Exam II - Solutions

Ben Hutz

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Problem 1 Consider the series $4 - 1 + \frac{1}{4} - \frac{1}{16} + \cdots$.

(a) Find a formula for the n-th term of the series.

(b) Find the sum of the series.

Solution:

(a): $4 - 1 + \frac{1}{4} - \frac{1}{16} + \cdots = 4\sum_{n=0}^{\infty}(-\frac{1}{4})^n$

(b): The sum of a geometric series is $\frac{a}{1-r}$. So we have $4\sum_{n=0}^{\infty}(-\frac{1}{4})^n = \frac{4}{1-(-\frac{1}{4})} = \frac{16}{5}$

Problem 2 Consider the series $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}$.

(a) Find a formula for the partial sums of the series.

(b) Find the sum of the series.

Solution:

(a): Splitting the terms into two fractions gives $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)} = \frac{1}{4}(\sum_{n=1}^{\infty}(\frac{1}{4n-3} - \frac{1}{4n+1}))$. So by writing out several terms(or solving algebraically) we see that $S_n = \frac{1}{4}(1 - \frac{1}{4n+1})$.

(b): The sum is the limit as $n \to \infty$ of S_n so we get $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)} = \frac{1}{4}$.

Problem 3 For which values of x is the series $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ convergent? What is its radius of convergence?

Solution: Using the ratio test we get $\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{x^n}\right| = \lim_{n\to\infty}\left|x\frac{n^3}{(n+1)^3}\right| = |x|$. For convergent we need this to be less than 1. So we get $|x| < 1$.

Now we check the end points ± 1. At $x = 1$ the series converges since it is a p-series with $p = 3$.

For $x = -1$, this is an alternating series which is decreasing and whose limit is 0, hence convergent. So it converges for $x \in [-1, 1]$. So $R = 1$.

Problem 4 Let $f(x) = \int_0^x e^{-y^2}dy$.

(a) Find the Maclaurin series of $f(x)$ and explain why it is convergent to $f(x)$.

(b) Use (a) to compute $f(1)$ to within 0.1.

Solution:

(a): $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$. So $e^{-y^2} = \sum_{n=0}^{\infty}\frac{(-y^2)^n}{n!}$. So $\int_0^x e^{-y^2}dy = \sum_{n=0}^{\infty}\int_0^x \frac{(-y^2)^n}{n!} = \sum_{n=0}^{x}[\frac{(-1)^n y^{2n+1}}{(2n+1)n!}]|_0^x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)n!}$.

For convergence to $f(x)$, we need to see that the remainder goes to 0. This is an alternating series so the remainder is bounded by the next term. $|R_{n-1}| < \left|\frac{(-1)^n x^{2n+1}}{(2n+1)n!}\right|$. Taking the limit as $n \to \infty$ yields, that for any value of x, $R_{n-1} \to 0$ since factorial groups faster than any fixed exponential function.

(b): Again this is an alternating series, so we just need to check the next term. Writing out the first few terms of the series gives $1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots$. $\frac{1}{42} < 0.1$ so we just need the first three terms. This gives a sum of $\frac{23}{30}$.

Problem 5 Find the Taylor series of the function $f(x) = \sin x$ around $x = -\frac{\pi}{2}$.

Solution:

n	$f^{(n)}(x)$	$f^{(n)}(-\frac{\pi}{2})$	c_n
0	$\sin x$	-1	-1
1	$\cos x$	0	0
2	$-\sin x$	1	$\frac{1}{2!}$
3	$-\cos x$	0	0
4	$\sin x$	-1	$-\frac{1}{4!}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

So we have $\sin x = -1 + \frac{1}{2!}(x+\frac{\pi}{2})^2 - \frac{1}{4!}(x+\frac{\pi}{2})^4 + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x+\frac{\pi}{2})^{2n}}{(2n)!}$.