Math 13 Fall 2008: Final Exam

Instructions: There are 10 questions on this exam of which you must do 8. Each problem is scored out of 8 points for a total of 64 points. You may not use any outside materials (eg. notes or calculators). You have 3 hours to complete this exam. Remember to fully justify your answers. Mark clearly in your blue book which problems are to be graded.

Problem 1. A particle moves through space with position $r(t) = \left\langle t, t^2, \frac{4}{3}t^{3/2} \right\rangle$.

(1) Find the symmetric and parametric equations for the tangent line at $\left(1, 1, \frac{4}{3}\right)$.
(2) Find the curvature at $\left(1, 1, \frac{4}{3}\right)$.

Proof.

(1) We have T is in the direction of $r'(t) = \left\langle 1, 2t, 2t^{1/2} \right\rangle$. At $t = 1$ this is $\langle 1, 2, 2 \rangle$ so we have the line

$$r(t) = \left\langle 1 + t, 1 + 2t, \frac{4}{3} + 2t \right\rangle$$

$$x - 1 = \frac{y - 1}{2} = \frac{z - \frac{4}{3}}{2}$$

(2) We know the curvature is given by

$$\kappa = \frac{|r' \times r''|}{|r'|^3}$$

So we compute

$$r'(t) = \left\langle 1, 2t, 2t^{1/2} \right\rangle$$

$$r''(t) = \left\langle 0, 2, t^{-1/2} \right\rangle$$

At $t = 1$ we have

$$r'(1) = \langle 1, 2, 2 \rangle$$

$$r''(1) = \langle 0, 2, 1 \rangle$$

$$\left| r'(t) \right| = 3.$$

Then we compute the cross product to get

$$\langle 1, 2, 2 \rangle \times \langle 0, 2, 1 \rangle = \langle -2, -1, 2 \rangle$$

So we have

$$\kappa = \frac{3}{27} = \frac{1}{9}.$$

□

Problem 2. For the two planes

$$S_1 : 4x + y - z = 4 \quad \text{and} \quad S_2 : 2x + 2y + z = 3.$$

(1) Find the line of intersection of S_1 and S_2.
(2) Find the angle between S_1 and S_2.

Proof.

(1) The direction of the line of intersection is given by the cross product

$$\langle 4, 2, -1\rangle \times \langle 2, 2, 1\rangle = \langle 3, -6, 6\rangle .$$

We can also write this direction as $\langle 1, -2, 2\rangle$.

A point of intersection is $(0, \frac{7}{3}, \frac{5}{3})$. So we have the line

$$r(t) = \left\langle t, \frac{7}{3} - 2t, \frac{5}{3} + 2t \right\rangle .$$

(2) The angle is given by

$$\cos\theta = \frac{8+2-1}{\sqrt{18}\sqrt{9}} = \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}}$$

so we have $\theta = \frac{\pi}{4}$.

□

Problem 3. Determine the following two limits (if they exist).

(1) $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2}$.
(2) $\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^4+y^2}$.

Proof.

(1) We have

$$\left|\frac{5x^2y}{x^2+y^2}\right| \le |5y|$$

and hence

$$\begin{array}{ccccc} -5\,|y| & \le & \frac{5x^2y}{x^2+y^2} & \le & 5\,|y| \\ \downarrow_{y\to 0} & & \downarrow & & \downarrow_{y\to 0} \\ 0 & \le & \lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^4+y^2} & \le & 0 \end{array}$$

so by the squeeze theorem the limit exists and takes the values 0.

(2) We take the path $x = 0$ to get

$$\lim_{(0,y)\to(0,0)} \frac{0}{0+y^2} = 0$$

and the path $y = x^2$ to get

$$\lim_{(x,x^2)\to(0,0)} \frac{5x^4}{2x^4} = \frac{5}{2}$$

Since these two paths give different limiting values, the limit does not exist.

□

Problem 4. The plane $8x - 8y - 2z = C$ is tangent to the surface $z = x^2 + 2y^2$ at a certain point (x_0, y_0, z_0). Find (x_0, y_0, z_0) and the constant C.

Proof. The tangent plane to the surface is given by $\langle 2x, 4y, -1\rangle$ so we must have

$$2x_0(x-x_0)+4y_0(y-y_0)-(z-z_0)=0$$

The normal to the plane is given by $\langle 8,-8,-2\rangle$ which is parallel to $\langle 4,-4,-1\rangle$ so we must have $x_0 = 2$ and $y_0 = -1$. Finding z_0 from the equation for the surface we $z_0 = 6$. We find C using the fact that (x_0, y_0, z_0) must also be on the plane, so we get

$$C = 8(2) - 8(-1) - 2(6) = 16 + 8 - 12 = 12.$$

So we have

$$C = 12 \quad \text{and} \quad (x_0, y_0, z_0) = (2, 1, 4).$$

□

Problem 5. Classify the critical points for $f(x, y) = x^4 + y^4 - 4xy + 1$.

Proof. We find the critical points by solving

$$f_x = 4x^3 - 4y = 0$$
$$f_y = 4y^3 - 4x = 0.$$

Which is the same as

$$x^3 = y$$
$$y^3 = x$$

Which has the three solutions $(1, 1)$, $(-1, -1)$, and $(0, 0)$.

