Math 13: Project 2
Due Friday 4/16/2010

1. Introduction

Many rockets such as the Pegasus XL currently used to launch satellites and the Saturn V that put men on the moon, are designed to use three stages in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit around the Earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages, which are to be designed in such a way as to minimize the total mass of the rocket while enabling it to reach a desired velocity.

Remember that you should fully justify all of your work. This project will be graded out of 25 points.

2. Background

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocker vehicle is modeled by

$$\Delta v = -c \ln \left(1 - \frac{(1-S)M_r}{P + M_r} \right)$$

where

$M_r =$ the mass of the rocket engine including initial fuel
$P =$ the mass of the payload
$S =$ a structural factor determine by the design of the rocket (specifically it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket vehicle with payload)
$c =$ the (constant) speed of exhaust relative to the rocket.

For a rocket with three stages and a payload mass of A. Assume that the outside forces are negligible and that c and S remain constant for each stage. If M_i is the mass of the ith stage, we can initially consider the rocket engine to have mass M_1 and its payload to have mass $M_2 + M_3 + A$.; the second and third stage can be handled similarly.

3. Problem

(1) Show that the velocity attained after all three stages have been jettisoned is given by

$$v_f = c \left[\ln \left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} \right) + \ln \left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A} \right) + \ln \left(\frac{M_3 + A}{SM_3 + A} \right) \right]$$

(2) We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocker subject to the constraint that the desired final velocity (calculated in (1)) v_f is obtained. However, the method of Lagrange multipliers is difficult to carry out with the current expressions. To simplify, we define variables N_i so that the constraint equation may be expressed as

$$v_f = c(\ln N_1 + \ln N_1 + \ln N_3).$$

Since M is now difficult to express in terms of the N_i we wish to use a simpler function that will be minimized at the same place as M. Show that

$$\begin{aligned}\frac{M_1+M_2+M_3+A}{M_2+M_3+A} &= \frac{(1-S)N_1}{1-SN_1}\\ \frac{M_2+M_3+A}{M_3+A} &= \frac{(1-S)N_2}{1-SN_2}\\ \frac{M_3+A}{A} &= \frac{(1-S)N_3}{1-SN_3}\end{aligned}$$

and conclude that

$$\frac{M+A}{A} = \frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)}.$$

(3) Argue that

$$\ln\left(\frac{M+A}{A}\right)$$

is minimized at the same location as M. Find expressions for the values of N_i where the minimum occurs subject to the final velocity constraint.

(4) Find an expression for the minimum value of M as a function of v_f.

(5) If we want to put a three-stage rocket into orbit 100miles above the Earth's surface, a final velocity of approximately $17,500$mph is required. Suppose that each stage is built with a structural constant $S = 0.2$ and an exhaust speed of $c = 6000$mph.
   (a) Find the minimum total mass M of the rocket engines as a function of A.
   (b) Find the mass of each individual stage as a function of A.

(6) The same rocket would require a final velocity of approximately 24,700mph in order to escape Earth's gravity. Find the mass of each individual stage that would minimize the total mass of the rocket engines and allow the rocket to propel a 500-pound probe into deep space.

4. Checklist for Your Writing Projects

Based on checklists by Annalisa Crannell at Franklin & Marshall and Tommy Ratliff at Wheaton College.

Does this paper:

(1) clearly (re)state the problem to be solved?
(2) provide a paragraph which explains how the problem will be approached?
(3) state the answer in a few complete sentences which stand on their own?
(4) give a precise and well-organized explanation of how the answer was found?
(5) clearly label diagrams, tables, graphs, or other visual representations of the math?
(6) define all variables, terminology, and notation used?
(7) clearly state the assumptions which underlie the formulas and theorems, and explain how each formula or theorem is derived, or where it can be found?
(8) give acknowledgment where it is due?
(9) use correct spelling, grammar, and punctuation?
(10) contain correct mathematics?
(11) solve the questions that were originally asked?