Solutions to the Analysis problems on the Comprehensive Examination of January 28, 2011

1. (a) State the Axiom of Completeness (also known as the Axiom of Continuity for the Real Numbers or Axiom C).

   Solution: Every nonempty set of real numbers that is bounded above has a least upper bound.

   (b) Use the Axiom of Completeness to prove that an increasing bounded above sequence of real numbers converges.

   Solution: Let (a_n) be an increasing bounded above sequence with $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Then by the Axiom of Completeness, we can find

   $$s = \sup\{a_n \mid n \in \mathbb{N}\}.$$

   Now we show that s is the limit of this sequence.

   Suppose $\epsilon > 0$. By the definition of supremum, $s - \epsilon$ is not an upper bound for $\{a_n \mid n \in \mathbb{N}\}$. Therefore we can find an $N \in \mathbb{N}$ such that $s - \epsilon < a_N$. Because a_n is an increasing sequence, $a_n \geq a_N$ for $n \geq N$. By the definition of s, we also have $a_n \leq s < s + \epsilon$. Hence $s - \epsilon < a_n < s + \epsilon$ for all $n \geq N$. Hence for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - s| < \epsilon$ for all $n \geq N$. By definition, the sequence (a_n) converges to s.

2. Consider the sequence defined by $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3 + a_n}$ for $n \geq 1$.

   (a) Prove that $a_n < a_{n+1}$ for all $n \geq 1$.

   Solution: We prove this by induction on n.

   When $n = 1$, $a_1 = \sqrt{3}, a_2 = \sqrt{3 + \sqrt{3}} > \sqrt{3} = a_1 > 0$, so the statement holds when $n = 1$.

   Suppose that for some n we have $0 < a_n < a_{n+1}$. Then

   $$\begin{aligned} & a_n < a_{n+1} \\ \Rightarrow\quad & 3 + a_n < 3 + a_{n+1} \\ \Rightarrow\quad & \sqrt{3 + a_n} < \sqrt{3 + a_{n+1}} \quad (\text{Note } 3 + a_n > 3 > 0) \\ \Rightarrow\quad & a_{n+1} < a_{n+2}, \end{aligned}$$

   where the last line follows by the recursive definition of the sequence.

   Therefore $a_n < a_{n+1}$ for all $n \geq 1$.

   (b) Prove that $a_n < 1 + \sqrt{3}$ for all $n \geq 1$

   Solution: Again we use induction on n.

   Clearly the statement is true when $n = 1$.

Now suppose $a_n < 1 + \sqrt{3}$ for some $n \in \mathbb{N}$. Then

$$(\star) \qquad a_{n+1} = \sqrt{3 + a_n} < \sqrt{3 + (1 + \sqrt{3})} = \sqrt{4 + \sqrt{3}}.$$

Hence it suffices to show that

$$(\star\star) \qquad \sqrt{4 + \sqrt{3}} < 1 + \sqrt{3}.$$

Since both sides are positive, $(\star\star)$ is equivalent to $4 + \sqrt{3} < (1 + \sqrt{3})^2$, which in turn is equivalent to

$$4 + \sqrt{3} < 1 + 2\sqrt{3} + 3 = 4 + 2\sqrt{3}.$$

This is clearly true, so $(\star\star)$ is true. Combining this with $(\star)$ gives $a_{n+1} < 1 + \sqrt{3}$ as desired.

Therefore $a_n < 1 + \sqrt{3}$ for all $n \geq 1$.

(c) Explain why $\lim\limits_{n\to\infty} a_n$ exists and find the limit.

By (a) we know (a_n) is an increasing sequence and by (b) we know this sequence is bounded above. Therefore by the Monotone Convergence Theorem (a_n) converges. Therefore $\lim\limits_{n\to\infty} a_n$ exists.

We show that $\lim\limits_{n\to\infty} a_n = \frac{1+\sqrt{13}}{2}$.

