Life on a Merry-Go-Round

An Examination of Relativistic Rotating Reference Frames

Nathaniel I. Reden

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Faculty advisor Kannan Jagannathan

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Contents

Abstract

When we measure the circumference and radius of a rapidly rotating disk in the inertial frame in which it is undergoing no translational motion, we find the ordinary relation $C = 2\pi R$. But in this frame the edge of the disk is moving in a direction parallel to itself, so this measurement must be of a length-contracted distance. The circumference in the rotating frame must therefore be greater than 2π times the radius, giving us a noneuclidean geometry in that frame. This is called Ehrenfest's paradox, and has troubled relativity almost since its inception.

The goal of my work is to illuminate the difficulties of relativistic rotating reference frames. I will analyze the system with general relativity and also develop a computer-aided numeric method to visualize a set of points from the point of view of a rotating observer.

Chapter 1

An Introduction to Relativistic Rotating Reference Frames

1.1 Overview

Consider a flat, circular disk floating in euclidean space. In a particular inertial reference frame, which we shall call the lab frame, the center point of the disk has zero velocity, but the disk itself is rotating around an axis through and perpendicular to this center point with a constant angular velocity ω. It is somewhat like a merry-go-round floating in deep space, far away from gravitating bodies.

Now, imagine standing on that merry-go-round at a distance R_0 from the center and rotating with it. We wish to know what the disk—and the rest of the universe—looks like from this perspective. What coordinate system should we use to define distance and time, and what is the shape of the disk? These seem like simple questions to answer, but a preliminary analysis shows them to be deeper and more difficult than we might expect. The edge of the disk, at radius R_0, has a velocity $v = R_0\omega$ in the inertial reference frame. This means that the entire edge has undergone Lorentz contraction in this frame, shortening it by a factor of $\sqrt{1 - v^2/c^2}$. When we step into the rotating frame, the edge is no longer moving, so the circumference we measured in the inertial frame is smaller than the circumference in the rotating frame. The radius, however, does not change, as it is moving in a direction perpendicular to the motion. The circumference of the disk in the rotating frame is greater than 2π times the radius, implying noneuclidean spatial geometry. Moreover, clocks on the disk are moving in the inertial frame and by virtue of time dilation are running slow.

This makes for a strange set of coordinates in the rotating frame, but general relativity often deals with strange coordinate systems. This set of coordinates also raises worries about general relativity, however. Einstein's theory says that the curvature of spacetime is absolute; that is, in any reference frame the Riemann curvature tensor should be the same.[1] Unfortunately, the geometry of the rotating frame is noneuclidean while the geometry of the inertial frame is euclidean, so different curvature tensors have been obtained in different reference frames. It seems as though we should be able to build up the geometry from a set of reasonable coordinates in the rotating frame, so this disagreement with general relativity must be resolved.

The aim of this work is to analyze carefully the relativistic rotating disk and to clarify its problems and properties to the satisfaction of both general relativists and those new to the field. We begin with an overview of the system and the problems it presents.

1.2 History

Ehrenfest was the first to consider the rotating disk from the point of view of Einstein's special relativity, and he wrote a paper pointing out the apparent paradox in the ratio between radius and circumference in 1909 [2]. Einstein read the paper and realized that it

[1] The components of the tensor may vary from coordinate system to coordinate system, but the tensor itself is an invariant object. A tensor is often defined as a set of values that satisfy specific transformation rules under changes in coordinates (reference [1] has a thorough development of tensor algebra). If all components of a tensor are zero in one coordinate system, however, they are zero in all coordinate systems.

implied a geometric cause for gravitation; he used this idea to create the general theory of relativity. He never explicitly solved the relativistic rotating disk problem, however. It was only used as a motivation for the new theory [3], and in its finished form general relativity declares that geometry is invariant. According to Einstein, the curvature tensor in the disk frame should be the same as that in the lab frame, and as we will see in sections 1.4 and 2.4, this is not the implication of Ehrenfest's paper.

When Arzeliès wrote his book on relativistic kinematics [4], general relativity was still in the process of being accepted by the physics community. Arzeliès was a supporter of the theory in name, although perhaps not in practice. He claimed to follow Einstein's theory in approaching the disk problem, but contradicted the general relativistic idea of invariant geometry. A variety of diverse thoughts on the subject of the relativistic rotating disk had sprung up by this point. Some considered it a proof of the invalidity of the theory. Others disagreed on how to resolve the apparent paradoxes: some (Arzeliès among them) by geometric calculation, others by hypothesizing relativistic warping of the disk material [5, 6], and still others by playing with the definitions of angular velocity, angle, radius, and so on [7, 8, 9, 10]. None of the methods at the time were entirely satisfactory, and so the controversy continued.

In the present day, general relativity and the curvature of spacetime is almost universally accepted as the non-quantum theory of gravity. Most general relativists use the generalized coordinate approach developed by Einstein, saying simply "write down some coordinate system for the disk that obeys the following constraints." Regularly, though, newcomers to the field take "Ehrenfest's paradox" to show general relativity to be incorrect. The relativists generally ignore these claims, stating (quite rightly from their point of view) that the approach is flawed. And yet, there is no clear explanation of Ehrenfest's paradox or a description of the disk that is universally regarded as correct [11, 12, 13]. The present work is intended to help ameliorate this situation.

Recently, some work has also been done on the practice of electrodynamics on the rotating disk [11, 14, 15]. Like the geometry thereof, many results in this area are apparently paradoxical, and no consensus has been reached on how to resolve them. My work does not touch on electrodynamics directly. It does have consequences for electrodynamics on the disk, though; the problems in electrodynamics stem from the geometry and especially the definitions of time and simultaneity on the disk.

1.3 The Metric

The spatial geometry of any surface may be found by constructing the metric of that surface. The disk has rotational symmetry around its axis of rotation, so cylindrical coordinates are the easiest ones to use. We must find expressions for a length element ds in terms of dr, $d\theta$, and dz (see Figure 1.1). Take a point with coordinates (r, θ, z) and a nearby point with coordinates $(r + dr, \theta + d\theta, z + dz)$. The distance between them is ds. The distance along the z-axis between the two points is trivially dz, and that along the r-axis is dr (the same

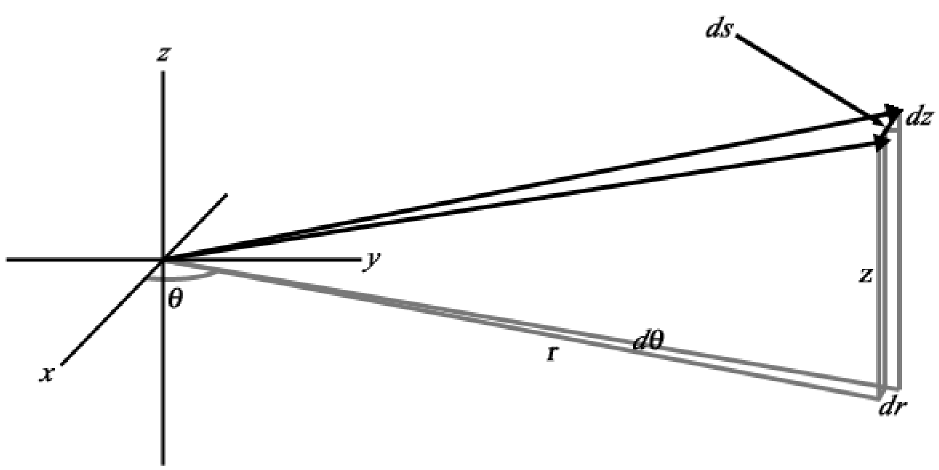


Figure 1.1: Construction of ds

as an inertial frame, since they are perpendicular to the motion). The distance between the points along the θ-axis, however, is contracted by a factor of $\sqrt{1-v^2/c^2}$ in the inertial frame (which is euclidean), so in the rotating frame we must re-extend it to $rd\theta/\sqrt{1-v^2/c^2}$. ds is then given by the expression

$$ds^2 = dr^2 + \frac{r^2 d\theta^2}{1-\frac{r^2\omega^2}{c^2}} + dz^2 \tag{1.1}$$

This gives us the metric tensor[2], which looks like

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{r^2}{1-\frac{r^2\omega^2}{c^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

1.4 The Curvature Tensor

Once we have the metric tensor, it is a simple matter of calculation to find the Riemann curvature tensor. Noting that

$$[g_{ij}][g^{jk}] = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.3}$$

[2] I am using the notational conventions of Foster and Nightingale's *A Short Course in General Relativity* [1] for my calculations of the curvature tensor and related quantities. Indices i, j, k, and so on run from 1 to 3, while Greek indices run from 0 to 3, with $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, and $x^3 = z$. Superscripted indices are contravariant suffixes, while subscripted indices are covariant suffixes. When an expression appears with the same suffix once as a superscript and once as a subscript, Einstein's summation convention is employed. There is one significant difference between my notation and theirs, and that is that I am using $\eta_{00} = -1$ and $\eta_{11} = \eta_{22} = \eta_{33} = 1$ rather than the opposite.

