Math 13 Fall 2008: Exam 2

Name:

Instructions: There are 5 questions on this exam of which you must do 4. Each problem is scored out of 8 points for a total of 32 points. You may not use any outside materials (eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Circle below the 4 problems you wish to be graded. Otherwise, I will grade the first 4 completed problems

1 2 3 4 5
Problem 1. Determine the following limits or show that they do not exist.

(a) \[ \lim_{(x,y) \to (0,0)} \frac{x^4 y^4}{(x^2 + y^4)^3} \]

(b) \[ \lim_{(x,y) \to (0,0)} \frac{2x^4 y}{x^4 + y^4} \]

Proof.

(a) Considering the path \( y = 0 \) we have

\[ \lim_{(x,y) \to (0,0)} \frac{x^4 y^4}{(x^2 + y^4)^3} = \lim_{(x,0) \to (0,0)} \frac{0}{x^2} = 0. \]

Considering the path \( x = y^2 \)

\[ \lim_{(y^2,y) \to (0,0)} \frac{y^{12}}{(2y^4)^3} = \lim_{y \to 0} \frac{y^{12}}{8y^{12}} = \frac{1}{8} \]

Since the limit along two different paths to \((0,0)\) do not agree, then the limit does not exist.

(b) We will show that this limit is 0.

Method 1: using the definition of limits:

Using the \( \epsilon - \delta \) definition of limits we choose \( \epsilon > 0 \) and need to find a \( \delta > 0 \) such that if \( \sqrt{x^2 + y^2} < \delta \) then \( \left| \frac{2x^4 y}{x^4 + y^4} \right| < \epsilon \). So we consider

\[ \left| \frac{2x^4 y}{x^4 + y^4} \right| < \epsilon. \]

Then since \( x^4 \) and \( y^4 \) are non-negative we have

\[ \frac{x^4}{x^4 + y^4} \leq 1 \]

and so

\[ \left| \frac{2x^4 y}{x^4 + y^4} \right| \leq 2|y| = 2\sqrt{y^2} \leq 2\sqrt{x^2 + y^2} \]

So we may choose \( \delta = \frac{\epsilon}{2} \).

Alternatively, we could use the Squeeze Theorem to have

\[ -2|y| \leq \frac{2x^4 y}{x^4 + y^4} \leq 2|y|. \]

Since \(-2|y|\) converges to 0 and \(2|y|\) converges to 0 as \((x, y) \to (0,0)\) we have that \( \frac{2x^4 y}{x^4 + y^4} \) must also converge to 0.
Method 2: using the Squeeze Theorem:

We have

\[ \left| \frac{2x^4y}{x^4 + y^4} \right| \leq |2y| \]

so we have

\[-2|y| \leq \frac{2x^4y}{x^4 + y^4} \leq 2|y|.

We have

\[ \lim_{(x,y) \to (0,0)} -2|y| = \lim_{y \to 0} -2|y| = 0 \]
\[ \lim_{(x,y) \to (0,0)} 2|y| = \lim_{y \to 0} 2|y| = 0 \]

so by the squeeze theorem we have

\[ \lim_{(x,y) \to (0,0)} \frac{2x^4y}{x^4 + y^4} = 0. \]
Problem 2. Let \( w = f(x, y, z) = x^3 + y^3 - z \).

(a) Find the rate of change of \( f \) at the point \((1, 1, 2)\) along the line \( \frac{x - 1}{3} = \frac{y - 1}{2} = \frac{z - 2}{2} \) in the direction of decreasing \( x \).

(b) Find \( \frac{\partial w}{\partial t} \) for

\[
\begin{align*}
x(s, t) &= \sin(st) \\
y(s, t) &= \ln(st) \\
z(s, t) &= se^t.
\end{align*}
\]

**Proof.**

(a) We have that the line is in the direction \( \langle 3, 2, -2 \rangle \) and so we have \( u = \frac{1}{\sqrt{17}} \langle -3, -2, 2 \rangle \). We also have \( \nabla f = \langle 3x^2, 3y^2, -1 \rangle \) and so we have

\[
D_u f(1, 1, 2) = \langle 3, 3, -1 \rangle \cdot \frac{1}{\sqrt{17}} \langle -3, -2, 2 \rangle = \frac{-9 - 6 - 2}{\sqrt{17}} = -\sqrt{17}.
\]

(b) We have

\[
\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} = (3x^2)(s \cos(st)) + 3y^2 \frac{st}{st} - se^t = 3s \sin^2(st) \cos(st) + \frac{3(\ln(st))^2}{t} - se^t.
\]

\( \square \)
Problem 3. Classify the critical points of \( f(x, y) = x^4 + y^3 - 32x - 27y - 1 \).

Proof. We have

\[
\begin{align*}
    f_x &= 4x^3 - 32 = 0 \\
    f_y &= 3y^2 - 27 = 0
\end{align*}
\]

so we have

\[
\begin{align*}
    x^3 &= 8 \\
    y^2 &= 9
\end{align*}
\]

Which gives the points \((2, 3), (2, -3)\).

Taking the second derivatives we have

\[
\begin{align*}
    f_{xx} &= 12x^2 \\
    f_{yy} &= 6y \\
    f_{xy} &= 0.
\end{align*}
\]

So we have

\[
\begin{align*}
    D(2, 3) &= 24 \cdot 18 > 0 \\
    D(2, -3) &= 24 \cdot (-18) < 0
\end{align*}
\]

Since \( D > 0 \) and \( f_{xx} > 0 \) we have \((2, 3)\) is a relative min.

Since \( D < 0 \) we have \((2, -3)\) is a saddle point.

