\pi/2} \\ &= 96 \int_0^{\pi/2} \frac{2}{3} d\theta = 32\pi. \end{aligned}$$

□