and using the definition of the *connection coefficients* Γ^i_{jk}:

$$\Gamma^l_{ki} \equiv \frac{1}{2} g^{lj} \left(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki} \right) \tag{1.4}$$

we can write the components of the curvature tensor as

$$R^l_{ijk} \equiv \partial_j \Gamma^l_{ik} - \partial_k \Gamma^l_{ij} + \Gamma^m_{ik} \Gamma^l_{mj} - \Gamma^m_{ij} \Gamma^l_{mk} \tag{1.5}$$

The nonzero connection coefficients are

$$\begin{aligned} \Gamma^1_{22} &= -\frac{r}{\left(1 - \frac{r^2\omega^2}{c^2}\right)^2} \\ \Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{r\left(1 - \frac{r^2\omega^2}{c^2}\right)} \end{aligned} \tag{1.6}$$

A routine calculation shows that the nonzero components of R^l_{ijk} are

$$R^1_{212} = -R^1_{221} = \partial_1 \Gamma^1_{22} - \Gamma^2_{21} \Gamma^1_{22} = -\frac{3r^2\omega^2}{c^2 \left(1 - \frac{r^2\omega^2}{c^2}\right)^3} \tag{1.7}$$

and

$$R^2_{112} = -R^2_{121} = \partial_1 \Gamma^2_{12} + \Gamma^2_{12} \Gamma^2_{21} = \frac{3\omega^2}{c^2 \left(1 - \frac{r^2\omega^2}{c^2}\right)^2} \tag{1.8}$$

The existence of nonzero components of $[R^l_{ijk}]$ indicates that the geometry is noneuclidean. In section 1.1, however, we assumed euclidean geometry in the lab frame. The spatial geometry is thus dependent on reference frame, according to Arzeliès.

1.5 Spatial Geodesics

From section 1.4, we know that the spatial geometry of the disk is not euclidean. Using the geodesic equation,

$$\frac{d^2u^i}{ds^2} + \Gamma^i_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 0 \tag{1.9}$$

finding the geodesic equations is fairly straightforward. From equations 1.6 and 1.9 we obtain three equations.

$$\frac{d^2z}{ds^2} = 0 \tag{1.10}$$

$$\frac{d^2r}{ds^2} = \frac{r}{\left(1 - \frac{r^2\omega^2}{c^2}\right)^2}\left(\frac{d\theta}{ds}\right) \tag{1.11}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r\left(1 - \frac{r^2\omega^2}{c^2}\right)}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) = 0 \tag{1.12}$$

From equation 1.1, we obtain a constraint equation:

$$\left(\frac{dr}{ds}\right)^2 + \frac{r^2}{1 - \frac{r^2\omega^2}{c^2}}\left(\frac{d\theta}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1 \tag{1.13}$$

Observe that

$$\frac{d}{ds}\left(\frac{r^2}{1 - \frac{r^2\omega^2}{c^2}}\frac{d\theta}{ds}\right) = \left(\frac{r^2}{1 - \frac{r^2\omega^2}{c^2}}\right)\left(\frac{d^2\theta}{ds^2} + \frac{2}{r\left(1 - \frac{r^2\omega^2}{c^2}\right)}\frac{dr}{ds}\frac{d\theta}{ds}\right) = 0 \tag{1.14}$$

From this we determine that

$$\frac{r^2}{1 - \frac{r^2\omega^2}{c^2}}\frac{d\theta}{ds} = A \tag{1.15}$$

for some constant A. Similarly, from equation 1.10 we obtain $\frac{dz}{ds} = B$ for some constant B.

Now we substitute into equation 1.13 for $\frac{d\theta}{ds}$ and $\frac{dz}{ds}$, giving

$$\left(\frac{dr}{ds}\right)^2 = 1 - \frac{A^2}{r^2} + \frac{A^2\omega^2}{c^2} - B^2 \tag{1.16}$$

Since $\frac{dr}{d\theta} = \frac{dr}{ds}\frac{ds}{d\theta}$, we can write

$$\frac{dr}{d\theta} = \pm \frac{r^2}{A\left(1 - \frac{r^2\omega^2}{c^2}\right)} \sqrt{K - \frac{A^2}{r^2}} \tag{1.17}$$

with the constants condensed into $K \equiv 1 + \frac{A^2\omega^2}{c^2} - B^2$. This is the equation for geodesics on the disk. Note that it does not include geodesics that pass through the center; it blows up when $r = 0$. For those geodesics that do pass through the center, we make use of equation 1.15. At some point, $r = 0$, so A must also be 0. However, r is not 0 everywhere along the geodesic. Therefore, $\frac{d\theta}{ds} = 0$ in these cases. Equation 1.11 becomes $\frac{d^2r}{ds^2} = 0$, and $\frac{dr}{ds}$ is thus constant. The radii are geodesics, as are the lines formed by "tilting" radial lines away from being parallel with the $r\theta$-axis.

For geodesics that do not pass through the center of the disk, some further examination of equation 1.17 is in order. First we rearrange it into

$$\frac{A}{r^2\sqrt{K - \frac{A^2}{r^2}}} \frac{dr}{d\theta} = \pm \frac{1}{1 - \frac{r^2\omega^2}{c^2}} \tag{1.18}$$

Then we observe from equation 1.17 that

$$\frac{dr}{d\theta} = \pm \left(\frac{\frac{r^2\omega^2}{c^2}}{1 - \frac{r^2\omega^2}{c^2}}\right)\left(\frac{\sqrt{K - \frac{A^2}{r^2}}}{\frac{A\omega^2}{c^2}}\right) = \pm \left(\frac{1}{1 - \frac{r^2\omega^2}{c^2}} - 1\right)\left(\frac{\sqrt{K - \frac{A^2}{r^2}}}{\frac{A\omega^2}{c^2}}\right) \tag{1.19}$$

and therefore that

$$\pm \frac{1}{1 - \frac{r^2\omega^2}{c^2}} = \pm 1 + \frac{A\omega^2}{c^2\sqrt{K - \frac{A^2}{r^2}}} \frac{dr}{d\theta} \tag{1.20}$$

We now rewrite equation 1.18 as

$$\frac{A}{r^2\sqrt{K - \frac{A^2}{r^2}}} \frac{dr}{d\theta} = \pm 1 + \frac{A\omega^2}{c^2\sqrt{K - \frac{A^2}{r^2}}} \frac{dr}{d\theta} \tag{1.21}$$

Multiply through by $d\theta$ and integrate:

$$A\int \frac{dr}{r\sqrt{Kr^2 - A^2}} = \pm \int d\theta + \frac{A\omega^2}{c^2}\int \frac{dr}{\sqrt{K - \frac{A^2}{r^2}}} \tag{1.22}$$

This gives us

$$\cos^{-1}\left(\frac{A}{r\sqrt{K}}\right) = \pm\theta + \frac{A\omega^2}{Kc^2}\sqrt{Kr^2 - A^2} \tag{1.23}$$

or, as a function for θ with respect to r:

$$\theta = \pm\cos^{-1}\left(\frac{A}{r\sqrt{K}}\right) \mp \frac{A\omega^2}{Kc^2}\sqrt{Kr^2 - A^2} \tag{1.24}$$

Equation 1.24 tells us that $\left|\frac{A}{r\sqrt{K}}\right| \leq 1$, which sets a lower bound on r ($r \geq \frac{A}{\sqrt{K}}$) at which $\frac{dr}{d\theta} = 0$. It already has an upper bound ($r \leq \frac{c}{\omega}$), at which point equation 1.17 shows that $\frac{dr}{d\theta} \to \infty$. At the limiting edge of the disk the geodesics are tangential to the radii, and they curve away from the center point, eventually reaching their closest approach and then moving on (see Figure 1.2). Curves that rise steadily in the z-direction while tracing such a

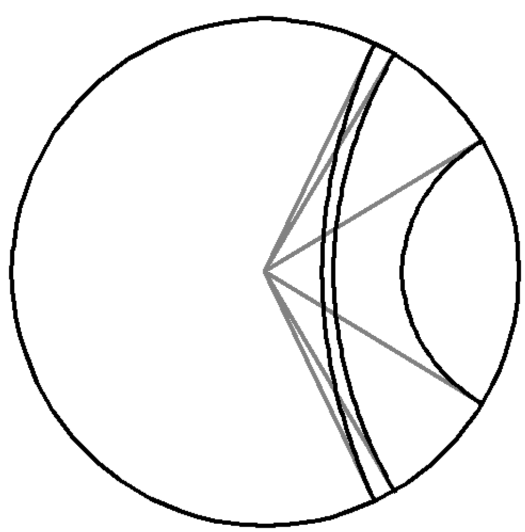


Figure 1.2: Some representative geodesics in the $r\theta$ plane

path in r and θ are also geodesics.

