Math 211, Multivariable Calculus, Fall 2011
Midterm II Solutions

1. *A stunt performer drives a car (horizontally) at* 40 *meters per second off the edge of a cliff that is* 80 *meters high.*

   (a) *What is the speed of the car when it lands?*

   (b) *At what angle does the car hit the ground?*

   (c) *How far from the base of the cliff does the car land?*

   (Assume that the acceleration due to gravity is 10 *meters per second squared and that no other forces act on the car. Also assume that the face of the cliff is vertical and the ground beyond the cliff is horizontal.)*

   (a) The car takes off horizontally so its initial velocity is $\langle 40, 0\rangle$. Choose the coordinates so the initial position is $\langle 0, 80\rangle$. Then the position at time t is given by

   $$\mathbf{r}(t) = \langle 0, 80\rangle + t\,\langle 40, 0\rangle + \left\langle 0, -5t^2\right\rangle = \left\langle 40t, 80 - 5t^2\right\rangle.$$

   We first find the time t_l when the car lands. This satisfies

   $$80 - 5t_l^2 = 0$$

   so

   $$\begin{aligned} 5t_l^2 &= 80 \\ t_l^2 &= 16 \\ t_l &= 4 \end{aligned}$$

   The velocity at time t is given by

   $$\mathbf{r}'(t) = \langle 40, -10t\rangle$$

   so the velocity at landing is

   $$\mathbf{r}'(4) = \langle 40, -40\rangle.$$

   The speed at landing is therefore $\sqrt{40^2 + (-40)^2} = 40\sqrt{2}$.

   (b) Since the velocity at landing is $\langle 40, -40\rangle$, the car is landing at an angle of 45 degrees.

   (c) The position at landing is

   $$\mathbf{r}(4) = \langle 160, 0\rangle$$

   so the car lands at a distance of 160 meters from the cliff.

2. *Find the linear approximation to the function*

   $$f(x, y) = x^2 \ln(y^2)$$

   at the point $(5, 1)$. *Use your answer to estimate the value of* $f(5.01, 0.998)$

The linear approximation is given by

$$l(x,y) = f(5,1) + f_x(5,1)(x-5) + f_y(5,1)(y-1).$$

We have

$$f_x = 2x\ln(y^2), f_y = \frac{2x^2y}{y^2} = \frac{2x^2}{y}$$

and so

$$f(5,1) = 0, \quad f_x(5,1) = 0, \quad f_y(5,1) = 50.$$

The linear approximation is therefore

$$l(x,y) = 0 + 0(y-5) + 50(y-1) = 50(y-1).$$

We therefore have

$$f(5.01, 0.998) \approx 50(0.998 - 1) = 50(-0.002) = -0.1.$$

3. *Let $f(x,y) = y\cos(2x) - \sin(2x)$.*

   (a) *Find the directional derivative of f at $(0,0)$ in the direction $\langle 1,-1\rangle$.*

   (b) *What is the value of the largest directional derivative of f at $(0,0)$?*

   (c) *Give a vector* **u** *that is tangent to the level curve of f that passes through the point $(0,0)$. (No explanation necessary.)*

   (a) We have

   $$\nabla f = \langle -2y\sin(2x) - 2\cos(2x), \cos(2x)\rangle$$

   so

   $$\nabla f(0,0) = \langle -2, 1\rangle .$$

   The directional derivative is then given by

   $$\begin{aligned} D_{\langle 1,-1\rangle} f(0,0) &= \frac{\langle -2,1\rangle \cdot \langle 1,-1\rangle}{|\langle 1,-1\rangle|} \\ &= \frac{-1}{\sqrt{2}} \end{aligned}$$

   (b) The value of the largest directional derivative is

   $$|\nabla f(0,0)| = \sqrt{5}.$$

   (c) The gradient vector $\nabla f(0,0)$ is perpendicular to the level curve, so we want a vector that is perpendicular to $\langle -2,1\rangle$ such as $\langle 1,2\rangle$.

4. *Let θ be some fixed angle (not a variable) and suppose that*

$$x(u,v) = u\cos\theta + v\sin\theta, \quad y(u,v) = -u\sin\theta + v\cos\theta.$$

(a) *(4 points) If* $z = f(x, y)$*, find expressions for* $\dfrac{\partial z}{\partial u}$ *and* $\dfrac{\partial z}{\partial v}$ *in terms of* $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$ *and* θ.

