Math 211, Multivariable Calculus, Fall 2011
Midterm III Solutions

1. *Use the Lagrange multiplier method to find the absolute maximum and absolute minimum of the function*

$$f(x,y) = x + y$$

subject to the constraint

$$x^2 + 2y^2 = 1.$$

(You must use the Lagrange multiplier method to get credit for this question.)

The region satisfying the constraint is closed and bounded, and the function f is continuous, so the Extreme Value Theorem implies that there is an absolute maximum and minimum.

We apply the Lagrange multiplier method with $g(x,y) = x^2 + 2y^2$. Since f and g are differentiable everywhere, we just need to check for points where $\nabla f = \lambda \nabla g$ or $\nabla g = \mathbf{0}$, where

$$\nabla f = \langle 1, 1 \rangle, \quad \nabla g = \langle 2x, 4y \rangle.$$

The only point where $\nabla g = \mathbf{0}$ is $(0,0)$ which does not satisfy the constraint.

The equation $\nabla f = \lambda \nabla g$ gives

$$1 = 2x\lambda, \quad 1 = 4y\lambda.$$

This tells us that

$$2y = 4xy\lambda = x.$$

Substituting into the constraint we have

$$(2y)^2 + 2y^2 = 1$$

so

$$6y^2 = 1$$

and hence

$$y = \pm 1/\sqrt{6}.$$

Since $x = 2y$, the two constrained critical points are

$$\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \quad \left(\frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right).$$

We now evaluate f at each of these:

$$f\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \frac{3}{\sqrt{6}}$$

which is the absolute maximum, and

$$f\left(\frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right) = \frac{-3}{\sqrt{6}}$$

which is the absolute minimum.

2. *A function $f(x, y)$ has*

$$\nabla f = \left\langle 2xe^y, x^2e^y + y^2 - 4 \right\rangle.$$

Find the critical points of f and classify them as local maxima, local minima, or saddle points.

The critical points satisfy $\nabla f = \mathbf{0}$ so

$$2xe^y = 0, \quad x^2e^y + y^2 - 4 = 0.$$

Since e^y cannot be zero, the first equation implies that $x = 0$. The second equation then tells us that $y^2 = 4$ so $y = \pm 2$. Therefore the two critical points are

$$(0, 2), \quad (0, -2).$$

To classify these, we use the Second Derivative Test. Since $f_x = 2xe^y$ and $f_y = x^2e^y + y^2 - 4$, we have

$$f_{xx} = 2e^y,\ f_{xy} = f_{yx} = 2xe^y,\ f_{yy} = x^2e^y + 2y.$$

At $(0, 2)$ we have

$$f_{xx}(0, 2) = 2e^2, f_{xy}(0, 2) = 0, f_{yy}(0, 2) = 4$$

so

$$D(0, 2) = (2e^2)(4) - (0)^2 = 8e^2 > 0.$$

Therefore, since $f_{xx}(0, 2) > 0$, $(0, 2)$ is a local minimum.

Also

$$f_{xx}(0, -2) = 2e^{-2}, \quad f_{xy}(0, -2) = 0, f_{yy}(0, -2) = -4$$

so

$$D(0, -2) = (2e^{-2})(-4) - (0)^2 = -8e^{-2} < 0$$

so $(0, -2)$ is a saddle point.

3. *Let R be the quarter-disc given by $x^2 + y^2 \leq 4$, $x \geq 0$ and $y \geq 0$. Find*

$$\iint_R x + y \, dA.$$

In polar coordinates, this region can be described as

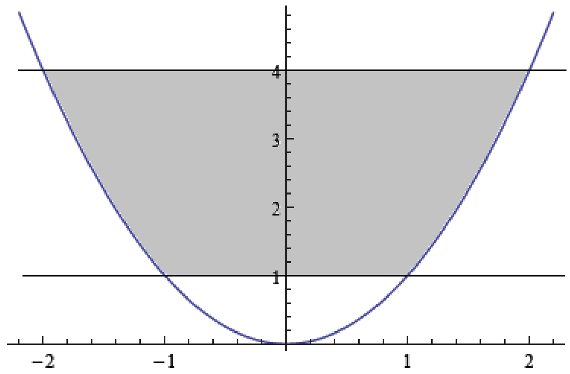
$$0 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

and so the integral is

$$\begin{aligned}
\int_{r=0}^{r=2} \int_{\theta=0}^{\theta=\pi/2} (r\cos\theta + r\sin\theta) r \, d\theta \, dr &= \int_{r=0}^{r=2} r^2 \left[\sin\theta - \cos\theta\right]_{\theta=0}^{\theta=\pi/2} \, dr \\
&= \int_{r=0}^{r=2} 2r^2 \, dr \\
&= \left[2r^3/3\right]_{r=0}^{r=2} \\
&= \frac{16}{3}
\end{aligned}$$

4. *Use a triple integral to find the volume of the region D inside the paraboloid $z = x^2 + y^2$ and between the planes $z = 1$ and $z = 4$.*

If we draw a cross-section through the region we are interested in, it looks like this:



It's easiest to describe this region using cylindrical coordinates. In this case it is given by

$$\begin{aligned} &0 \le \theta \le 2\pi \\ &1 \le z \le 4 \\ &0 \le r \le \sqrt{z} \end{aligned}$$

The key part here is the range of values for r given a fixed z. The minimum value for r is 0 and the maximum is on the surface $z = x^y + y^2 = r^2$ so $r = \sqrt{z}$.