The curves of constant radius, interestingly, are *not* geodesics. At points where $\frac{dr}{d\theta} = 0$, $\frac{d^2r}{ds^2} = \frac{\sqrt{K}}{1-\frac{A^2\omega^2}{Kc^2}}$, and $B^2 \leq 1$ (by equation 1.13), so $K \neq 0$. The radius is at a minimum, but will not remain constant. Curves with constant radius and constant θ are the only exception; they are geodesics and correspond to curves solely in the z-direction.

1.6 Light Paths

The path taken by a light ray in the lab frame is a straight line. Described in cylindrical coordinates, the general equation is[3]

$$r\cos\theta = a \qquad z = b\sin\theta + z_0 \tag{1.25}$$

To examine this in the rotating frame, we make the coordinate substitutions $r \to r$, $\theta \to \theta \pm \omega t$,[4] and $z \to z$. The new equation is

$$r\cos(\theta \pm \omega t) = a \qquad z = b\sin(\theta \pm \omega t) + z_0 \tag{1.26}$$

For the case of a light ray, we can eliminate t by means of the equation $ct = \sqrt{r^2 - a^2 - z^2}$, giving

$$\theta = \pm\cos^{-1}\frac{a}{r} \pm \frac{\omega}{c}\sqrt{r^2 - a^2 - z^2} = \pm\sin^{-1}\frac{z - z_0}{b} \pm \frac{\omega}{c}\sqrt{r^2 - a^2 - z^2} \tag{1.27}$$

Note that unlike in equation 1.24, all four combinations of sign are permitted. Figure 1.3 shows a pair of typical light paths in the $r\theta$ plane, with arrows indicating the direction the

[3]Assuming that a is the closest approach of the line to $r = 0$ and choosing axes such that $\theta = 0$ at that point. This is technically not a general equation for a given set of axes, but it describes a typical line, so we will use it to illustrate our point.

[4]t is the central time. See section 2.1.

light is travelling. The disk is spinning counterclockwise.

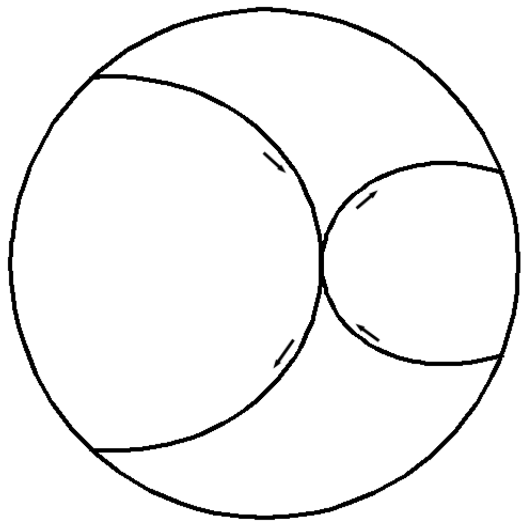


Figure 1.3: Typical light paths in the $r\theta$ plane

Comparing figure 1.2 with figure 1.3, we see that even if $a = \frac{A}{\sqrt{K}}$, light does not generally follow the spatial geodesics in the disk frame. Equations 1.24 and 1.27 confirm that this is the case. In general relativity, light always follows the geodesics, so this is a definite problem for the theory.

1.7 Summary

We have examined the spatial geometry of the rotating disk, and found a number of worrisome properties. It appears to be noneuclidean, implying a nonzero Riemann curvature tensor. This contradicts Einstein's interpretation of general relativity, which states that if the curvature tensor is zero in one frame (in this case, the euclidean lab frame), then it should be zero in all other frames. Furthermore, we've found that light does not generally follow the geodesics of the disk as general relativity says it should.

There is one thing we have not included in our analysis so far, though, and that is time.

General relativity deals with four-dimensional spacetime, and depending on how we define time in our coordinate system, the problems in the spatial analysis of the disk may go away. The nonzero components of the curvature tensor may become zero with the addition of the time dimension, and the light paths may actually trace out four-dimensional geodesics even if their three-dimensional projections are not spatial geodesics. We will add time to the mix in the next chapter.

Chapter 2

Time on the Disk

2.1 Central Time

According to the theory of special relativity, moving clocks run slow. From the inertial reference frame, clocks sitting on the rotating disk are moving with a velocity proportional to their distance from the axis of rotation. Thus, they are seen to slow down by a factor of $(1 - v^2/c^2)^{-1/2}$. An observer on the disk will see his own clock ticking normally, but after one full rotation of the disk his clock will have advanced more than identical clocks farther from the axis of rotation, and less than clocks closer to the axis. This is a serious problem for anyone who wants to time events that take place over a nonlocalized area of the disk; for example, timing a particle's journey from a point on the disk to the axis and back. Observers at the two endpoints can synchronize their clocks at the start of the experiment and will still measure different ending times. We would like a consistent way to define time on the disk so that we can use it concretely.

In fact, there are several such ways, each with its own advantages and disadvantages. Perhaps the simplest is what Arzeliès [4] calls "central time". It is, as the name implies, what one would read on the clock at the axis of rotation. That clock can be set to send out a light signal every time it ticks, and every other clock on the disk can be synchronized to it. Time on the disk is thus measured precisely as time in the inertial frame. The advantages to this definition are that (a) clocks do not get "ahead" of each other (if one observer compares his time with another and finds that their clocks read times offset a certain amount from each other, then at any later time if he tries the same experiment he will get the same offset) and (b) it is easy to calculate, since we can just transfer the problem in question to the inertial frame and find the time there. The principal disadvantages to the definition are (a) the lack of synchronization between clocks (we will return to this) and (b) the fact that clocks at different radii must be constructed differently, such that if we move a clock from one radius to another, it will tick at a different rate from the one native to that radius.[1] This lack of portability makes it difficult to define the *proper time* of a particle travelling along a path in terms of the central time.

The lack of synchronization in central time is easy to prove. Suppose we have two clocks near each other on the disk, one at some point $A = (r, \theta)$ and the other at $B = (r, \theta + \Delta\theta)$ (assume $\Delta\theta$ is small). We will synchronize B to A using the methods of special relativity, by sending a light pulse from A to B at time $t = 0$ and measuring the distance between them with a meterstick. Since light moves at a constant velocity c, we can use the distance to figure out how much time has elapsed on the clock at A during the journey, and set the clock at B to that value. The distance between the two points on the disk is $\frac{r\Delta\theta}{\sqrt{1-r^2\omega^2/c^2}}$ (from equation 1.1), so A should display a time

$$\Delta t_A = \frac{r\Delta\theta}{\sqrt{c^2 - r^2\omega^2}} \tag{2.1}$$

when the signal reaches B.

[1] It is possible to remedy this by some mechanism within the clocks that senses the strength of the rotation and adjusts the clock's rate accordingly, but this makes the clocks into complex machines, not the delightfully simple light-clocks imagined by Einstein.

Now, from the lab frame we see the light emitted at A moving toward B, but B moving away from A with speed $r\omega$ as the disk rotates. The time it takes for the journey (as measured by B, as well as the rest of the clocks) is

$$\Delta t_B = \frac{r\Delta\theta}{c - r\omega} \tag{2.2}$$

B will read Δt_B when the signal reaches it, but to synchronize with A it must be adjusted to Δt_A. The offset is

$$\Delta t_A - \Delta t_B = \frac{r\Delta\theta}{c^2 - r^2\omega^2}\left(\sqrt{c^2 - r^2\omega^2} - c - r\omega\right) \tag{2.3}$$

Thus, while both A and B are synchronized to the clock at the center, they are not synchronized to each other. If we relax one of our assumptions—that the speed of light is constant in the disk frame when we use central time—we can allow clocks on the disk reading central time to be synchronized to each other as well as to the central clock, but we would no longer be doing relativity if we permit that.

2.2 Local Time

Perhaps we have decided that we really want portable clocks instead of these clumsy central time clocks that behave differently at different radii. The thing to do then is just to place ordinary clocks from our inertial frame on the disk and let them rotate. Sure, they'll see each other ticking at different rates, but we know what those rates are, so we can adjust our calculations accordingly. Plus, we can synchronize them to their neighbors, which we couldn't do with central time. We will again follow Arzeliès and call this "local time".

