Math 13 Fall 2008: Exam 2

Name:

Instructions: There are 5 questions on this exam of which you must do 4. Each problem is scored out of 8 points for a total of 32 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Circle below the 4 problems you wish to be graded. Otherwise, I will grade the first 4 completed problems

1 2 3 4 5

Problem 1. Determine the following limits or show that they do not exist.

(a)
$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{2x^4y}{x^4+y^4}$$

Proof.

(a) Considering the path $y = 0$ we have

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0.$$

Considering the path $x = y^2$

$$\lim_{(y^2,y)\to(0,0)} \frac{y^{12}}{(2y^4)^3} = \lim_{y\to 0} \frac{y^{12}}{8y^{12}} = \frac{1}{8}$$

Since the limit along two different paths to $(0, 0)$ do not agree, then the limit does not exist.

(b) We will show that this limit is 0.

Method 1: using the definition of limits:

Using the $\epsilon-\delta$ definition of limits we choose $\epsilon > 0$ and need to find a $\delta > 0$ such that if $\sqrt{x^2 + y^2} < \delta$ then $\left|\frac{2x^4y}{x^4+y^4}\right| < \epsilon$. So we consider

$$\left|\frac{2x^4y}{x^4+y^4}\right| < \epsilon.$$

Then since x^4 and y^4 are non-negative we have

$$\frac{x^4}{x^4+y^4} \leq 1$$

and so

$$\left|\frac{2x^4y}{x^4+y^4}\right| \leq 2\left|y\right| = 2\sqrt{y^2} \leq 2\sqrt{x^2+y^2}$$

So we may choose $\delta = \frac{\epsilon}{2}$.

Alternatively, we could use the Squeeze Theorem to have

$$-2\left|y\right| \leq \frac{2x^4y}{x^4+y^4} \leq 2\left|y\right|.$$

Since $-2|y|$ converges to 0 and $2|y|$ converges to 0 as $(x, y) \to (0, 0)$ we have that $\frac{2x^4y}{x^4+y^4}$ must also converge to 0.

Method 2: using the Squeeze Theorem:

We have

$$\left|\frac{2x^4y}{x^4+y^4}\right| \le |2y|$$

so we have

$$-2\left|y\right| \le \frac{2x^4y}{x^4+y^4} \le 2\left|y\right|.$$

We have

$$\lim_{(x,y)\to(0,0)} -2\left|y\right| = \lim_{y\to 0} -2\left|y\right| = 0$$
$$\lim_{(x,y)\to(0,0)} 2\left|y\right| = \lim_{y\to 0} 2\left|y\right| = 0$$

so by the squeeze theorem we have

$$\lim_{(x,y)\to(0,0)} \frac{2x^4y}{x^4+y^4} = 0.$$

□

Problem 2. Let $w = f(x, y, z) = x^3 + y^3 - z$.

(a) Find the rate of change of f at the point $(1, 1, 2)$ along the line $\frac{x-1}{3} = \frac{y-1}{2} = \frac{2-z}{2}$ in the direction of decreasing x.

(b) Find $\frac{\partial w}{\partial t}$ for

$$\begin{aligned} x(s,t) &= \sin(st) \\ y(s,t) &= \ln(st) \\ z(s,t) &= se^t. \end{aligned}$$

Proof.

(a) We have that the line is in the direction $\langle 3, 2, -2\rangle$ and so we have $u = \frac{1}{\sqrt{17}}\langle -3, -2, 2\rangle$. We also have $\nabla f = \left\langle 3x^2, 3y^2, -1\right\rangle$ and so we have

$$D_u f(1,1,2) = \langle 3, 3, -1\rangle \cdot \frac{1}{\sqrt{17}}\langle -3, -2, 2\rangle = \frac{-9-6-2}{\sqrt{17}} = -\sqrt{17}.$$

(b) We have

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \\ &= (3x^2)(s\cos(st)) + 3y^2\frac{s}{st} - se^t \\ &= 3s\sin^2(st)\cos(st) + \frac{3(\ln(st))^2}{t} - se^t. \end{aligned}$$

□

Problem 3. Classify the critical points of $f(x, y) = x^4 + y^3 - 32x - 27y - 1$.

Proof. We have

$$f_x = 4x^3 - 32 = 0$$
$$f_y = 3y^2 - 27 = 0$$

so we have

$$x^3 = 8$$
$$y^2 = 9$$

Which gives the points $(2, 3), (2, -3)$.

Taking the second derivatives we have

$$f_{xx} = 12x^2$$
$$f_{yy} = 6y$$
$$f_{xy} = 0.$$

So we have

$$D(2, 3) = 24 \cdot 18 > 0$$
$$D(2, -3) = 24 \cdot (-18) < 0$$

Since $D > 0$ and $f_{xx} > 0$ we have $(2, 3)$ is a relative min.

Since $D < 0$ we have $(2, -3)$ is a saddle point. □

Problem 4. Find the maximum and minimum for $3x^2 + 4xy$ for points on the disk $x^2 + y^2 \leq 14$.

Proof. First we find the critical points. So we have

$$6x + 4y = 0 \quad \text{and} \quad 4x = 0$$

so we have $x = 0$ and hence $y = 0$ which is the critical point $(0, 0)$. This point is within the disk.

