Math 12 Fall 2009: Exam 3

Name:

Instructions: There are 4 questions on this exam each of which is scored out of 8 points for a total of 32 points. You may not use any outside materials (eg. notes or books). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1. Determine whether or not the sequence converges and find its limit if it does converge.

(a) $a_n = \frac{e^n}{n^2}$, $n \geq 1$.

(b) $a_n = \frac{2n+\ln n}{n+3}$, $n \geq 1$.

Proof. To find the convergence of a sequence, we take the limit of the terms. If the limit exists, then the sequence converges to that value.

(a) We apply L'Hopital's Rule twice to get

$$\lim_{n\to\infty} \frac{e^n}{n^2} = \lim_{n\to\infty} \frac{e^n}{2n} = \lim_{n\to\infty} \frac{e^n}{2} = \infty$$

so the sequence diverges.

(b) We use L'Hopital's rule.

$$\lim_{n\to\infty} \frac{2n+\ln n}{n+3} = \lim_{n\to\infty} \frac{2+1/n}{1} = 2.$$

so the sequence converges to 2.

□

Problem 2. Determine whether or not the following series converge absolutely, converge conditionally, or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n}{(n+1)^2-n}$

(b) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$.

Proof.

(a) We apply the limit comparison test to the divergent p-series $\sum_{n=1}^{\infty}\frac{1}{n}$.

$$\lim_{n\to\infty}\frac{n^2}{(n+1)^2-n} = 1.$$

Since this is a finite nonzero value, we know from the limit comparison test that $\sum_{n=1}^{\infty}\frac{n}{(n+1)^2-n}$ must also diverge.

(b) We apply the alternating series test.

(a) We first check the limit of the terms to get

$$\lim_{n\to\infty}\frac{n}{n^2+1} = 0.$$

(b) We then need to see that the sequence is decreasing so we compare

$$a_{n+1} = \frac{(n+1)}{(n+1)^2+1} \quad \text{to} \quad a_n = \frac{n}{n^2+1}$$

by cross multiplying to get

$$(n+1)(n^2+1) = n^3+n^2+n+1 \quad ?? \quad n((n+1)^2+1) = n(n^2+2n+2) = n^3+2n^2+2n$$

Since we have n^2 on the left-hand side and $2n^2$ on the right-hand side we have

$$a_{n+1} < a_n$$

for n large enough.

Therefore the series converges by the alternating series test.

To check conditional or absolute convergence we consider

$$\sum_{n=0}^{\infty}\frac{n}{n^2+1}$$

Using the limit comparison to the Harmonic Series we have

$$\lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$$

so the absolute value series diverges.

Therefore, the original series is conditionally convergent.

□

Problem 3. Find the interval and radius of convergence of

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{2^n\sqrt{n}}.$$

Proof. Applying the ratio test to the given series gives

$$|x| \leq 2$$

We have that $R = 2$ and convergence for $-2 < x < 2$. We now need to check the endpoints.

Checking $x = -2$ we get the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{\sqrt{n}} = -\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}.$$

This is a divergent p-series with $p = 1/2$.

Checking $x = 2$ we get the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}.$$

This is an alternating series. The limit of its terms goes to 0 and they are decreasing since the numerator is a constant and the denominator is increasing. Therefore, this is a convergent series.

Therefore, we have

$$R = 2$$
$$I = (-2, 2].$$

□

Problem 4. Use power series to approximate $\int_0^{1/2} \frac{2x}{1+x^2} dx$ to within $\frac{1}{100}$.

Proof. We know $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$ so we have

$$\begin{aligned} \frac{2x}{1+x^2} &= 2x(1 - x^2 + x^4 - x^6 + \cdots) \\ &= 2x - 2x^3 + 2x^5 - 2x^7 + \cdots = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \end{aligned}$$

Integrating term-by-term we have

$$\begin{aligned} \int_0^{1/2} \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} dx &= \sum_{n=0}^{\infty} (-1)^n \frac{2x^{2n+2}}{2n+2}]_0^{1/2} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+2}(n+1)} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n+2}(n+1)} \\ &= \frac{1}{2^2} - \frac{1}{2^4(2)} + \frac{1}{2^6(3)} - \cdots \\ &= \frac{1}{4} - \frac{1}{32} + \frac{1}{192} - \cdots \end{aligned}$$

Or just integrating the first few terms we have

$$\begin{aligned} \int_0^{1/2} \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} dx &= \int_0^{1/2} 2x - 2x^3 + 2x^5 - 2x^7 + \cdots dx \\ &= \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots \right)_0^{1/2} \\ &= \frac{1}{4} - \frac{1}{32} + \frac{1}{192} - \cdots \end{aligned}$$

Since this is an alternating series, the remainder is at most the next term, so we have the approximation

$$\int_0^{1/2} \frac{2x}{1+x^2} dx \approx \frac{1}{4} - \frac{1}{32} = \frac{7}{32}.$$

□