The problem with local time becomes apparent when one begins synchronizing neighboring clocks around a circumference. As with central time, we will synchronize a clock at the same radius r and some small angle $d\theta$ away. The time displayed on this new clock at the moment of synchronization is

$$\bar{dt} = dt_{lab}\sqrt{1-\frac{r^2\omega^2}{c^2}} = \frac{rd\theta}{c-r\omega}\sqrt{1-\frac{r^2\omega^2}{c^2}} \tag{2.4}$$

Now we'll take another clock another $d\theta$ along the circumference and synchronize it to this one, using the same procedure. As we continue synchronizing clocks around the circumference of the disk, the synchronization times $\bar{dt}$ add up.

$$\bar{t} = \int_0^\theta \frac{rd\theta'}{c-r\omega}\sqrt{1-\frac{r^2\omega^2}{c^2}} = \frac{r\theta}{c-r\omega}\sqrt{1-\frac{r^2\omega^2}{c^2}} \tag{2.5}$$

To cover the whole circumference $2\pi r/\sqrt{1-r^2\omega^2/c^2}$, we need $\theta = 2\pi$, so the synchronization time is

$$\bar{t}_1 = \frac{2\pi r}{c-r\omega}\sqrt{1-\frac{r^2\omega^2}{c^2}} \tag{2.6}$$

Now, let's try the same procedure, but send our light signal in the opposite direction around the circumference. In this case, equation 2.4 becomes

$$\bar{dt} = dt_{lab}\sqrt{1-\frac{r^2\omega^2}{c^2}} = \frac{rd\theta}{c+r\omega}\sqrt{1-\frac{r^2\omega^2}{c^2}} \tag{2.7}$$

and the synchronization time becomes

$$\bar{t}_2 = \frac{2\pi r}{c+r\omega}\sqrt{1-\frac{r^2\omega^2}{c^2}} \tag{2.8}$$

Comparing with equation 2.6, we can see that the original clock is not synchronized with itself![2] The difference between the clockwise and counterclockwise synchronizations is

$$\Delta\bar{t} = \bar{t}_1 - \bar{t}_2 = 2\pi r\sqrt{1 - \frac{r^2\omega^2}{c^2}}\left(\frac{1}{c - r\omega} - \frac{1}{c + r\omega}\right) = \frac{4\pi r^2\omega}{c\sqrt{c^2 - r^2\omega^2}} \tag{2.9}$$

This is clearly a problem for local time, since if a clock isn't synchronized with itself then it's fairly useless[3].

Moreover, suppose we synchronize a clock at some radius R_1 with one at a different radius R_2. These clocks tick at different rates, so if we wait some finite amount of time (measured by either clock) and then check the synchronization, they will no longer be synchronized; the longer we wait, the larger the discrepancy. In local time, synchronization between two points is in general only an instantaneous thing.

2.3 Natural Time

We can patch up local time somewhat by employing an arbitrary convention. Since the synchronization problem doesn't appear unless we fully circumnavigate the disk, we can draw a ray from the axis of rotation outwards, and say that there is a time difference between clocks on opposite sides of the ray that just cancels out the circumnavigation effect. Arzeliès calls this the "natural time", and while I think it is a misnomer I will stick to his nomenclature. The natural time is defined by

$$d\tau = \sqrt{1 - \frac{r^2\omega^2}{c^2}}\left(dt_{lab} - \frac{r^2\omega d\theta}{c^2 - r^2\omega^2}\right) \tag{2.10}$$

Apparently, if $d\theta = 0$ this simplifies to local time, which means that clocks sitting still in the disk frame behave just as special relativity says they should. We would also like to know the

[2] This is called the *Sagnac effect* and is discussed in references [16], [17], and [18].

[3] Note that we can still use local time provided that we limit the area over which it is valid so that a full circuit of the disk cannot be made. The problem with radial discrepancies won't go away so easily, though.

indications of the various natural clocks at a single time as measured in the lab frame; to do this we select an arbitrary point in the rotating frame that indicates τ_0 at time $t_{lab} = 0$ and look at paths from it to other points. In particular, we will examine the case in which the path is at a constant radius. To find the indication of any other clock on the path, we simply integrate equation 2.10 over the angle θ between the two clocks.

$$\tau(\theta) = \tau_0 \pm \frac{r^2\omega}{c\sqrt{c^2 - r^2\omega^2}} \int_0^\theta d\theta' = \tau_0 \pm \frac{r^2\omega\theta}{c\sqrt{c^2 - r^2\omega^2}} \tag{2.11}$$

A problem clearly arises from this: what happens when you go all the way around the disk and come back to the original clock? According to equation 2.11, the clock reads $\tau_0 \pm \frac{2\pi r^2\omega}{c\sqrt{c^2-r^2\omega^2}}$, but we originally assumed that it reads τ_0. It cannot read both, so we introduce the convention mentioned at the beginning of this section. Whenever we pass through the "cut" formed by the ray at our arbitrarily-chosen θ, we adjust the time by the period $T = \frac{2\pi r^2\omega}{c\sqrt{c^2-r^2\omega^2}}$. When passing clockwise we subtract T, and when passing counterclockwise we add T. This also solves the problem of synchronizing a clock with itself that we saw with local time. Equation 2.9 becomes

$$\Delta\tau = \tau_1 - \tau_2 = \frac{4\pi r^2\omega}{c\sqrt{c^2 - r^2\omega^2}} - 2T = 0 \tag{2.12}$$

The cut convention of natural time adds just the right value to allow clocks to be synchronized with themselves. The final problem of local time also goes away; clocks at a larger radius tick more slowly, but are also adjusted by a larger T, so a clock at one radius will always read something close to one at a different radius even though they are ticking at different rates. This preserves time translation invariance, at least on a long time scale. The price of all of these benefits is breaking the disk's symmetry. There are clocks right next to each other that are not synchronized, and if we want to use natural time we just have to accept that.

2.4 Spacetime Metrics

Now that we have definitions of time, we can write down complete metric tensors for spacetime on the disk instead of just the spatial metric in equation 1.2. We do this by means of the invariant spacetime interval, which in the lab frame is[4]

$$dr_0^2 + r_0^2 d\theta_0^2 + dz_0^2 - c^2 dt_0^2 = \text{constant} \tag{2.13}$$

We then use the relations between the lab and disk coordinates ($dr_0 = dr$, $d\theta_0 = d\theta + \omega dt_0$, $dz_0 = dz$) to write this in terms of the latter.

$$dr^2 + r^2 d\theta^2 + 2r^2\omega d\theta dt_0 + r^2\omega^2 dt_0^2 + dz^2 - c^2 dt_0^2 = \text{constant} \tag{2.14}$$

Central time is defined by $dt = dt_0$, so plugging this into equation 2.14 gives us the following metric tensor:

$$[g_{\alpha\beta}] = \begin{pmatrix} -\sqrt{1-\frac{r^2\omega^2}{c^2}} & 0 & r^2\omega & 0 \\ 0 & 1 & 0 & 0 \\ r^2\omega & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.15}$$

Note that the spatial portion of this metric looks nothing like equation 1.2, and moreover that there are a pair of cross terms in g_{20} and g_{02}. The metric produced by central time does not allow for complete separation between space and time.

[4]In this section, subscripted zeros indicate lab frame coordinates, while a lack of subscripts indicates disk frame coordinates.

Equation 2.4 defines local time, so equation 2.14 becomes

$$dr^2 + r^2 d\theta^2 + \frac{2r^2\omega d\theta d\bar{t}}{\sqrt{1 - \frac{r^2\omega^2}{c^2}}} + dz^2 - c^2 d\bar{t}^2 = \text{constant} \tag{2.16}$$

The metric tensor is then

$$[g_{\alpha\beta}] = \begin{pmatrix} -1 & 0 & \frac{r^2\omega}{\sqrt{1-\frac{r^2\omega^2}{c^2}}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{r^2\omega}{\sqrt{1-\frac{r^2\omega^2}{c^2}}} & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.17}$$

Once again, the spatial portion of this metric does not look like equation 1.2, and two cross terms appear. Local time does not produce an easily separable metric either.

Natural time is defined by equation 2.10. When this is plugged into equation 2.14 it makes a bit of a mess, but after simplification it becomes

$$dr^2 + \frac{r^2 d\theta^2}{1 - \frac{r^2\omega^2}{c^2}} + dz^2 - c^2 d\tau^2 = \text{constant} \tag{2.18}$$

The metric tensor is thus

$$[g_{\alpha\beta}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{r^2}{1-\frac{r^2\omega^2}{c^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.19}$$

Unlike the metrics for central and local time, this metric has no cross terms and separates neatly into spatial and temporal pieces. The spatial piece matches perfectly equation 1.2. Natural time provides the metric we expect for the rotating disk coordinates.

The spacetime curvature of these three metrics may also be found, though it is a long procedure by hand. The computer program Maxima[5] handles tensors nicely, and provides a list of nonzero components of the curvature tensor for a given metric tensor.[6] For the central time metric (equation 2.15) it obtains nonzero values for two components, for the local time metric (equation 2.17) it obtains nonzero values for four components, and for the natural time metric (equation 2.19) it finds one nonzero component.

2.5 Summary

We examined three promising definitions of time, all of which have flaws. Central time does not allow us to synchronize clocks with each other and requires them to function differently at different radii. Clocks measuring local time cannot be synchronized with themselves due to the Sagnac effect, and they don't have time translation invariance. Natural time requires us to make an arbitrary cut along the disk in order to define it properly, breaking rotational symmetry. Moreover, none of the three results in a Riemann curvature tensor with all components zero, so we have not solved the problems mentioned in section 1.7.

