Math 211: Fall 2011
Differentiability and linear approximation for functions of two variables

Suppose f is a function of two variables that is defined in some region containing (a, b). (More precisely, we assume that f is defined at all points in some disc centered at (a, b).)

Definition. We say that f **is differentiable at** (a, b) if:

- f is continuous at (a, b); and
- there is a linear function $l(x, y)$ such that

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - l(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

Fact. If f is differentiable at (a, b) then the linear function $l(x, y)$ must be given by

$$l(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b).$$

This function is also referred to as **the linear approximation to f at** (a, b). The idea is that, when (x, y) is close to (a, b), $f(x, y)$ is very close to $l(x, y)$, so l is a good approximation to f.

Theorem. If f is continuous at (a, b), and the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both defined and continuous at (a, b), then f is differentiable at (a, b).

This theorem is very useful because we know many functions that are continuous. Unless specified otherwise, you may use the fact that any function built from polynomials, trig functions, exponentials, logarithms, by adding, subtracting, multiplying, dividing, or taking powers, is continuous wherever it is defined.

Example. Show that $f(x, y) = x^3 y$ is differentiable at every point (a, b). Find the linear approximation to $f(x, y)$ at the point $(1, 2)$ and use this to estimate $f(1.01, 1.99)$.

Solution. The function f is continuous at every point (a, b). The partial derivatives are given by

$$\frac{\partial f}{\partial x} = 3x^2 y, \quad \frac{\partial f}{\partial y} = x^3$$

which are both continuous at every point (a, b). Therefore (by the Theorem above), f is differentiable at every point (a, b).

To find the linear approximation, we just use the formula for $l(x, y)$ above with $(a, b) = (1, 2)$. This gives us

$$l(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$$

which gives

$$l(x,y) = 2 + 6(x-1) + (y-2).$$

(It makes sense to leave the answer in this form because it is only a good approximation when (x,y) is close to $(1,2)$.) We then have estimate

$$f(1.01, 1.99) \approx l(1.01, 1.99) = 2 + 6(0.01) + (-0.01) = 2.05.$$

□

One consequence of the Fact above is that if f is differentiable at (a,b), then the partial derivatives must exist at (a,b) (though they need not be continuous at (a,b)). This gives us a good way to show a function is not differentiable at a particular point.

Example. Show that $f(x,y) = \sqrt{x^2+y^2}$ is not differentiable at $(0,0)$.

Solution. If f were differentiable then the partial derivative $\frac{\partial f}{\partial x}(0,0)$ would exist. But that partial derivative is the limit

$$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2}}{h}.$$

Now notice that $\frac{\sqrt{h^2}}{h}$ is $+1$ if $h > 0$ and -1 if $h < 0$. Therefore, the limit as $h \to 0$ does not exist and so the partial derivative does not exist. □

Remark. The graph of $f(x,y) = \sqrt{x^2+y^2}$ is (the upper-half of) a cone with apex at $(0,0,0)$. It should be clear intuitively that there is no well-defined tangent plane at the apex. This example is the higher-dimensional version of the fact that the absolute value function $|x|$ is not differentiable at $x = 0$.

Sometimes the partial derivatives of a function do exist, but the function is still not differentiable. We saw the following example of this in class.

Example. Show that the function

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0); \end{cases}$$

is not differentiable at $(0,0)$.

Solution. The partial derivatives are given by

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} 0 = 0$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} 0 = 0.$$

Therefore, the linear approximation would have to be

$$l(x,y) = 0 + 0(x-0) + 0(y-0) = 0.$$

For f to be differentiable at $(0,0)$, we would need the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-l(x,y)}{\sqrt{x^2+y^2}}$$

to be equal to 0. In our case, that limit is

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{(x^2+y^2)^{\frac{3}{2}}}.$$

If this limit exists, it must be equal to the limit as (x,y) approaches the origin along the line $y=x$, which would be

$$\lim_{x\to 0} \frac{2x^3}{(2x^2)^{\frac{3}{2}}}$$

which is equal to

$$\lim_{x\to 0} \frac{2x^3}{2\sqrt{2}x^3} = \lim_{x\to 0} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

In particular, this value is not 0, so the original limit could not be equal to zero. (In fact, that limit does not exist though you don't need to show that, just that it does not equal zero.) Therefore, f is not differentiable at $(0,0)$. $\square$

Finally, there is the possibility that the function is not defined by a single formula, but is still differentiable.

Example. Show that the function

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y)\neq(0,0); \\ 0 & \text{if } (x,y)=(0,0); \end{cases}$$

is differentiable at $(0,0)$.

Solution. By a similar calculation to the previous example, the linear approximation, if it exists, would have to be

$$l(x,y) = 0.$$

Therefore, we have to show that the following limit is equal to zero:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}.$$

There are no tricks to showing this, as far as I know, so we use an ϵ/δ-proof.

Given $\epsilon > 0$, let $\delta = \epsilon$. We then have that: **if**

$$0 < \sqrt{x^2+y^2} < \delta,$$

then

$$\begin{aligned}\left|\frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}-0\right| &= \left|\frac{x^2}{x^2+y^2}\right|\frac{|y|}{\sqrt{x^2+y^2}}|y| \\ &\le 1(1)\sqrt{x^2+y^2} \\ &< \delta = \epsilon\end{aligned}$$

which means that the limit is equal to 0 as required. So f is differentiable at $(0,0)$. □

Using the ϵ/δ definition is really a last resort, to be used when you can't think of another reason why the function should be differentiable. But, as you can see, the actual proof is not that long or complicated.

Challenge Problem. Find a function f that is differentiable at $(0,0$ but for which the partial derivatives are **not** continuous at $(0,0)$. (Hint: first you might think about the corresponding one-variable problem. Do you know a function $f(x)$ that is differentiable at $x=0$ but for which the derivative $f'(x)$ is not continuous at $x=0$?)