(There are other possibilities: for example, one can work out the volume of the whole paraboloid from $z = 0$ to $z = 4$ and then subtract the part from $z = 0$ to $z = 1$.)

The volume is given by integrating the constant function 1 over the solid region. This gives

$$\begin{aligned} \iiint_D 1 \, dV &= \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=4} \int_{r=0}^{r=\sqrt{z}} r \, dr \, dz \, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=4} \left[r^2/2 \right]_{r=0}^{r=\sqrt{z}} dz \, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=4} \frac{z}{2} \, dz \, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \left[z^2/4 \right]_{z=1}^{z=4} d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \frac{15}{4} \, d\theta \\ &= \frac{15\pi}{2} \end{aligned}$$

5. (a) *Find the Jacobian* $\dfrac{\partial(x,y)}{\partial(u,v)}$ *related to the change of variables*

$$u = e^{x+y}, \quad v = e^{x-y}.$$

(b) *Calculate the integral*

$$\iint_R e^{2x}\, dA$$

where R is the region bounded by the straight lines

$$y = x, \quad y = -x, \quad x = \frac{\ln 2}{2}.$$

(a) We need to find x and y in terms of u and v. The two equations give

$$x + y = \ln(u)$$
$$x - y = \ln(v)$$

so

$$x = \frac{\ln(u) + \ln(v)}{2}, \quad y = \frac{\ln(u) - \ln(v)}{2}.$$

We therefore have

$$\frac{\partial x}{\partial u} = \frac{1}{2u}, \frac{\partial x}{\partial v} = \frac{1}{2v}, \frac{\partial y}{\partial u} = \frac{1}{2u}, \frac{\partial y}{\partial v} = \frac{-1}{2v}$$

and so

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= \frac{1}{2u}\frac{-1}{2v} - \frac{1}{2u}\frac{1}{2v} \\ &= \frac{-1}{4uv} - \frac{1}{4uv} \\ &= \frac{-1}{2uv}\end{aligned}$$

(b) The line $y = x$ is given by $x - y = 0$ and so $v = e^0 = 1$. The line $y = -x$ is given by $x + y = 0$ and so $u = e^0 = 1$. The line $x = (\ln 2)/2$ is

$$\frac{\ln(u) + \ln(v)}{2} = \frac{\ln 2}{2}$$

so

$$\ln(uv) = \ln 2$$

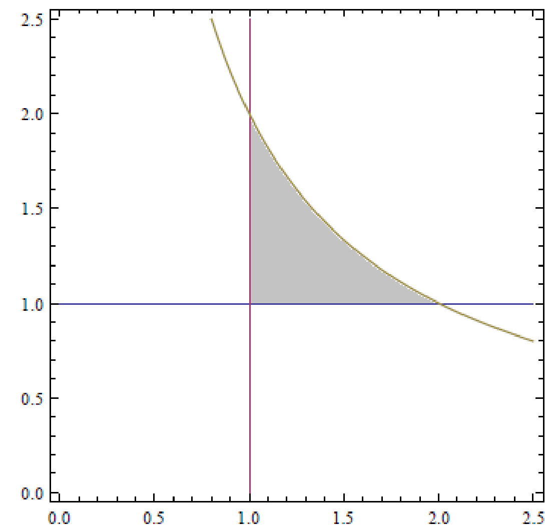
so

$$uv = 2$$

or

$$v = \frac{2}{u}.$$

Therefore the region corresponding to R in the uv-plane looks like



This is described by

$$1 \le u \le 2 \; 1 \le v \le \frac{2}{u}.$$

We also have

$$e^{2x} = \exp(\ln(u) + \ln(v)) = uv$$

and so the integral becomes

$$\begin{aligned} \int_{u=1}^{u=2} \int_{v=1}^{v=2/u} uv \left| \frac{-1}{2uv} \right| \, dv \, du &= \int_{u=1}^{u=2} \int_{v=1}^{2/u} \frac{1}{2} \, dv \, du \\ &= \int_{u=1}^{u=2} \frac{1}{u} - \frac{1}{2} \, du \\ &= \left[\ln(u) - \frac{u}{2} \right]_{u=1}^{u=2} \\ &= \ln(2) - \frac{1}{2} \end{aligned}$$