It may not be possible to define a general time coordinate on the rotating disk. Einstein certainly takes this view, stating "... it is not possible to obtain a reasonable definition of time with the aid of clocks which are arranged at rest with respect to the [rotating disk]." [3] Without a well-defined time coordinate, a spatial metric is not very useful. It is, however, possible to define time at one point on the disk and to examine the disk as an observer from that single point. It is this that we attempt next.

[5]Maxima is an open-source implementation of the computer algebra program Macsyma. It is available under the GNU Public License at `http://maxima.sourceforge.net`.

[6]Transcripts of my Maxima sessions may be found in appendix B.

Chapter 3

The Numeric Method

3.1 Introduction

One of the problems with analyzing the rotating disk geometrically is that the steps are somewhat ambiguous. When we talk about "length elements" and "natural time", it is not instantly clear what is being described. It would be useful to analyze the rotating disk using a method that is clear and unambiguous at every step, and special relativity gives us a limited way of doing that. Instead of examining the rotating frame as a whole, we will consider only a single inertial frame at one moment of time. The equations involved are transcendental, unfortunately, so no analytic result is possible, but they may be solved numerically, and thus I will refer to this technique as the "numeric method". It was originally explored by Kannan Jagannathan and David Griffiths in the spring of 2002 [19], and I have used their work as a starting point for my own.

3.2 A Ring of Points

We begin by dispensing with the physical disk. Instead, we will consider a collection of n points, distributed evenly around a circle of radius R_0 centered at the origin (figure 3.1). They each bear a unique label: 0 through $n-1$. Now we will suppose that each of these points is moving at a constant angular velocity ω around the origin. (We must stipulate that $R_0\omega < c$.) Each point j is given by two coordinates in the lab frame, x_j and y_j, and they are given by the equations

$$x_j = R_0 \sin(\omega t + \frac{2\pi j}{n}) \tag{3.1}$$

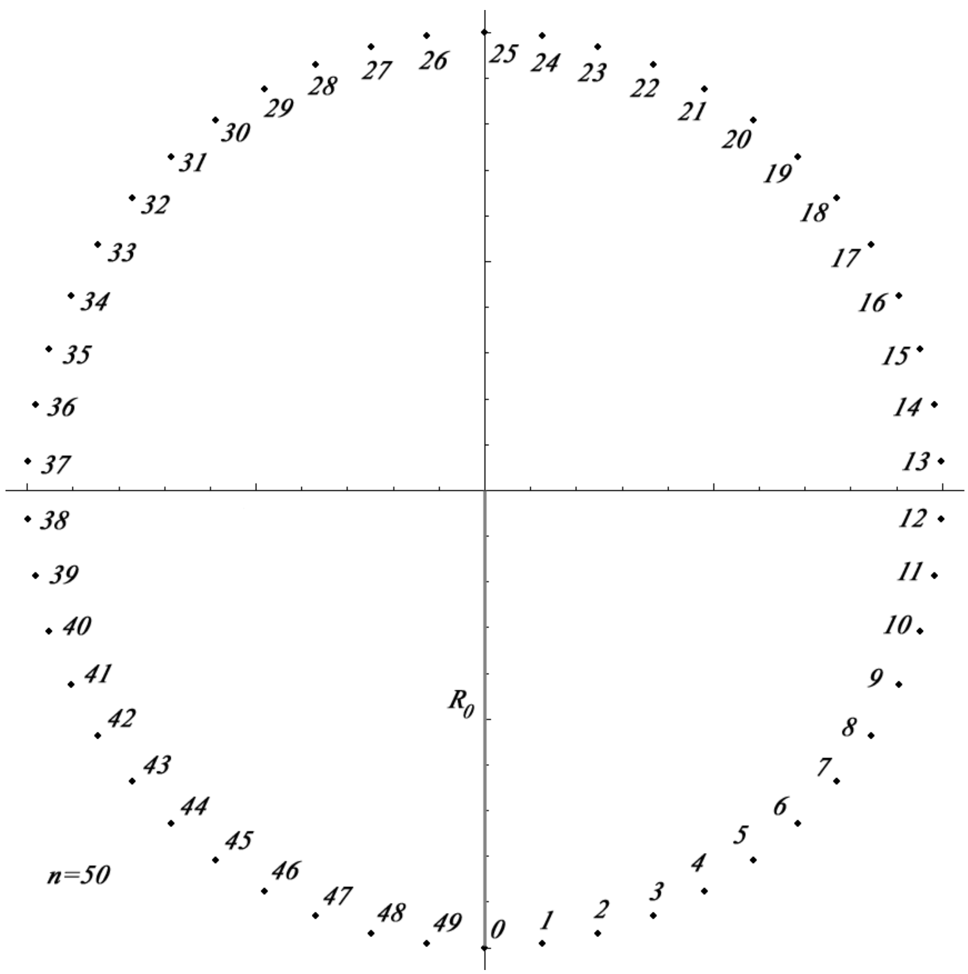


Figure 3.1: A ring of labelled points in the lab frame

$$y_j = -R_0 \cos(\omega t + \frac{2\pi j}{n}) \tag{3.2}$$

where t is the time in the lab frame. At time $t = 0$, the $j = 0$ point is at $(0, -R_0)$, and is moving in the x direction with velocity $v = R_0\omega$.

We will now consider the reference frame in which the zeroth point is instantaneously at rest. The Lorentz transformations are given by

$$\begin{array}{lcl} x' & = & \gamma\,(x - vt) \\ y' & = & y \\ z' & = & z \\ t' & = & \gamma\left(t - \frac{vx}{c^2}\right) \end{array} \qquad \begin{array}{lcl} x & = & \gamma\,(x' + vt') \\ y & = & y' \\ z & = & z' \\ t & = & \gamma\left(t' + \frac{vx'}{c^2}\right) \end{array} \tag{3.3}$$

γ is $\frac{1}{\sqrt{1-\frac{\omega^2 R_0^2}{c^2}}}$, and we may use the variable $\beta = v/c$. At the zeroth point, $t = 0$ implies that $t' = 0$ as well, so we solve the equations for $t' = 0$. This leads us to the following equations for x' and y':

$$x'_j = \frac{R_0}{\gamma} \sin\left(\frac{\gamma}{R_0}\beta^2 x'_j + \frac{2\pi j}{n}\right) \tag{3.4}$$

$$y'_j = -R_0 \cos\left(\frac{\gamma}{R_0}\beta^2 x'_j + \frac{2\pi j}{n}\right) \tag{3.5}$$

These equations are transcendental and cannot be solved analytically, but they may be solved numerically. Mathematica produces graphs that look like those in figure 3.2.[1]

The "clustering" of points on the far side of the stationary point is due to the difference in simultaneity between the lab frame and the instantaneous rest frame. Equation 3.3 gives $t = \gamma\left(t' + \frac{vx'}{c^2}\right)$, so when $t' = 0$ we are left with $t = \gamma\left(\frac{vx'}{c^2}\right)$. When $x' > 0$, $t > 0$, and when $x' < 0$, $t < 0$. This means that although all points are in their locations simultaneously in the instantaneous rest frame, in the lab frame those with positive x values reach those locations after time $t = 0$, while those with negative x values reach their locations before $t = 0$. In the

[1]The Mathematica code used to generate this image may be found in appendix A.1.

instantaneous rest frame this means that the points with positive x have advanced farther along their rotation, and thus are farther away from the stationary point, while the points with negative x have not advanced as far along their rotation, and thus are also farther away from the stationary point. This effect, combined with the contraction along the direction of motion that we expect from a Lorentz transformation, gives the point distribution seen in figure 3.2.

