Math 12 Spring 2009: Exam 1

Name:

Instructions: There are 4 questions on this exam each of which is scored out of 8 points for a total of 32 points. You may not use any outside materials (eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1. Evaluate the following integrals.

(a) $\int \frac{\sqrt{1-\sin^2 x}}{\sec^2 x \csc^4 x} dx$

(b) $\int \frac{4x^4-x+1}{x^3+1} dx$

Proof.

(a) Using $\sin^2 x + \cos^2 x = 1$ we have

$$\begin{aligned}\int \frac{\sqrt{1-\sin^2 x}}{\sec^2 x \csc^4 x} dx &= \int \frac{\sqrt{\cos^2 x}}{\sec^2 x \csc^4 x} dx \\ &= \int \frac{\cos x}{\sec^2 x \csc^4 x} dx \\ &= \int \cos^3 x \sin^4 x dx \\ &= \int \cos x(1-\sin^2 x) \sin^4 x dx\end{aligned}$$

making the substitution $u = \sin x, du = \cos x dx$ we have

$$\begin{aligned}\int \frac{\sqrt{1-\sin^2 x}}{\sec^2 x \csc^4 x} dx &= \int (1-u^2)u^4 du \\ &= \int u^4 - u^6 du = \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.\end{aligned}$$

(b) We first apply polynomial long division to get

$$\begin{array}{r|l} & \quad 4x \\ \hline x^3+1 & 4x^4 - x + 1 \\ & 4x^4 + 4x \\ \hline & \quad -5x + 1 \end{array}$$

which tells us that

$$\frac{4x^4 - x + 1}{x^3 + 1} = 4x + \frac{-5x+1}{x^3+1}.$$

Examining the denominator we see that -1 is a root, so we know $x+1$ divides the denominator.

We again apply long division to get

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x+1 & x^3 \qquad\quad + 1 \\
 & x^3 + x^2 \\
\hline
 & -x^2 + x + 1 \\
 & -x^2 - x \\
\hline
 & x + 1 \\
 & x + 1 \\
\hline
 & 0
\end{array}
$$

So we know that

$$\frac{4x^4 - x + 1}{x^3 + 1} = 4x + \frac{-5x + 1}{(x+1)(x^2 - x + 1)}.$$

Apply partial fractions to the right hand side we need to find constants A, B, C such that

$$\frac{-5x + 1}{(x+1)(x^2 - x + 1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 - x + 1}.$$

So we need to solve

$$
\begin{aligned}
(x^2): &\quad A + B = 0 \\
(x): &\quad -A + B + C = -5 \\
(1): &\quad A + C = 1.
\end{aligned}
$$

We get the solution

$$A = 2 \quad B = -2 \quad C = -1.$$

So we must perform the integration

$$
\begin{aligned}
\int \frac{4x^4 - x + 1}{x^3 + 1} dx &= \int 4x + \frac{2}{x+1} + \frac{-2x - 1}{x^2 - x + 1} dx \\
&= 2x^2 + 2\ln|x+1| - \int \frac{2x + 1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx.
\end{aligned}
$$

For the last integral we make the substitution $x - 1/2 = \frac{\sqrt{3}}{2}\sec\theta$ to get

$$
\begin{aligned}
\int \frac{2x + 1}{(x - \frac{1}{2})^2 + \frac{3}{4}} dx &= \int \frac{\sqrt{3}\sec\theta + 2}{3/4\tan\theta} \frac{\sqrt{3}}{2} \sec\theta\tan\theta d\theta \\
&= 2\int \sec^2\theta d\theta + \frac{4}{\sqrt{3}} \int \sec\theta d\theta \\
&= 2\tan\theta + \frac{4}{\sqrt{3}} \ln|\sec\theta + \tan\theta|.
\end{aligned}
$$

Drawing the triangle for $\frac{2x-1}{\sqrt{3}} = \sec\theta$ and making the substitutions we then have

$$\int \frac{4x^4 - x + 1}{x^3 + 1}\,dx =$$
$$2x^2 + 2\ln|x+1| - 2\frac{\sqrt{(2x-1)^2 - 3}}{\sqrt{3}} + \frac{4}{\sqrt{3}} \ln\left|\frac{2x-1}{\sqrt{3}} + \frac{\sqrt{(2x-1)^2-3}}{\sqrt{3}}\right| + C.$$

□

Problem 2. Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{2e^x-2x-2}{x^2}$

(b) $\lim_{x\to\infty} x(e^{1/x}-1)$.

