

Math 13 Fall 2009: Exam 3
Wednesday December 2, 2009

Name:

Instructions: There are 4 questions on this exam each of which is scored out of 8 points for a total of 32 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1. Evaluate

$$\int_{-2}^2 \int_{x^2}^4 \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dy dz dx$$

Proof. This is the volume inside the paraboloid $z = x^2 + y^2$ below $z = 4$. In Cylindrical coordinates we have

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 r dz dr d\theta = \int_0^{2\pi} \int_0^2 (4r - r^3) dr d\theta = \int_0^{2\pi} \left(2r^2 - \frac{r^4}{4} \right)_0^2 d\theta = \int_0^{2\pi} 4 d\theta = 8\pi.$$

□

Problem 2. Given the region inside $r = 1 + 2 \cos \theta$ and outside $r = 2$ with density $\delta = r$.

1. Write but do not evaluate integrals for mass and center of mass.
2. Write but do not evaluate integrals for the moments of inertia I_x and I_y .

Proof. We solve for the points of intersection as $2 = 1 + 2 \cos \theta$ to get $\cos \theta = \frac{1}{2}$ and so $\theta = \pm \frac{\pi}{3}$.

$$\begin{aligned}M &= \iint_R \delta dA = \int_{-\pi/3}^{\pi/3} \int_2^{1+2 \cos \theta} r^2 dr d\theta \\M_x &= \iint_R y \delta dA = \int_{-\pi/3}^{\pi/3} \int_2^{1+2 \cos \theta} r^3 \sin \theta dr d\theta \\M_y &= \iint_R x \delta dA = \int_{-\pi/3}^{\pi/3} \int_2^{1+2 \cos \theta} r^3 \cos \theta dr d\theta\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right).$$

$$\begin{aligned}I_x &= \iint_R y^2 \delta dA = \int_{-\pi/3}^{\pi/3} \int_2^{1+2 \cos \theta} r^4 \sin^2 \theta dr d\theta \\I_y &= \iint_R x^2 \delta dA = \int_{-\pi/3}^{\pi/3} \int_2^{1+2 \cos \theta} r^4 \cos^2 \theta dr d\theta\end{aligned}$$

□

Problem 3. Find the volume of the solid bounded by $z = x^2$, $y + z = 4$, and $y = 0$.

Proof. We have

$$\begin{aligned}
 V &= \int_{-2}^2 \int_{x^2}^4 \int_0^{4-z} dydzdx = \int_{-2}^2 \int_{x^2}^4 (4-z)dzdx \\
 &= \int_{-2}^2 \left(4z - \frac{z^2}{2}\right)_{x^2}^4 dx = \int_{-2}^2 (16-8) - \left(4x^2 - \frac{x^4}{2}\right) dx \\
 &= \int_{-2}^2 8 - 4x^2 + \frac{x^4}{2} dx = \left(8x - \frac{4x^3}{3} + \frac{x^5}{10}\right)_{-2}^2 \\
 &= 2 \left(16 - \frac{32}{3} + \frac{16}{5}\right) \\
 &= 2 \left(16 - 10\frac{2}{3} + 3\frac{1}{5}\right) \\
 &= 2 \left(5\frac{1}{3} + 3\frac{1}{5}\right) \\
 &= 16\frac{16}{15} = 17\frac{1}{15} = \frac{256}{15}.
 \end{aligned}$$

or

$$\begin{aligned}
 V &= \int_{-2}^2 \int_0^{4-x^2} \int_{x^2}^{4-y} dzdydx \\
 V &= \int_0^4 \int_0^{4-y} \int_{-\sqrt{x}}^{\sqrt{z}} dx dz dy \\
 V &= \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_0^{4-z} dy dx dz \\
 V &= \int_0^4 \int_0^{4-z} \int_{x^2}^{4-y} dx dy dz \\
 V &= \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{x^2}^{4-y} dz dy dx
 \end{aligned}$$

□

Problem 4. Find the mass inside the ellipsoid $\frac{x^2}{25} + y^2 + \frac{z^2}{9} = 4$ in the first octant if $\delta = \frac{x^2}{25} + y^2 + \frac{z^2}{9}$.

Proof. We make the change of variables $x = 5u, y = v, z = 3w$ to get Jacobian

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 15.$$

Notice that points in the first octant stay in the first octant, since the change of coordinates sends positive values to positive values and negative values to negative values and zero to zero. So we are integrating $\delta = \rho^2$ over the sphere of radius 2 in the first octant. We get

$$\begin{aligned} M &= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 15\rho^4 \sin \phi d\rho d\theta d\phi \\ &= \int_0^{\pi/2} \int_0^{\pi/2} 15 \frac{32}{5} \sin \phi d\theta d\phi \\ &= \int_0^{\pi/2} 48\pi \sin \phi d\phi \\ &= 48\pi \end{aligned}$$

□