3.3 The Whole Disk

The previous section describes a procedure for plotting the locations of a ring of points as viewed from the zeroth point. However, we want to know what the entire disk looks like from that point. To do this, we must modify our original equations slightly to include variations

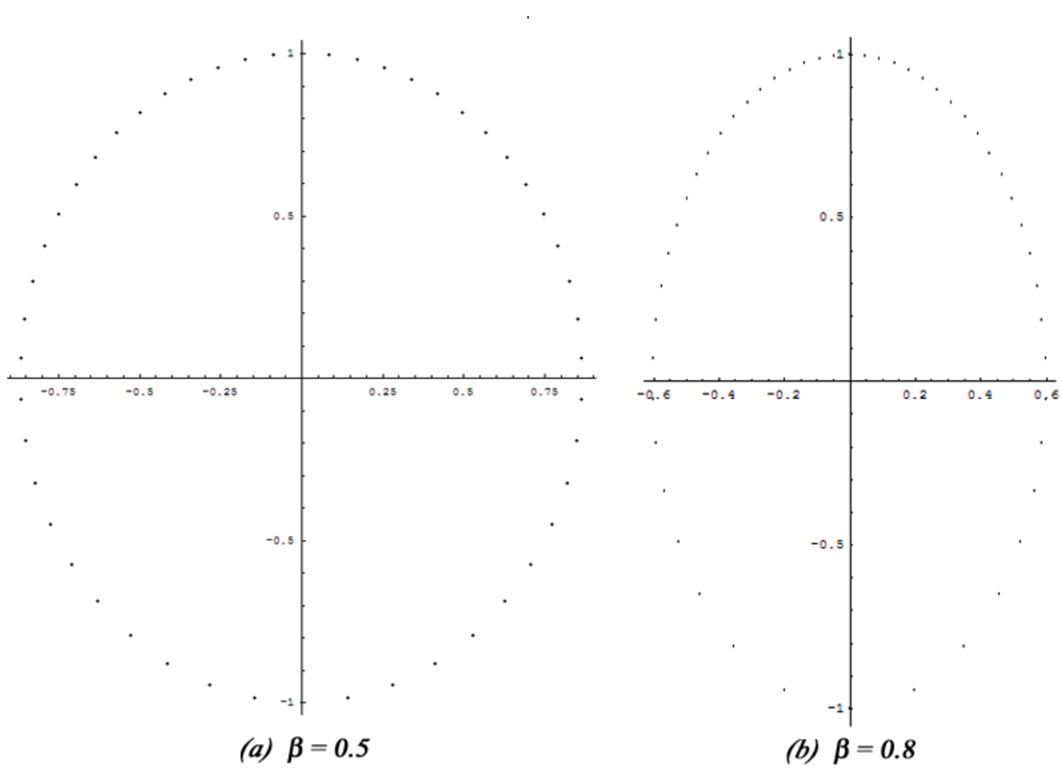


Figure 3.2: The ring in the comoving frame. $R_0 = 1$

in the radius. Equations 3.1 and 3.2 then become

$$x_j(R) = R\sin(\omega t + \frac{2\pi j}{n}) \tag{3.6}$$

$$y_j(R) = -R\cos(\omega t + \frac{2\pi j}{n}) \tag{3.7}$$

where R is the distance from the origin of the point in question. γ, v, and β still depend on R_0, not R. Once again using the Lorentz transformations, we obtain the equations

$$x'_j(R) = \frac{R}{\gamma}\sin\left(\frac{\gamma}{R_0}\beta^2 x'_j(R) + \frac{2\pi j}{n}\right) \tag{3.8}$$

$$y'_j(R) = -R\cos\left(\frac{\gamma}{R_0}\beta^2 x'_j(R) + \frac{2\pi j}{n}\right) \tag{3.9}$$

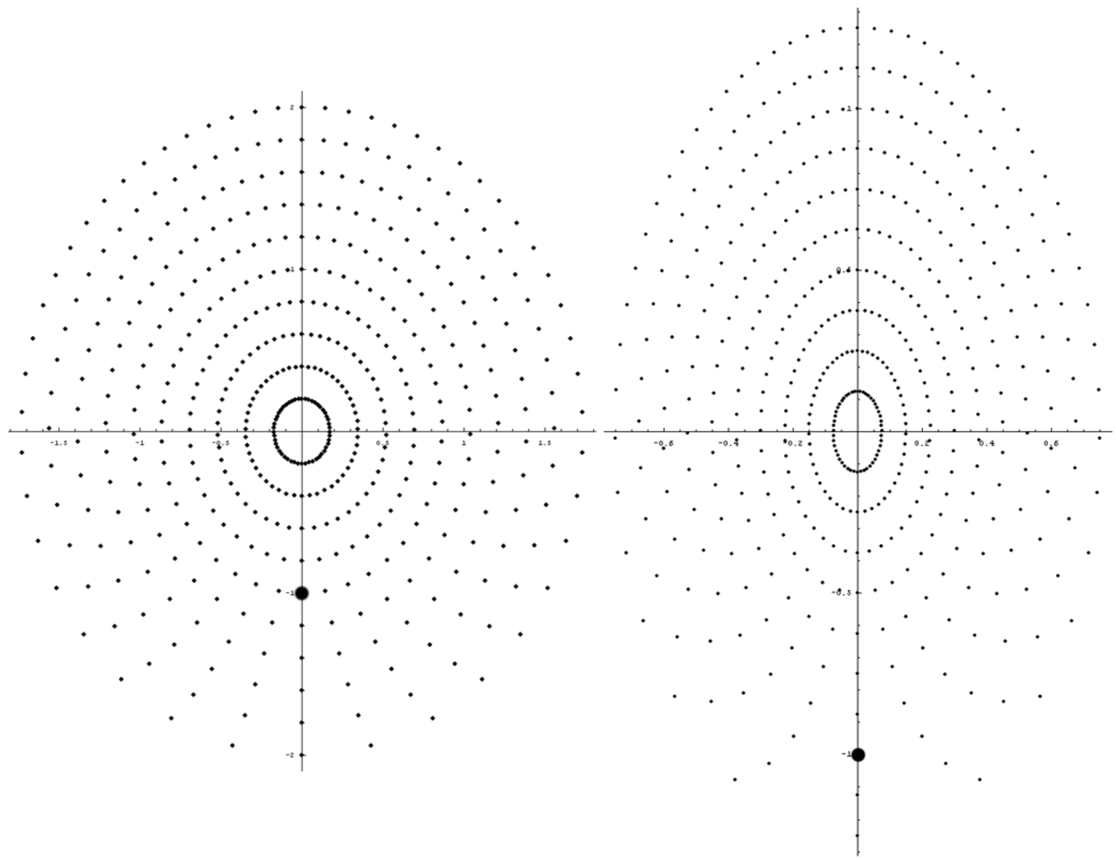
Again, Mathematica allows us to plot the disk to arbitrary accuracy, as shown in figure 3.3.[2]

The same influences that were seen on the ring of points are seen here. We can also see that it is only the speed of the stationary point and not the speed of the observed point that affects the latter's placement. However, the distance of a given point from the origin still alters the effect upon its coordinates, as we can see from the fact that radial lines are now curved. Griffiths also showed [19] that the surface area of the disk is given by $A = \frac{\pi R^2}{\gamma}$ for a disk of radius R, but the calculation will not be shown here.

3.4 Time Evolution

The numeric method is all well and good, but all it has done so far is provide a snapshot of the rotating disk. The snapshot is not even necessarily from the point of view of an observer on the disk; it is from the point of view of an observer moving past the disk who happens

[2]The Mathematica code generating this image is in appendix A.2.



(a) $\beta = 0.5$ *(b)* $\beta = 0.8$

Figure 3.3: The disk from the comoving frame. The instantaneous rest point is marked.

to see one point of the disk standing still at a given moment. Thus, what we've actually examined is not the rotating frame itself, but rather what (for example) a stationary observer would see watching a wheel rolling past him.

We would like to know whether we can describe the rotating frame in terms of a sequence of instantaneous inertial frames. One thing we do know about the rotating frame is that *the disk should look the same in it at any point during the rotation.*[3] We can evolve the disk in time in the "snapshot" inertial frame, and then perform another Lorentz transformation into

[3] If this were not the case, then the rotating frame would not be invariant under time translation, and we would lose the ability to choose an arbitrary zero for time on the disk.

the frame where the zeroth point is once again instantaneously at rest. Taking a snapshot of this will give us two pictures of the disk at different times, both of which from reference frames in which the zeroth point is at rest. If they are not the same, then there is a problem with the numeric method's ability to describe the disk at any time other than $t = 0$, and it is only useful for taking pictures from outside the frame. Deriving a frame for a rotating observer with the numeric method would be impossible.

To get the new snapshot of the disk, we must first rederive the equations for x' and y' with arbitrary t'. This is not difficult, and leaves us with two equations quite similar to the originals (equations 3.8 and 3.9):

$$x'_j(R,t') = \frac{R}{\gamma}\sin\left(\frac{\gamma}{R_0}\beta^2 x'_j(R,t') + \frac{2\pi j}{n} + \omega\gamma t'\right) - vt' \tag{3.10}$$

$$y'_j(R,t') = -R\cos\left(\frac{\gamma}{R_0}\beta^2 x'_j(R,t') + \frac{2\pi j}{n} + \omega\gamma t'\right) \tag{3.11}$$

These are equivalent to equations 3.4 and 3.5 except for the addition of two terms. Both equations 3.10 and 3.11 have an additional $\omega\gamma t'$ in the angle expression, which corresponds to the angle the disk has turned through in the time elapsed since $t' = 0$. In addition, equation 3.10 has an extra distance $-vt'$ tacked on, which comes from the linear motion of the disk in this frame.

