Math 111, Introduction to the Calculus, Fall 2011
Midterm III Practice Exam 1 Solutions

1. *Find the critical points of the function*

$$f(x) = x^2(x+1)^2$$

For each critical point, decide if it is a local maximum or local minimum, or neither.

The critical points are where $f'(x) = 0$. We have

$$f'(x) = 2x(x+1)^2 + 2(x+1)x^2$$

which can be factored as

$$f'(x) = 2x(x+1)((x+1)+x) = 2x(x+1)(2x+1).$$

So the critical points satisfy

$$2x(x+1)(2x+1) = 0$$

which means that there are three critical points:

$$x = 0, \quad x = -1, \quad x = -1/2.$$

To decide if these are local maxima or local minima, we could try the Second Derivative Test. We have

$$f''(x) = 2(x+1)^2 + 4x(x+1) + 2x^2 + 4(x+1)x.$$

Then $f''(-1) = 2$ so $x = -1$ is a local minimum, $f''(-1/2) = 2(1/2)^2 + 8(-1/2)(1/2) + 2(-1/2)^2 = -1$ so $x = -1/2$ is a local maximum, and $f''(0) = 2$ so $x = 0$ is a local minimum.

2. (a) *Show that the function*

$$f(x) = x^3 + 6x$$

is increasing on all of $\mathbb{R}$.

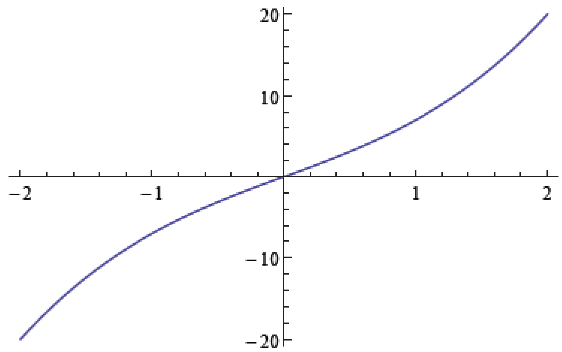
(b) *Find the intervals on which $f(x)$ is concave up or concave down, and find the inflection points.*

(c) *Sketch a graph of $f(x)$ based on the information from parts (a) and (b).*

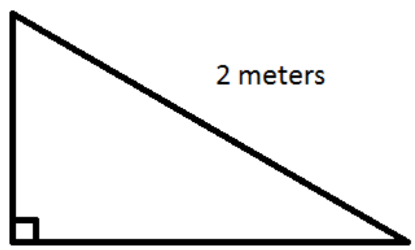
(a) We have $f'(x) = 3x^2 + 6$ which is positive for any x so $f(x)$ is increasing on all of $\mathbb{R}$.

(b) We have $f''(x) = 6x$. Therefore $f''(x)$ is negative for $x < 0$ (so f is concave down for $x < 0$) and $f''(x)$ is positive for $x > 0$ (so f is concave up for $x > 0$). At $x = 0$, $f''(x) = 0$ and f changes from concave down to concave up, so $x = 0$ is an inflection point.

(c) The graph looks as follows:



3. A right-angled triangle has hypoteneuse 2 meters, as in the diagram below:



What is the largest possible area of the triangle?

Let x and y be the other sides of the triangle. Then we want to maximize the function

$$A = \frac{1}{2}xy.$$

The relationship between x and y is that

$$x^2 + y^2 = 2^2 = 4$$

so

$$y = \sqrt{4 - x^2}.$$

Therefore

$$A(x) = \frac{1}{2}x\sqrt{4 - x^2}.$$

We know that $0 \le x \le 2$ so we want to find the absolute maximum of $A(x)$ on this interval.

To do this, we find

$$A'(x) = \frac{1}{2}\sqrt{4 - x^2} + \frac{1}{2}x(-2x)\frac{1}{2\sqrt{4 - x^2}}$$

which simplifies to

$$\frac{(4-x^2)-x^2}{2\sqrt{4-x^2}} = \frac{2-x^2}{\sqrt{4-x^2}}.$$

The critical points are therefore where $2-x^2=0$ which is $x=\sqrt{2}$ (since x must be positive).

The other possible points where the maximum could be is at the endpoints of the interval (that is, $x=0$ and $x=2$) and where $A(x)$ is not differentiable. From the formula we see that $A(x)$ is not differentiable when the denominator is zero which occurs when $x=\pm 2$. Since we are already considering $x=2$ and negative values for x are not allowed, there are no new possibilities here.

So the maximum must occur at one of the three points $x=0$, $x=\sqrt{2}$, $x=2$. To decide which, we just have to find the value of A at each of these: $A(0)=0$, $A(\sqrt{2})=1$, $A(2)=0$. The maximum area is therefore when $x=\sqrt{2}$. Answering the question we can say that the largest possible area of the triangle is 1 meter squared.

4. *Calculate each of the following integrals:*

   (a) $\int_0^2 x^2\,dx$;

   (b) $\int_1^2 x^{-2}(\sqrt{x}+1)\,dx$;

   (c) $\int_0^{\frac{\pi}{2}} \sin(2t)\,dt$.

   (a) $[x^3/3]_{x=0}^{x=2} = \frac{8}{3}$

   (b) $x^{-2}(\sqrt{x}+1) = x^{-3/2}+x^{-2}$ so the integral becomes

   $$\left[-2x^{-1/2}-x^{-1}\right]_{x=1}^{x=2} = (-2/\sqrt{2}-1/2)-(-2-1) = -\sqrt{2}+\frac{5}{2}.$$

   (c) We could guess the answer has something to do with $\cos(2t)$. The derivative of $\cos(2t)$ is $-2\sin(2t)$, so the derivative of $-\cos(2t)/2$ is $\sin(2t)$. Therefore the integral is

   $$[-\cos(2t)/2]_{t=0}^{t=\pi/2} = (-\cos(\pi)/2)-(-\cos(0)/2) = 1.$$

5. *A bathtub holds 54 gallons of water. If the tub is filled at a rate of 3t gallons per minute at the time t minutes after starting, how long does it take to fill up the whole tub? Explain your answer fully.*

   The amount of water in the bathtub at time t is given by the integral of the rate, that is

   $$\int_0^t 3x\,dx = \left[3x^2/2\right]_{x=0}^{x=t} = 3t^2/2.$$

We want to know the value of t when this equals 54, that is, we want to solve

$$\frac{3t^2}{2} = 54$$

so

$$t^2 = 36$$

and hence $t = 6$. Therefore, it takes 6 seconds to fill the tub completely.