\(\square\)
**Problem 4.** Find the maximum and minimum for \(3x^2 + 4xy\) for points on the disk \(x^2 + y^2 \leq 14\).

**Proof.** First we find the critical points. So we have

\[
6x + 4y = 0 \quad \text{and} \quad 4x = 0
\]

so we have \(x = 0\) and hence \(y = 0\) which is the critical point \((0, 0)\). This point is within the disk.

Using Lagrange Multipliers we have

\[
\begin{align*}
6x + 4y &= \lambda 2x \\
4x &= \lambda 2y \\
x^2 + y^2 &= 14
\end{align*}
\]

**Method 1: solving first for \(\lambda\):**

If \(y = 0\) we have \(x = \pm\sqrt{14}\) by the constraint equation, but \(x = 0\) by the second equation, so \(y = 0\) is not possible. So we have, \(\lambda = \frac{2x}{y}\) and

\[
y(6x + 4y) = 2x(2x) = 4x^2 \\
x^2 + y^2 = 14.
\]

Which is

\[
6xy + 4y^2 = 4x^2 \\
x^2 + y^2 = 14.
\]

and the first equation factors as \(2(y + 2x)(2y - x) = 0\), in other words, \(x = 2y\) or \(y = -2x\).

If \(x = 2y\) we have

\[
5y^2 = 14
\]

and hence \(y = \pm\sqrt{\frac{14}{5}}\), which gives the two points \((2\sqrt{14/5}, \sqrt{14/5}), (-2\sqrt{14/5}, -\sqrt{14/5})\).

If \(y = -2x\) then we have

\[
5x^2 = 14
\]

and hence \(x = \pm\sqrt{\frac{14}{5}}\), which gives the two points \((-\sqrt{14/5}, 2\sqrt{14/5}), (\sqrt{14/5}, -2\sqrt{14/5})\).

**Method 2: solving first for \(x\):**

We have \(x = \lambda y/2\) and so

\[
\begin{align*}
3\lambda y + 4y &= \lambda^2 y \\
\lambda^2 y^2/4 + y^2 &= 14
\end{align*}
\]

We can factor the first equation

\[
y(\lambda - 4)(\lambda + 1) = 0
\]

This gives the three solutions \(y = 0, \lambda = 4, \lambda = -1\). If \(y = 0\), then the second equation cannot be satisfied, so that does not yeild any solution. If \(\lambda = 4\) we get

\[
5y^2 = 14
\]
and so \( y = \pm \sqrt{14/5} \), which gives the two points \((2\sqrt{14/5}, \sqrt{14/15}), (-2\sqrt{14/5}, -\sqrt{14/5})\). If \( \lambda = -1 \) then we have

\[
5y^2/4 = 14
\]
and hence \( y = 2\sqrt{14/5} \), which gives the two points \((-\sqrt{14/5}, 2\sqrt{14/5}), (\sqrt{14/5}, -2\sqrt{14/5})\).

So we have

\[
\begin{align*}
f(0,0) &= 0 \\
f\left(\sqrt{\frac{14}{5}}, -2\sqrt{\frac{14}{5}}\right) &= -14 \\
f\left(-\sqrt{\frac{14}{5}}, 2\sqrt{\frac{14}{5}}\right) &= -14 \\
f\left(2\sqrt{\frac{14}{5}}, \sqrt{\frac{14}{5}}\right) &= 56 \\
f\left(-2\sqrt{\frac{14}{5}}, -\sqrt{\frac{14}{5}}\right) &= 56
\end{align*}
\]

So we have

Min at \( \left(\sqrt{\frac{14}{5}}, -2\sqrt{\frac{14}{5}}\right) \) and \( -\left(\sqrt{\frac{14}{5}}, 2\sqrt{14/5}\right) \) is \(-14\)

Max at \( \left(2\sqrt{\frac{14}{5}}, \sqrt{\frac{14}{5}}\right) \) and \( -\left(2\sqrt{14/5}, -\sqrt{14/5}\right) \) is \(56\)
Problem 5. Let $a < 0$. Consider the surface $xyz = a$.

(a) Find the tangent plane to the surface at $(x_0, y_0, z_0)$.

(b) The tangent plane forms a tetrahedron (pyramid with a triangular base) in the first octant. Show that the volume of this tetrahedron is $\frac{9}{2}a$. (Hint: the volume of a tetrahedron is given by $\frac{1}{3}$ (area of the base triangle) (height).)

Proof.

(a) We have $\nabla f = (yx, xz, xy)$ and so $\nabla f(x_0, y_0, z_0) = (y_0x_0, x_0z_0, x_0y_0)$ and the tangent plane is given by

$y_0z_0(x - x_0) + x_0z_0(y - y_0) + x_0y_0(z - z_0) = 0.$

which is

$y_0z_0x + x_0z_0y + x_0y_0z = 3a$.

(b) We know that $(0, 0, 0)$ is a vertex of the tetrahedron. To find the other three, we set two of $x, y, z$ equal to zero and solve for the third. So for the point on the $x$-axis we get

$y_0z_0x = 3a = 3x_0y_0z_0$

which is $x = 3x_0$. Similarly for the $y$-axis and $z$-axis. So the points are the coordinates axes are given by $(0, 0, 0), (0, 0, 3z_0), (0, 3y_0, 0), (3x_0, 0, 0)$. So we have the volume as

$\frac{1}{3} \left( \frac{1}{2}(3x_0)(3y_0) \right) (3z_0) = \frac{27x_0y_0z_0}{6} = \frac{9}{2}a.$

\qed