Now we need to know the velocity of this new reference frame with respect to the old one. To obtain it we use the velocity of the zeroth point in the first comoving frame, which are given by the derivatives of equations 3.10 and 3.11. Implicit derivation is necessary here. For the case of 3.10, we rewrite the equation as

$$x'_j(R,t') = \frac{R}{\gamma}\sin\theta'_j(R,t') - vt' \tag{3.12}$$

where $\theta'_j(R,t')$ is simply a placeholder for the longer expression shown below:

$$\theta'_j(R,t') = \frac{\gamma}{R_0}\beta^2 x'_j(R,t') + \frac{2\pi j}{n} + \omega\gamma t' \tag{3.13}$$

It will be useful to know the time derivative (with respect to t', of course; dot notation will imply this) of $\theta'_j(R,t')$ as well.

$$\dot{\theta}'_j(R,t') = \frac{\gamma}{R_0}\beta^2 \dot{x}'_j(R,t') + \omega\gamma \tag{3.14}$$

Knowing this, we can easily write $\dot{x}'_j(R,t')$ using the chain rule:

$$\dot{x}'_j(R,t') = \frac{R}{\gamma}\cos\theta'_j(R,t')\dot{\theta}'_j(R,t') - v = \frac{R}{\gamma}\cos\theta'_j(R,t')\left(\frac{\gamma}{R_0}\beta^2\dot{x}'_j(R,t') + \omega\gamma\right) - v \tag{3.15}$$

Then we solve for $\dot{x}'_j(t')$ and simplify to get

$$\dot{x}'_j(R,t') = \omega R_0\left(\frac{R\cos\theta'_j(R,t') - R_0}{R_0 - R\beta^2\cos\theta'_j(R,t')}\right) \tag{3.16}$$

Since we only need the velocity of the reference point, we set $R = R_0$ and $j = 0$ to get

$$v'_{x0}(t') = \dot{x}'_0(t') = \frac{\omega R_0\left(\cos\theta'_0(t') - 1\right)}{1 - \beta^2\cos\theta'_0(t')} \tag{3.17}$$

(where $\theta'_0(t') \equiv \frac{\gamma}{R_0}\beta^2 x'_0 + \omega\gamma t'$). A similar procedure for equation 3.11 yields

$$\dot{y}'_j(R,t') = \frac{\omega R R_0}{\gamma}\frac{\sin\theta'_j(R,t')}{R_0 - R\beta^2\cos\theta'_j(t')} \tag{3.18}$$

$$v'_{y0}(t') = \dot{y}'_0(t') = \frac{\omega R_0}{\gamma}\frac{\sin\theta'_0(t')}{1 - \beta^2\cos\theta'_0(t')} \tag{3.19}$$

The total velocity is the sum of v_{x0} and v_{y0} in quadrature:

$$v_0'(t')^2 = \omega^2 R_0^2 \frac{2 - 2\cos\theta_0'(t') - \beta^2 \sin^2\theta_0'(t')}{\left(1 - \beta^2 \cos\theta_0'(t')\right)^2} \tag{3.20}$$

The Lorentz transformations for a general boost in the xy plane are given by

$$\begin{aligned}
\bar{x} &= \bar{\gamma}\left(\tfrac{v_x(xv_x+yv_y)}{v^2} - v_x t\right) - \tfrac{v_y}{v^2}\left(yv_x - xv_y\right) \\
\bar{y} &= \bar{\gamma}\left(\tfrac{v_y(xv_x+yv_y)}{v^2} - v_y t\right) + \tfrac{v_x}{v^2}\left(yv_x - xv_y\right) \\
\bar{t} &= \bar{\gamma}\left(t - \tfrac{xv_x+yv_y}{c^2}\right)
\end{aligned} \tag{3.21}$$

In the new frame, we will need the events simultaneous in that frame; that is, $t'' = \bar{t}$ is constant, while t' may vary. What value t'' takes as a function of time elapsed before switching to the new comoving frame (that is, t') is found by solving the last equation of 3.21 at the reference point. First, though, we need to know what $\bar{\gamma}$ is. Using the definition of γ in special relativity and equation 3.20, we find that

$$\bar{\gamma} \equiv \frac{1}{\sqrt{1 - \frac{v_0'^2}{c^2}}} = \gamma^2\left(1 - \beta^2\cos\theta_0'(t')\right) \tag{3.22}$$

Now we use equations 3.17, 3.19, 3.10, and 3.11 in conjunction with equation 3.22 to find a simplified expression for t''.

$$t'' = t_0' + \frac{\gamma\beta^2}{\omega}\sin\theta_0'(t_0') \tag{3.23}$$

t_0' is the time elapsed in the old comoving frame before shifting to the new frame; t'' gives the corresponding time in the new comoving frame. While equation 3.23 still relies on $\theta_0'(t_0')$, we can adapt the methods of section 3.2 to obtain a numeric value of $\theta_0'(t_0')$ (and therefore of t'') for any given t_0'.

Likewise, we can obtain numeric values for $\bar{\gamma}$, $v_{x0}'(t_0')$, $v_{y0}'(t_0')$, and $v_0'^2(t_0')$. These may be treated as constants when working with the Lorentz transformation, and as such we will drop the function notation when using them in the future. The time in the first comoving frame is not constant from point to point, however. The general boost equations 3.21 yield

$$t' = \frac{t''}{\bar{\gamma}} + \frac{v'_{x0}\left(\frac{R}{\gamma}\sin\theta'_j(R,t') - \omega R_0 t'\right) - v'_{y0} R\cos\theta'_j(R,t')}{c^2} \tag{3.24}$$

Again, $\theta'_j(R,t')$ is a placeholder for a longer expression, $\frac{\gamma}{R_0}\beta^2 x'_j(R,t') + \frac{2\pi j}{n} + \omega\gamma t'$, so equation 3.24 must be solved numerically for every point with a unique R and j. This gives us a time to plug into the x and y equations of the transformation, which are

$$x''_j(R,t') = \bar{\gamma}\left(\frac{v'^2_{x0} x'_j(R,t') + v'_{x0} v'_{y0} y'_j(R,t')}{v'^2_0} - v_{x0} t'\right) - \frac{v'_{y0}}{v'^2_0}\left(v'_{x0} y'_j(R,t') - v'_{y0} x'_j(R,t')\right) \tag{3.25}$$

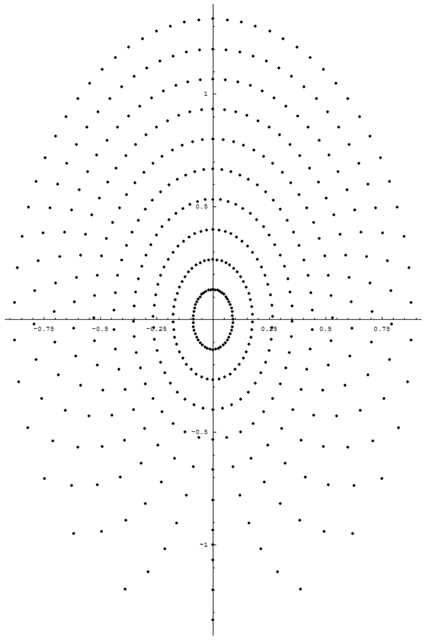
$$y''_j(R,t') = \bar{\gamma}\left(\frac{v'_{x0} v'_{y0} x'_j(R,t') + v'^2_{y0} y'_j(R,t')}{v'^2_0} - v_{y0} t'\right) + \frac{v'_{x0}}{v'^2_0}\left(v'_{x0} y'_j(R,t') - v'_{y0} x'_j(R,t')\right) \tag{3.26}$$

$x'_j(R,t')$ and $y'_j(R,t')$ are given by equations 3.10 and 3.11, which are solved numerically with the value of t' found with equation 3.24.[4] The end result is a collection of ordered pairs of points in the new comoving frame, which may be graphed to examine the disk from this frame. Figure 3.4 shows the disk at a number of times from the instantaneous comoving frame of the reference point.

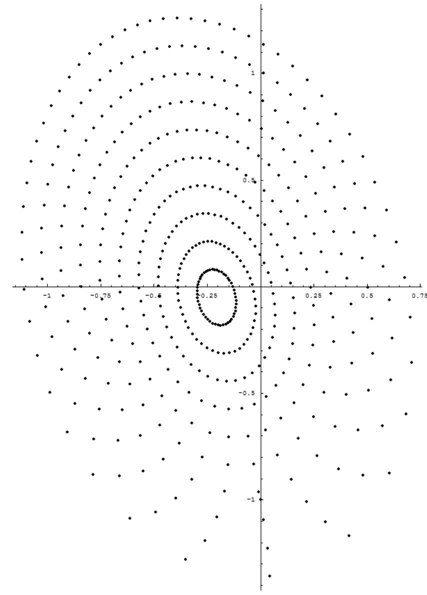
Apparently, shifting from one comoving frame to another does not change the shape of the disk.[5] It does move both the center and the orientation of the disk, but that has to do with the rotation and lateral movement of the disk before shifting into the second comoving frame and could be corrected for if one desired to do so. These results were expected, and they are a strong confirmation of the feasibility of the numeric method.

[4]The Mathematica code for this is in appendix A.3.