Using the second derivative test we compute

$$f_{xx} = 12x^2$$
$$f_{xy} = -4$$
$$f_{yy} = 12y^2$$

So we have discriminants

$$D(1, 1) = 144 - 16 > 0$$
$$D(-1, -1) = 144 - 16 > 0$$
$$D(0, 0) = 0 - 16 < 0.$$

So we have $(0, 0)$ is a saddle point. For the other two points was have $f_{xx} > 0$ so we have a local minimums at $(1, 1)$ and $(-1, -1)$. □

Problem 6. Find the point(s) on the surface $2x^2+3y^2+6z^2 = 36$ the sum of whose coordinates is maximal.

Proof. This is a Lagrange Multiplier problem with function $f(x, y, z) = x+y+z$ and constraint $x^2 + y^2 + z^2 - 36 = 0$. So we have the system

$$1 = \lambda 4x$$
$$1 = \lambda 6y$$
$$1 = \lambda 12z$$
$$36 = 2x^2 + 3y^2 + 6z^2.$$

Since λ cannot be 0 and still satisfy the equations we must have

$$4x = 6y = 12z.$$

Hence we have

$$x = \frac{3}{2}y \quad \text{and} \quad z = \frac{1}{2}y.$$

Now using the constraint equation we have

$$36 = \frac{9}{2}y^2 + 3y^2 + \frac{3}{2}y^2$$

Which is

$$72 = 18y^2$$

and hence

$$y = \pm 2.$$

So we have the two point $(3, 2, 1)$ and $(-3, -2, -1)$. So the maximum occurs for $(3, 2, 1)$. □

Problem 7. Find the moment of inertia about the x-axis for the triangle bounded by $3x+4y = 24$, $x = 0$, and $y = 0$ and having constant density 1.

Proof.

$$I_x = \int_0^8 \int_0^{6-(3/4)x} y^2 dydx = \int_0^8 -\frac{9}{64}(-8+x)^3 dx = \int_0^8 72 - 27x + \frac{27x^2}{8} - \frac{9x^3}{64} dx = 144$$
$$I_x = \int_0^6 \int_0^{8-(4/3)y} y^2 dxdy = \int_0^6 8y^2 - \frac{4}{3}y^3 dy = (\frac{8}{3}y^3 - \frac{y^4}{3})_0^6 = 72(8) - 36(12) = 72(2) = 144$$

□

Problem 8. Find the volume bounded by $y = x^2 + 2z^2$ and $y = 4 - x^2$.

Proof. The intersection is along $x^2 + z^2 = 2$ so we have in (alternative) polar coordinates

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} 4r - 2r^3 drd\theta = 2\pi(2r^2 - \frac{1}{2}r^4)_0^{\sqrt{2}} = 4\pi.$$

□

Problem 9. Evaluate

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{3/2} dzdydx.$$

Proof. Changing to Cylindrical we have

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{3/2}dzdydx = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2} r^3 r dz dr d\theta$$
$$= \int_0^{2\pi}\int_0^1 r^4(2-r^2-r^2)drd\theta$$
$$= \int_0^{2\pi}\int_0^1 2r^4-2r^6drd\theta$$
$$= \int_0^{2\pi}(\frac{2}{5}-\frac{2}{7})d\theta$$
$$= \frac{8\pi}{35}.$$

□

Problem 10. A closed path C is traversed counter-clockwise beginning at $(-1,-1)$ and moving to $(2,2)$ along the curve $y = x^2 - 2$ then returning to the point $(-1,-1)$ along the curve $y = x$. Find

$$\int_C -xydx + \frac{x^2}{2}dy$$

in two ways.

(1) Parameterizing the curves and computing the line integral directly.
(2) Using Green's Theorem.

Proof.

(1) We parameterize $C_1 : \langle t, t^2-2\rangle, -1 \le t \le 2$ and $C_2 : \langle 2-t, 2-t\rangle, 0 \le t \le 3$. So we have

$$\int_C -xydx + \frac{x^2}{2}dy = \int_{-1}^{2} -t(t^2-2) + \frac{t^2}{2}(2t)dt + \int_0^3 -(2-t)^2(-1) + \frac{(2-t)^2}{2}(-1)dt$$
$$= \int_{-1}^{2} -t^3+2t+t^3dt + \int_0^3 4-4t+t^2-(2-2t+t^2/2)dt$$
$$= \left[t^2\right]_{-1}^{2} + \int_0^3 2-2t+\frac{t^2}{2}dt$$
$$= 3 + \left[2t - t^2 + \frac{t^3}{6}\right]_0^3$$
$$= 3 + (6 - 9 + \frac{9}{2})$$
$$= \frac{9}{2}.$$

(2) Using Green's Theorem we have

$$\begin{aligned}
\int_C -xydx + \frac{x^2}{2}dy &= \int_{-1}^{2}\int_{x^2-2}^{x} x-(-x)dydx \\
&= \int_{-1}^{2}\int_{x^2-2}^{x} 2xdydx \\
&= \int_{-1}^{2} 2x(x-x^2+2)dx \\
&= \int_{-1}^{2} -2x^3+2x^2+4xdx \\
&= \left[-\frac{x^4}{2}+\frac{2x^3}{3}+2x^2\right]_{-1}^{2} \\
&= (-8+\frac{16}{3}+8)-(-\frac{1}{2}-\frac{2}{3}+2) \\
&= \frac{16}{3}+\frac{2}{3}-\frac{3}{2} \\
&= 6-\frac{3}{2}=\frac{9}{2}.
\end{aligned}$$

□