Solutions to the Algebra problems on the Comprehensive Examination of February 1, 2008

1. Let G be a group, let $H \subseteq G$ be a subgroup, and define the **normalizer** of H to be

$$N(H) = \{x \in G : x^{-1}Hx = H\}.$$

(a) Prove that $N(H)$ is a subgroup of G.

Solution: ($N(H)$ nonempty) Trivially $e^{-1}He = H$, so $e \in N(H)$.✓
($N(H)$ closed under group operation) Given $x, y \in N(H)$, using associativity and the property of elements of $N(H)$:

$$(xy)^{-1}H(xy) = y^{-1}x^{-1}Hxy = y^{-1}Hy = H$$

so $xy \in N(H)$.✓
($N(H)$ closed under $^{-1}$) Given $a \in N(H)$, $a^{-1}Ha = H$, so $H = a(a^{-1}Ha)a^{-1} = aha^{-1} = (a^{-1})^{-1}Ha^{-1}$, so $a^{-1} \in N(H)$.✓
Thus $N(H)$ is a subgroup of G as desired. QED

(b) Prove that H is a subgroup of $N(H)$.

Solution: It is given that H is a subgroup of G, so we already know that H forms a group under the group operation. Thus, it is sufficient to show that $H \subseteq N(H)$. Given $a \in H$ (note that since H is closed under inverses, $a^{-1} \in H$), we show that $a^{-1}Ha = H$.
$\subseteq$: Given $a^{-1}ha \in a^{-1}Ha$, since H is closed under group operation, $a^{-1}ha \in H$.✓
$\supseteq$: Given $h \in H$, since H is closed under group operation, $aha^{-1} \in H$ so $h = a^{-1}(aha^{-1})a \in a^{-1}Ha$.✓
Thus $a \in H \Rightarrow a \in N(H)$ so $H \subseteq N(H)$, so H is a subgroup of $N(H)$ as desired. QED

(c) Prove that H is a **normal** subgroup of $N(H)$.

Solution: Given $x \in N(H)$, $h \in H$: by the property of $N(H)$, $x^{-1}Hx \subseteq H$. Thus since $x^{-1}hx \in x^{-1}Hx$, we have $x^{-1}hx \in H$ as desired. QED

2. Recall that S_n denotes the group of permutations on n symbols.

(a) Find an element of S_{10} of order 21.

Solution: The order of a permutation σ is the lcm of the orders of each individual disjoint cycle comprising σ (and the order of an n-cycle is n). Since $21 = \text{lcm}(3, 7)$, the element $\sigma = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9\ 10)$ has order 21.

(b) Prove that no element of S_{10} has order 11.

Solution: Suppose there exists a permutation σ of order 11. Then σ must have a disjoint cycle structure such that the lcm of the cycle lengths is 11. If $X_\sigma = \{x_1, \ldots, x_n\}$ is the set of cycle lengths of σ, this means that

$$11 = \text{lcm}(x_1, \ldots, x_n) = \frac{x_1 \times \cdots \times x_n}{\gcd(x_1, \ldots, x_n)}$$

But since 11 is prime, this means that $11 \big| x_i$ for some i; for that i, therefore, $x_i \geq 11$. Thus σ has a cycle that is (at least) an 11-cycle. But that's impossible, since σ is a permutation on only 10 symbols! $\Rightarrow\Leftarrow$ Thus no element of S_{10} has order 11. QED

3. Let R be a ring.

   (a) Define what it means for a subset $I \subseteq R$ to be an **ideal** of R.

   Solution: $I \subseteq R$ is an ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and $\forall x \in I,\ r \in R$, $xr \in I$ and $rx \in I$.

   (b) Let S be another ring, and let $\phi : R \to S$ be a ring homomorphism. Let $I \subseteq R$ be an ideal of R, and set

   $$J = \{x \in I : \phi(x) = 0_S\},$$

   where 0_S denotes the zero element of S. Prove that J is an ideal of R.

   Solution: First, we show that $(J, +)$ is a subgroup of $(R, +)$:
   (J nonempty) Pick some $x \in I$ (such an x must exists because I ideal $\Rightarrow J \neq \emptyset$). Then since I is an ideal, $0_R = 0_R x \in I$. Since ϕ is a ring homomorphism, $\phi(0_R) = 0_S$ so $0_R \in J$.✓
   (J closed under $+$) Given $x, y \in J \subseteq I$, $x + y \in I$ because I is closed under $+$. Also, since ϕ is a homomorphism, $\phi(x + y) = \phi(x) + \phi(y) = 0_S + 0_S = 0_S$. Thus, $x + y \in J$.✓
   (J closed under negatives) Given $x \in J \subseteq I$, $-x \in I$ because I is closed under negatives. Also, since ϕ is a homomorphism, $\phi(-x) = -\phi(x) = -0_S = 0_S$. Thus, $-x \in J$.✓
   Thus $(J, +)$ is a subgroup of $(R, +)$ ✓
   Now given $r \in R$, $x \in J \subseteq I$: $rx, xr \in I$ since I is an ideal. Now since ϕ is a homomorphism, $\phi(rx) = \phi(r)\phi(x) = \phi(r)0_S = 0_S$ and $\phi(xr) = \phi(x)\phi(r) = 0_S\phi(r) = 0_S$, so $xr, rx \in J$.✓
   Thus, J is an ideal of R. QED

4. Let $R = \mathbb{R}[x]$ be the ring of polynomials with coefficients in the field $\mathbb{R}$ of real numbers. Let $I \subseteq R$ be the subset

   $$I = \{f \in \mathbb{R}[x] : f(1) = f(2) = 0\}.$$

   (a) Prove that I is an ideal of R.

   Solution: First, we show that $(I, +)$ is a subgroup of $(R, +)$:
   (I nonempty:) Let $0(x)$ be the zero polynomial (all coefficients are zero). Then $0(1) = 0(2) = 0$ so $0(x) \in I$.✓
   (I closed under $+$) Given $f, g \in I$, $(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$ and $(f + g)(2) = f(2) + g(2) = 0 + 0 = 0$, so $(f + g) \in I$.✓
   (I closed under negatives) Given $f \in I$, $(-f)(1) = -f(1) = -0 = 0$ and

$(-f)(2) = -f(2) = -0 = 0$ so $-f \in I$.✓
Thus $(I, +)$ is a subgroup of $(R, +)$ ✓
Now given $r \in R$, $f \in I$: $(rf)(1) = r(1)f(1) = r(1)(0) = 0$ and $(rf(2) = r(2)f(2) = r(2)(0) = 0$. Similarly, $(fr)(1) = 0$ and $(fr)(2) = 0$. Thus, $rf, fr \in I$.✓
Thus, I is an ideal of R. QED

(b) Prove that R/I has zero-divisors. That is, show that there are two nonzero elements of R/I whose product is zero in R/I.

Solution: Let $f(X) = X - 1$ and $g(X) = X - 2$. Then $f(2) = 1 \neq 0$ and $g(1) = -1 \neq 0$, so $f, g \notin I$. Thus, $I + f$ and $I + g$ are nonzero. However, $(I + f)(I + g) = I + h$, where $h(X) = (X - 1)(X - 2)$. Since $h(1) = h(2) = 0$, $h \in I$ which means that $I + h = I + 0$ (0 here is the zero polynomial $0(X)$). Thus, f and g are zero-divisors as desired. QED