Math 13 Fall 2010: Exam 3
Tuesday April 27, 2010

Name:

Instructions: There are 4 questions on this exam each of which is scored out of 8 points for a total of 32 points. You may not use any outside materials(eg. notes or calculators). You have 50 minutes to complete this exam. Remember to fully justify your answers.

Score:

Problem 1. Evaluate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2} dxdy$$

Proof. Changing to polar we have

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2} dxdy &= \int_0^{\pi/2} \int_0^1 \frac{r}{1+r^2} drd\theta = \int_0^{\pi/2} \frac{1}{2} \ln \left|1+r^2\right| \Big|_0^1 d\theta \\ &= \int_0^{\pi/2} \frac{\ln 2}{2} d\theta = \frac{\pi \ln 2}{4}. \end{aligned}$$

□

Problem 2. Consider the region R that lies outside the cone $z^2 = 3(x^2 + y^2)$ and inside the sphere $x^2 + y^2 + z^2 = 4$ in the first octant. Set up the integral for the volume of R in rectangular, cylindrical, and spherical coordinates.

Proof. The curve of intersection is given by $4(x^2 + y^2) = 4$ and hence $x^2 + y^2 = 1$. We also need to know the cone equation is spherical coordinates, which is $\tan\phi = 1/\sqrt{3}$ and so $\phi = \frac{\pi}{6}$. So we have

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{3(x^2+y^2)}} dzdydx + \int_0^2 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} dzdydx$$

$$V = \int_0^{\pi/2} \int_0^1 \int_0^{r\sqrt{3}} rdzdrd\theta + \int_0^{\pi/2} \int_1^2 \int_0^{\sqrt{4-r^2}} rdzdrd\theta$$

$$V = \int_0^{\pi/2} \int_{\pi/6}^{\pi/2} \int_0^2 \rho^2 \sin\phi d\rho d\phi d\theta.$$

or we can subtract the area inside the cone from the sphere. We have

$$V = \frac{4}{3}\pi - \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} dzdydx$$

$$V = \frac{4}{3}\pi - \int_0^{\pi/2} \int_0^1 \int_{r\sqrt{3}}^{\sqrt{4-r^2}} rdzdrd\theta$$

$$V = \frac{4}{3}\pi - \int_0^{\pi/2} \int_0^{\pi/6} \int_0^2 \rho^2 \sin\phi d\rho d\phi d\theta.$$

□

Problem 3. Consider the region R above the upper sheet of the hyperboloid $x^2+y^2=z^2-1$ and below the plane $z=3$ with density $\delta(x,y,z)=2z$.

1. Find the mass of R.

2. Set-up but do not evaluate the integral for the moment of inertia about the z-axis for R.

Proof. We have that the vertex of the upper sheet is given by $z=1$ at $(0,0,1)$ and at $z=3$ we have $x^2+y^2=8$.

1. For the mass (using cylindrical coordinates) we have

$$\begin{aligned}
M &= \int_0^{2\pi}\int_0^{\sqrt{8}}\int_{\sqrt{1+r^2}}^{3} 2rz dz dr d\theta = \int_0^{2\pi}\int_0^{\sqrt{8}} 9r - r(1+r^2) dr d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{8}} 8r - r^3 dr d\theta \\
&= \int_0^{2\pi} 4r^2 - r^4/4]_0^{\sqrt{8}} d\theta = \int_0^{2\pi} 32 - 16 d\theta \\
&= 32\pi.
\end{aligned}$$

2. For the moment of inertia we have

$$I_z = \int_0^{2\pi}\int_0^{\sqrt{8}}\int_{\sqrt{1+r^2}}^{3} 2r^3 z dz dr d\theta.$$

□

Problem 4. Let R be the parallelogram bounded by the lines $x+y=1, x+y=2, 2x-3y=2, 2x-3y=4$. Evaluate

$$\iint_R 5x dA.$$

Proof. Set $u = x+y$ and $v = 2x-3y$ and so we have $x = \frac{1}{5}(3u+v)$ and $y = \frac{1}{5}(2u-v)$. The jacobian of the transformation is given by

$$J = \left|\det \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix}\right| = \frac{1}{5}.$$

So we have

$$\iint_R 5x dA = \int_1^2 \int_2^4 \frac{1}{5}(3u+v) dv du$$
$$\frac{1}{5}\int_1^2 6u + 6du$$
$$= \frac{1}{5}(9+6) = 3.$$

□