(b) *(4 points) Suppose that* $z = x^2 + y^2$*. Use your answer to part (a) to find* $\dfrac{\partial z}{\partial u}$ *and* $\dfrac{\partial z}{\partial v}$*. Give your answers in terms of* u *and* v.

(c) *(2 points) Suppose again that* z *is any function of* x *and* y*. Find an expression for* $\dfrac{\partial^2 z}{\partial u^2}$ *in terms of the second-order partial derivatives of* z *with respect to* x *and* y*, and* θ.

(a) We have

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= (\cos\theta)\frac{\partial z}{\partial x} - (\sin\theta)\frac{\partial z}{\partial y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= (\sin\theta)\frac{\partial z}{\partial x} + (\cos\theta)\frac{\partial z}{\partial y}\end{aligned}$$

(b) We have

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y$$

so

$$\begin{aligned}\frac{\partial z}{\partial u} &= 2x\cos\theta - 2y\sin\theta \\ &= 2(u\cos\theta + v\sin\theta)\cos\theta - 2(-u\sin\theta + v\cos\theta)\sin\theta \\ &= 2u(\cos^2\theta + \sin^2\theta) + 2v(\sin\theta\cos\theta - \sin\theta\cos\theta) \\ &= 2u\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial z}{\partial v} &= 2x\sin\theta + 2y\cos\theta \\ &= 2(u\cos\theta + v\sin\theta)\sin\theta + 2(-u\sin\theta + v\cos\theta)\cos\theta \\ &= 2u(\cos\theta\sin\theta + \sin\theta\cos\theta) + 2v(\sin^2\theta + \cos^2\theta) \\ &= 2v\end{aligned}$$

(c) We want to find

$$\begin{aligned}\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) &= \frac{\partial}{\partial u}\left(\cos\theta\frac{\partial z}{\partial x} - \sin\theta\frac{\partial z}{\partial y}\right) \\ &= \cos\theta\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) - \sin\theta\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)\end{aligned}$$

We use the chain rule to find these derivatives:

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) = \cos\theta\frac{\partial^2 z}{\partial x^2} - \sin\theta\frac{\partial^2 z}{\partial y \partial x}$$

and

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right) = \cos\theta\frac{\partial^2 z}{\partial x \partial y} - \sin\theta\frac{\partial^2 z}{\partial y^2}$$

Therefore

$$\frac{\partial^2 z}{\partial u^2} = \cos^2\theta\frac{\partial^2 z}{\partial x^2} - \cos\theta\sin\theta\frac{\partial^2 z}{\partial y \partial x} - \cos\theta\sin\theta\frac{\partial^2 z}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 z}{\partial y^2}.$$

5. *In each of the following cases, decide if $f(x, y)$ is differentiable at $(0, 0)$. Explain your answers.*

   (a) *(3 points)* $f(x, y) = xe^{xy}$

   (b) *(3 points)* $f(x, y) = |x + y|$

   (c) *(4 points)* $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

   (a) The partial derivatives are

   $$f_x = e^{xy} + xye^{xy}, \quad f_y = x^2e^{xy}.$$

   These are continuous everywhere so f is differentiable everywhere.

   (b) The partial derivative $f_x(0, 0)$ would be given by

   $$\lim_{h\to 0}\frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0}\frac{|h|}{h}.$$

   This limit does not exist because the limit from below is given by

   $$\lim_{h\to 0^-}\frac{-h}{h} = -1$$

   and the limit from above is given by

   $$\lim_{h\to 0^+}\frac{h}{h} = 1.$$

   Therefore f is not differentiable at $(0, 0)$.

   (c) The partial derivatives of f at $(0, 0)$ exist and are both equal to 0. The linear approximation would therefore be

   $$l(x, y) = f(0, 0) + 0x + 0y = 0.$$

   To see if f is differentiable, we have to check if

   $$\lim_{(x,y)\to(0,0)}\frac{f(x, y) - l(x, y)}{\sqrt{x^2 + y^2}}$$

is equal to 0 or not. This is the limit of

$$\frac{xy}{x^2 + y^2}$$

as $(x, y) \to (0, 0)$. Approaching along $y = x$, this is equal to $\frac{1}{2}$, and along $y = -x$ it is equal to $-\frac{1}{2}$. Therefore the limit does not exist and so f is not differentiable at $(0, 0)$.