Math 13 Fall 2008: Exam 1

Name:

Instructions: There are 5 questions on this exam of which you must do 4. Each problem is scored out of 10 points for a total of 40 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Circle below the 4 problems you wish to be graded. Otherwise, I will grade the first 4 completed problems

1 2 3 4 5

Problem 1. Consider motion along the curve

$$\vec{r}(t) = \left\langle \sin^2 t, \sin t \cos t, \cos t \right\rangle, \quad 0 \leq t \leq 2\pi.$$

(a) Find the velocity and acceleration as functions of t.

(b) At $(0, 0, -1)$ find $\vec{T}$ and κ.

Proof.

(a) We have

$$\begin{aligned} \vec{v}(t) &= \left\langle 2 \sin t \cos t, \cos^2 t - \sin^2 t, -\sin t \right\rangle \\ \vec{a}(t) &= \left\langle -2 \sin^2 t + 2 \cos^2 t, -2 \cos t \sin t - 2 \sin t \cos t, -\cos t \right\rangle \end{aligned}$$

(b) At $t = \pi$ we have

$$\begin{aligned} \vec{v}(\pi) &= \langle 0, 1, 0 \rangle \\ \vec{a}(\pi) &= \langle 2, 0, 1 \rangle \end{aligned}$$

so we have

$$\begin{aligned} T &= \frac{\vec{v}}{|\vec{v}|} = \langle 0, 1, 0 \rangle \\ \kappa &= \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{|\langle 1, 0, -2 \rangle|}{1} = \sqrt{5}. \end{aligned}$$

□

Problem 2. Consider the two lines

$$L1 : \frac{x-1}{6} = y - 1 = \frac{z}{2}$$
$$L2 : \langle 5 + 15t, 1 + 2t, -2 + 6t \rangle$$

and the point

$$P = (1, 0, -1).$$

(a) Show that $L1$ and $L2$ are skew.

(b) Find the equation of the plane containing P and $L2$.

Proof.

(a) The directions of the two lines are $\langle 6, 1, 2 \rangle$ and $\langle 15, 2, 6 \rangle$. These are not multiples of one another since $\frac{6}{15} \neq \frac{1}{2} \neq \frac{2}{6}$. So the lines are not parallel. Next we check if they are intersecting. We have to solve

$$\begin{aligned} 1 + 6s &= 5 + 15t \\ 1 + s &= 1 + 2t \\ 2s &= -2 + 6t \end{aligned}$$

From the z coordinates we get $s = -1 + 3t$ and then from the y coordinates we get $t = 1$ and hence $s = 2$. This gives the points $(13, 3, 4)$ and $(20, 3, 4)$ which are not the same points. So the lines also do not intersect. Hence they are skew.

(b) We have one direction in the plane $\langle 15, 2, 6 \rangle$ and to get another we pick any point on the line, say $(5, 1, -2)$ and make the displacement vector with P to get $\langle 4, -1, 1 \rangle$. Taking the cross product we get a normal vector:

$$\vec{n} = \langle 15, 2, 6 \rangle \times \langle 4, 1, -1 \rangle = \langle 8, -39, -7 \rangle .$$

So we have the plane

$$8(x - 1) - 39y - 7(z + 1) = 0 \quad \text{or} \quad 8x - 39y - 7z = 15.$$

□

Problem 3. Find the parametric equations for the line that is tangent to the curve of intersection of the surfaces

$$z = x^2 + y^2, \qquad 4x^2 + 4y^2 + z^2 = 12$$

at the point $(-1, 1, 2)$.

Proof. Substituting for z we get $(x^2 + y^2)^2 + 4(x^2 + y^2) - 12 = 0$ and hence $x^2 + y^2 = \{-4, 2\}$. Since this clearly must be a positive quantity we have $x^2 + y^2 = 2$ and hence $x = \sqrt{2}\cos t$ and $y = \sqrt{2}\sin t$ and $z = 2$. So we get

$$\vec{r}(t) = \left\langle \sqrt{2}\cos t, \sqrt{2}\sin t, 2 \right\rangle$$

Which has tangent direction

$$\vec{r'}(t) = \left\langle -\sqrt{2}\sin t, \sqrt{2}\cos t, 0 \right\rangle.$$

At $t = \frac{3\pi}{4}$ we get $\vec{r'}(t) = \langle -1, -1, 0 \rangle$ and hence the tangent line is given by

$$\langle -1 - t, 1 - t, 2 \rangle.$$

Note that you could also solve for z as $z^2 + 4z = 12$ to get $z = -6, 2$ and again $x^2 + y^2 = -6$ is not possible, so you have $z = 2$. $\square$

Problem 4. Suppose you start at the point $(0,0,3)$ and move 5π units along the curve $\vec{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$ in the positive direction. Where are you now?

Proof. We parameterize with respect to arc length. The point $(0,0,3)$ occurs at $t=0$ so we have

$$\begin{aligned} s(t) &= \int_0^t \sqrt{9\cos^2 t + 16 + 9\sin^2 t}dt \\ &= \int_0^t \sqrt{25}dt = 5t \end{aligned}$$

So we have $t = \frac{s}{5}$. So the parametrization with respect to arc length is

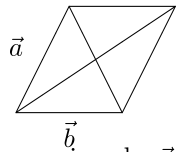
$$\vec{r}(s) = \left\langle 3\sin\frac{s}{5}, \frac{4}{5}s, 3\cos\frac{s}{5}\right\rangle;$$

So we have

$$\vec{r}(5\pi) = \langle 0, 4\pi, -3\rangle.$$

□

Problem 5. Given two vectors $\vec{a}$ and $\vec{b}$ which form a rhombus: a parallelogram whose sides are the same length ($|\vec{a}| = |\vec{b}|$). Use vectors to show that its two diagonals are perpendicular.



Proof. We first notice that the two diagonals are given by $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$. To show they are perpendicular we need to show $(\vec{a}+\vec{b}) \cdot (\vec{a}-\vec{b}) = 0$. So we compute

$$(\vec{a}+\vec{b}) \cdot (\vec{a}-\vec{b}) = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b} + \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} = |\vec{a}|^2 - |\vec{b}|^2 = 0$$

where the last equality comes from the assumption that $\vec{a}$ and $\vec{b}$ are the same length. $\square$