Using Lagrange Multipliers we have

$$\begin{aligned} 6x + 4y &= \lambda 2x \\ 4x &= \lambda 2y \\ x^2 + y^2 &= 14 \end{aligned}$$

Method 1: solving first for λ:

If $y = 0$ we have $x = \pm\sqrt{14}$ by the constraint equation, but $x = 0$ by the second equation, so $y = 0$ is not possible. So we have, $\lambda = \frac{2x}{y}$ and

$$\begin{aligned} y(6x + 4y) &= 2x(2x) \\ x^2 + y^2 &= 14. \end{aligned}$$

Which is

$$\begin{aligned} 6xy + 4y^2 &= 4x^2 \\ x^2 + y^2 &= 14. \end{aligned}$$

and the first equation factors as $2(y + 2x)(2y - x) = 0$, in other words, $x = 2y$ or $y = -2x$.

If $x = 2y$ we have

$$5y^2 = 14$$

and hence $y = \pm\sqrt{\frac{14}{5}}$, which gives the two points $(2\sqrt{14/5}, \sqrt{14/15}), (-2\sqrt{14/5}, -\sqrt{14/5})$.

If $y = -2x$ then we have

$$5x^2 = 14$$

and hence $x = \pm\sqrt{\frac{14}{5}}$, which gives the two points $(-\sqrt{14/5}, 2\sqrt{14/5}), (\sqrt{14/5}, -2\sqrt{14/5})$.

Method 2: solving first for x:

We have $x = \lambda y/2$ and so

$$\begin{aligned} 3\lambda y + 4y &= \lambda^2 y \\ \lambda^2 y^2/4 + y^2 &= 14. \end{aligned}$$

We can factor the first equation

$$y(\lambda - 4)(\lambda + 1) = 0$$

This gives the three solutions $y = 0, \lambda = 4, \lambda = -1$. If $y = 0$, then the second equation cannot be satisfied, so that does not yeidl any solutiosn. If $\lambda = 4$ we get

$$5y^2 = 14$$

and so $y = \pm\sqrt{14/5}$, which gives the two points $(2\sqrt{14/5}, \sqrt{14/15}), (-2\sqrt{14/5}, -\sqrt{14/5})$. If $\lambda = -1$ then we have

$$5y^2/4 = 14$$

and hence $y = 2\sqrt{14/5}$, which gives the two points $(-\sqrt{14/5}, 2\sqrt{14/5}), (\sqrt{14/5}, -2\sqrt{14/5})$.

So we have

$$\begin{aligned} f(0,0) &= 0 \\ f\left(\sqrt{\frac{14}{5}}, -2\sqrt{\frac{14}{5}}\right) &= -14 \\ f\left(-\sqrt{\frac{14}{5}}, 2\sqrt{\frac{14}{5}}\right) &= -14 \\ f\left(2\sqrt{\frac{14}{5}}, \sqrt{\frac{14}{5}}\right) &= 56 \\ f\left(-2\sqrt{\frac{14}{5}}, -\sqrt{\frac{14}{5}}\right) &= 56 \end{aligned}$$

So we have

$$\begin{aligned} \text{Min at } \left(\sqrt{\frac{14}{5}}, -2\sqrt{\frac{14}{5}}\right) \text{ and } \left(-\sqrt{\frac{14}{5}}, 2\sqrt{\frac{14}{5}}\right) \text{ is } -14 \\ \text{Max at } \left(2\sqrt{\frac{14}{5}}, \sqrt{\frac{14}{5}}\right) \text{ and } \left(-2\sqrt{\frac{14}{5}}, -\sqrt{\frac{14}{5}}\right) \text{ is } 56 \end{aligned}$$

□

Problem 5. Let $a < 0$. Consider the surface $xyz = a$.

(a) Find the tangent plane to the surface at (x_0, y_0, z_0).

(b) The tangent plane forms a tetrahedron (pyramid with a triangular base) in the first octant. Show that the volume of this tetrahedron is $\frac{9}{2}a$. (Hint: the volume of a tetrahedron is given by $\frac{1}{3}$(area of the base triangle)(height).)

Proof.

(a) We have $\nabla f = \langle yx, xz, xy \rangle$ and so $\nabla f(x_0, y_o, z_0) = \langle y_0 x_0, x_0 z_0, x_0 y_0 \rangle$ and the tangent plane is given by

$$y_0 z_0 (x - x_0) + x_0 z_0 (y - y_0) + x_0 y_0 (z - z_0) = 0.$$

which is

$$y_0 z_0 x + x_0 z_0 y + x_0 y_0 z = 3a$$

(b) We know that $(0, 0, 0)$ is a vertex of the tetrahedron. To find the other three, we set two fo x, y, z equal to zero and solve for the third. So for the point on the x-axis we get

$$y_0 z_0 x = 3a = 3x_0 y_0 z_0$$

which is $x = 3x_0$. Similarly for the y-axis and z-axis. So the points are the coordinates axes are given by $(0, 0, 0), (0, 0, 3z_0), (0, 3y_0, 0), (3x_0, 0, 0)$. So we have the volume as

$$\frac{1}{3}\left(\frac{1}{2}(3x_0)(3y_0)\right)(3z_0) = \frac{27x_0 y_0 z_0}{6} = \frac{9}{2}a.$$

□