Math 13 Fall 2009: Final Exam
December 19, 2009

Instructions: There are 8 questions on this exam for a total of 100 points. You may not use any outside materials (e.g., notes, calculators, or other devices). Please turn off your cell phone. You have 3 hours to complete this exam. Remember to fully justify your answers.

Problem 1 (15 Points).

(1) Find a vector function describing the line through $P = (2, 5, 7)$ and $Q = (4, 3, 8)$.

(2) Find the equation of the plane through the point $P = (2, 1, 5)$ and containing the line described by $\mathbf{r}(t) = \langle 3t, 2 + t, 2 - t\rangle$.

(3) Determine if the following two lines are parallel, intersecting, or skew:

$$x = 5 + 2t \quad y = -2 - 3t \quad z = 3 + t$$

and

$$x = 3 + 2s \quad y = -1 - 5s \quad z = 2 + s$$

Problem 2 (16 Points). Find the unit tangent vector $\mathbf{T}(t)$, unit normal vector $\mathbf{N}(t)$, unit binormal vector $\mathbf{B}(t)$, and curvature κ for the helix $\mathbf{r}(t) = (a\cos t)\hat{\imath} + (a\sin t)\hat{\jmath} + bt\hat{k}$, where $a, b \geq 0$.

Problem 3 (12 Points).

(1) Show that

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous at the origin.

(2) Show that

$$f(x,y) = \begin{cases} \frac{2x^2y+3y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous at the origin and calculate $\frac{\partial f}{dx}(0,0)$ and $\frac{\partial f}{dy}(0,0)$.

Problem 4 (12 Points).

(1) Let $f(x,y) = \sqrt{x^2+y^2}$. Find the tangent plane to the surface $z = f(x,y)$ at $(3, -4, 5)$. Linearly approximate the value of $f(3.1, -4.1)$.

(2) Find the directional derivative of $f(x,y) = 2e^x \sin y$ at $(0, \pi/4)$ in the direction of $v = \langle 1, -1\rangle$. In what direction is the maximal rate of change at $(0, \pi/4)$?

Problem 5 (10 Points). Find the absolute maximum and minimum values of $f(x,y) = x^2 + xy + y^2$ on the disk $x^2 + y^2 \leq 4$.

Problem 6 (10 Points). Find the volume of the solid that is inside the cylinder $x^2 + y^2 = 16$ and inside the sphere $x^2 + y^2 + z^2 = 25$.

Problem 7 (10 Points). Evaluate

$$\iint_D \frac{x-y}{x+y}\, dA$$

using the change of variables $u = x + y$ and $v = x - y$, where D is the region bounded by the lines $y = x$, $y = x - 1$, $y = 1 - x$, and $y = 2 - x$.

Problem 8 (15 Points).

(1) Calculate the line integral $\int_C (e^y + ye^x)dx + (e^x + xe^y)dy$, where C is a path that begins at (0,0) and ends at (1,-1).

(2) Evaluate the line integral $\int_C (y + e^x)dx + (2x^2 + \cos y)dy$, where C is the boundary of the triangle with vertices (0,0), (1,1), and (2,0) traversed once counterclockwise.

(3) Calculate $\int_C f ds$ where $f(x,y) = x + y$ and C is the curve $x^2 + y^2 = 4$ in the first quadrant from $(2, 0)$ to $(0, 2)$.