Suppose $\lim\limits_{n\to\infty} a_n = L$. Standard properties of limits imply that $\lim\limits_{n\to\infty} a_n^2 = L^2$. Then

$$L^2 = \lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} (3 + a_n) = \lim_{n\to\infty} 3 + \lim_{n\to\infty} a_n = 3 + L.$$

Solving the equation $L^2 - L - 3 = 0$ gives $L = \frac{1\pm\sqrt{13}}{2}$. Since $\sqrt{3} < a_n < 1 + \sqrt{3}$ for all $n \geq 1$, we must choose the positive answer, hence $\lim\limits_{n\to\infty} a_n = L = \frac{1+\sqrt{13}}{2}$.

3. Consider the sequence of functions on $[0, \pi]$ defined by $f_n(x) = (\sin x)^n$ for $x \in [0, \pi]$ and $n \geq 1$.

(a) Compute $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for all $x \in [0, \pi]$.

Solution:

When $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, $0 \leq \sin x < 1$, so $f_n(x) = (\sin x)^n \to 0$ as $n \to \infty$.

Now consider when $x = \frac{\pi}{2}$. When $x = \frac{\pi}{2}$, $f_n(x) = (\sin \frac{\pi}{2})^n = 1$, hence $f_n(x) \to 1$ as $n \to \infty$. Combining both we get the function

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi], \\ 1 & \text{if } x = \frac{\pi}{2}, \end{cases}$$

which satisfies $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for all $x \in [0, \pi]$.

(b) Does $\{f_n(x)\}_{n=1}^{\infty}$ converge uniformly to $f(x)$? Explain your reasoning.

Solution 1: Suppose that $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly to $f(x)$. A theorem proved in class states that a uniform limit of continuous functions is continuous. Since each $f_n(x)$ is continuous, this theorem and uniform convergence would imply that $f(x)$ would be continuous. But the formula for $f(x)$ from part (a) shows that $f(x)$ is not continuous at $x = \frac{\pi}{2}$. Hence uniform convergence must fail.

Solution 2: Let $\epsilon = 1/2$. For each $N \in \mathbb{N}$, because $f_N(x)$ is continuous on $[0, \pi]$, the range of $f_N(x)$ is $[0, 1]$ by the Intermediate Value Theorem. Hence we can find $x_N \in [0, \pi]$ such that $f_N(x_N) = (\sin x_N)^N = 1/2$. Note that $x_N \neq \pi/2$ since $f_N(\pi/2) = 1$, so $f(x_N) = 0$.

Thus for every $N \in \mathbb{N}$, we do not have $|f_N(x) - f(x)| < 1/2$ for all $x \in [0, \pi]$ since $f_N(x_N) = 1/2$ and $f(x_N) = 0$. Hence $f(x)$ does not satisfy the conditions for uniform convergence. We conclude that $\{f_n(x)\}_{n=1}^{\infty}$ does not converge uniformly to $f(x)$ on $x \in [0, \pi]$.

4. (a) State the definition of $\lim_{x \to 0} f(x) = L$.

Solution: $\lim_{x \to 0} f(x) = L$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$, it follows that $|f(x) - L| < \epsilon$.

(b) Assume that $\lim_{x \to 0} f(x) = L$ for some real number $L > 0$. Prove that there is $\delta > 0$ with the property that $f(x) > \frac{1}{2}L$ for all $x \in (-\delta, \delta)$, $x \neq 0$.

Solution: Let $\epsilon = \frac{L}{2}$. By the definition given in (a), we know there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$, it follows that $|f(x) - L| < \epsilon = \frac{L}{2}$. Then for all $x \in (-\delta, \delta)$, $x \neq 0$ we get $\frac{1}{2}L < f(x) < \frac{3}{2}L$. In particular, we have $f(x) > \frac{1}{2}L$ for $x \in (-\delta, \delta)$, $x \neq 0$. This is exactly what we wanted to prove.