Proof.

(a) This is of the form $\frac{0}{0}$ we so apply L/Höpital's Rule to get

$$\lim_{x\to 0} \frac{2e^x-2x-2}{x^2} = \lim_{x\to 0} \frac{2e^x-2}{2x}.$$

This is again of the form $\frac{0}{0}$ so we again apply L'Höpital's Rule to get

$$\lim_{x\to 0} \frac{2e^x-2}{2x} = \lim_{x\to 0} \frac{2e^x}{2} = 1.$$

So we know that

$$\lim_{x\to 0} \frac{2e^x-2x-2}{x^2} = 1.$$

(b) This is of the form $\infty \cdot 0$ so we make the fraction

$$\lim_{x\to\infty} x(e^{1/x}-1) = \lim_{x\to\infty} \frac{e^{1/x}-1}{1/x}$$

to have the form $\frac{0}{0}$. We apply L/Höpital's Rule to get

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}e^{1/x}}{-1/x^2} = \lim_{x\to\infty} e^{1/x} = 1.$$

□

Problem 3.

(a) Find the derivative of $f(x) = \sinh(\ln x)$.

(b) Evaluate $\int_0^1 2\sinh x dx$.

Proof.

(a) We make use of the definition

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

to get

$$f(x) = \frac{1}{2}\left(x - \frac{1}{x}\right).$$

We compute the derivative as

$$f'(x) = \frac{1}{2}\left(1 + \frac{1}{x^2}\right).$$

(b) We make use of the definition

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

to get

$$\int_0^1 2\sinh x dx = \int_0^1 e^x - e^{-x} dx.$$

We integrate to get

$$\begin{aligned}\int_0^1 2\sinh x dx &= [e^x + e^{-x}]_0^1 \\ &= \left(e + \frac{1}{e}\right) - (1+1) = e + \frac{1}{e} - 2.\end{aligned}$$

□

Problem 4. If the slope of the tangent line to a curve is given by $\frac{dy}{dx} = \frac{\arcsin x}{x^2}$ and we know $y(1) = 0$, find the curve $y(x)$.

Proof. So we must integrate $\frac{dy}{dx}$ to get $y(x)$. We start by using integration by parts to get

$$\int \frac{\arcsin x}{x^2} dx = \frac{-\arcsin x}{x} + \int \frac{1}{x\sqrt{1-x^2}} dx.$$

Now we make the trigonometric substitution $x = \sin\theta$ to get

$$\begin{aligned}\frac{-\arcsin x}{x} + \int \frac{1}{x\sqrt{1-x^2}} dx &= \frac{-\arcsin x}{x} + \int \frac{\cos\theta}{\sin\theta\cos\theta} d\theta \\ &= \frac{-\arcsin x}{x} + \int \csc\theta d\theta \\ &= \frac{-\arcsin x}{x} + \ln|\csc\theta - \cot\theta| + C.\end{aligned}$$

Drawing the triangle, we make the reverse substitution to get

$$\frac{-\arcsin x}{x} + \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + C$$

Using the initial point $y(1) = 0$ we have

$$0 = \frac{-\arcsin 1}{1} + \ln|1 - 0| + C = -\arcsin 1 + C.$$

We know $\arcsin 1 = \frac{\pi}{2}$ so we have

$$0 = -\frac{\pi}{2} + C$$

and hence $C = \frac{\pi}{2}$. So the final function is

$$y(x) = \frac{-\arcsin x}{x} + \ln\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + \frac{\pi}{2}.$$

□