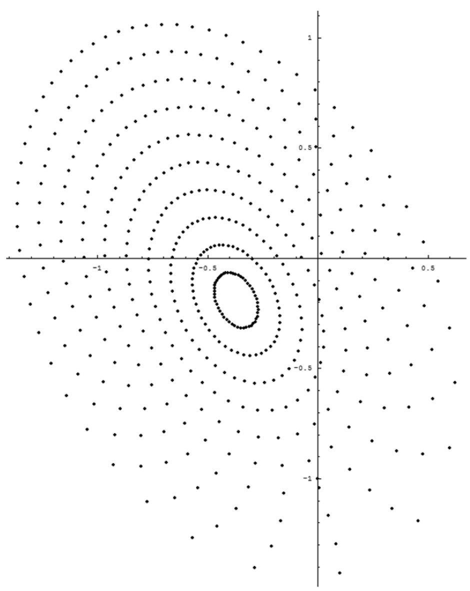
[5]There are slight imperfections in figure 3.4, but they are due to our use of a numeric algorithm rather than an analytic solution. Much higher values of t' result in the same shape of the disk, so the artifact does not increase with time.



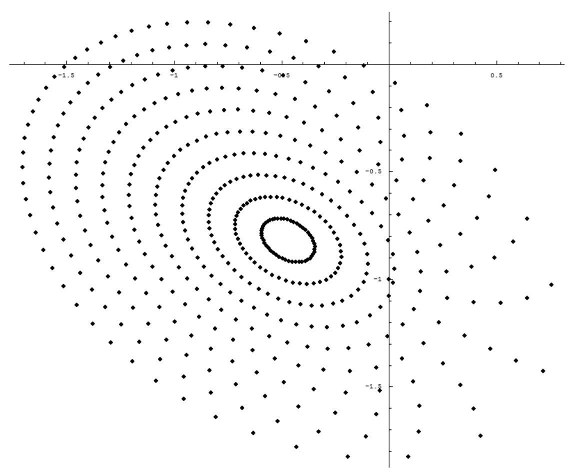
(a) t'=0



(b) t'=0.1



(c) t'=0.2



(d) t'=0.5

Figure 3.4: Time evolution of the disk as seen from a single reference point. $\beta = 0.75$

3.5 Taylor Expansion of Time Evolution

The results of section 3.4 do not precisely describe the disk as seen by an observer on it, however. They confirm that the disk appears the same in consecutive comoving frames of a given reference point, but we have not yet found a description of the disk that follows the disk observer continuously. We can approximate one with the methods of section 3.4 by allowing time to advance a small amount in one comoving frame and then transform to the new comoving frame of the reference point, then repeating *ad infinitum.* This results in a rather choppy frame, however, jumping from inertial frame to inertial frame.

One way to smooth out the curve is to take the limit as the time elapsed between Lorentz transformations goes to zero. If we integrate the infinitesimal transformations obtained over some finite time, we should get some continuous transformation that allows us to describe the disk at any moment from the point of view of a single observer on it.

To obtain the infinitesimal transformations, we must evaluate equations 3.10 and 3.11 at some small time dt'. The change in those equations between $t' = 0$ and $t' = dt'$ is very small, though, so we must examine them by way of Taylor approximations around $t' = 0$. The first derivative of equation 3.10 is given by equation 3.16, and the second derivative is also obtained by implicit differentiation:

$$\ddot{x}'_j(R, t') = \frac{RR_0^3\omega^2 \sin\theta'_j(R, t')}{\gamma^3 \left(R\beta^2 \cos\theta'_j(R, t') - R_0\right)^3} \tag{3.27}$$

The third time derivative and higher derivatives have negligible effects in the Taylor expansion near $t' = 0$, so they are not shown here.

Equation 3.18 gives the first derivative of equation 3.11. The second derivative of equation 3.11 is then

$$\ddot{y}'_j(R, t') = \frac{RR_0\omega \left(R\beta^2 - R_0 \cos\theta'_j(R, t')\right)}{\gamma^2 \left(R\beta^2 \cos\theta'_j(R, t') - R_0\right)^3} \tag{3.28}$$

We may now write out the Taylor expansions for equations 3.10 and 3.11, to second order:

$$x'_j(R,t') \approx \frac{R}{\gamma}\sin\theta'_j(R,0) + \omega R_0\left(\frac{R\cos\theta'_j(R,0) - R_0}{R_0 - R\beta^2\cos\theta'_j(R,0)}\right)t' \\ + \frac{RR_0^3\omega^2\sin\theta'_j(R,0)t'^2}{2\gamma^3\left(R\beta^2\cos\theta'_j(R,0) - R_0\right)^3} \quad (3.29)$$

$$y'_j(R,t') \approx -R\cos\theta'_j(R,0) + \frac{\omega RR_0}{\gamma}\frac{\sin\theta'_j(R,0)t'}{R_0 - R\beta^2\cos\theta'_j(R,0)} \\ + \frac{RR_0\omega\left(R\beta^2 - R_0\cos\theta'_j(R,0)\right)t'^2}{2\gamma^2\left(R\beta^2\cos\theta'_j(R,0) - R_0\right)^3} \quad (3.30)$$

When we graph this and compare it to the graph generated with equations 3.10 and 3.11, we can see how good the approximation is.[6] Apparently, it is very good. There is virtually no detectable difference between the two on visual inspection. The Taylor approximation is only good for small values of t' (in this case, 0.1), so if we were to take a larger value such as 1 the two graphs would differ substantially, but close to zero the approximation is excellent.

For some time dt' very close to 0, the points on the disk are given by coordinates

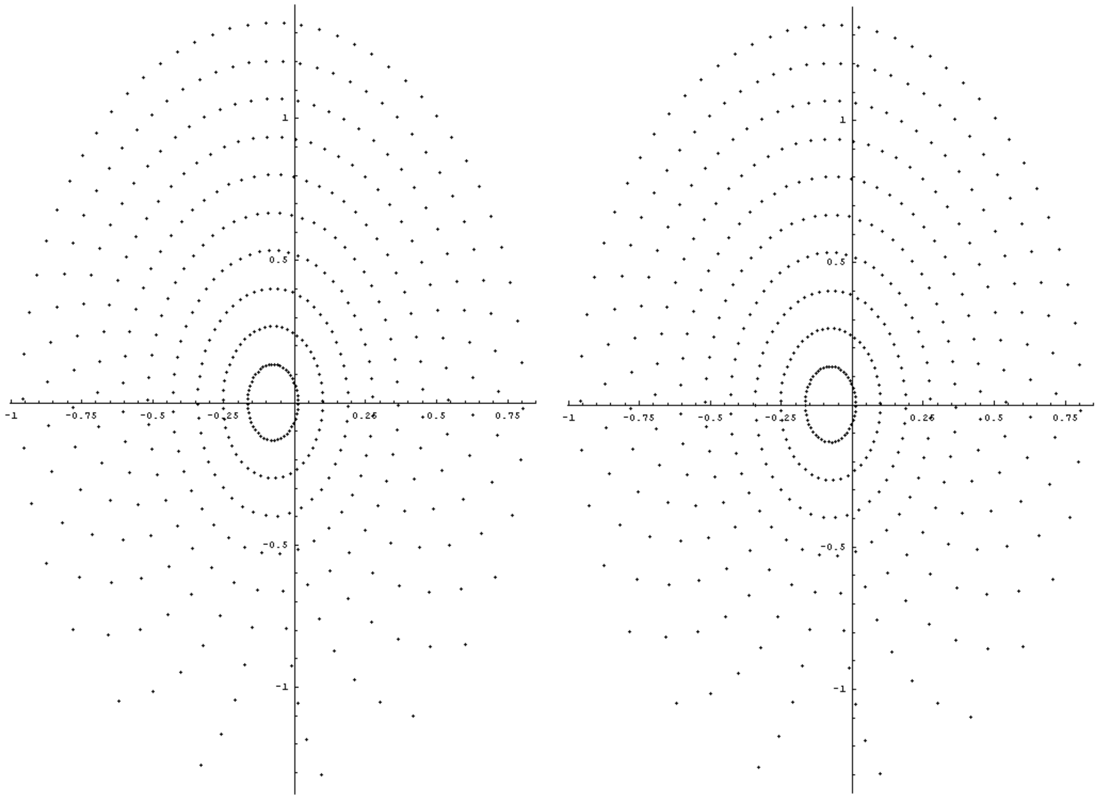
$$x'_j(R,dt') = x'_j(R,0) + \dot{x}'_j(R,0)dt' + \frac{\ddot{x}'_j(R,0)}{2}dt'^2 \quad (3.31)$$

$$y'_j(R,dt') = y'_j(R,0) + \dot{y}'_j(R,0)dt' + \frac{\ddot{y}'_j(R,0)}{2}dt'^2 \quad (3.32)$$

for the various derivatives of x and y. The x and y velocities of the particles are given by the time derivatives of 3.31 and 3.32:

$$v'_{xj}(R,dt') = \dot{x}'_j(R,0) + \ddot{x}'_j(R,0)dt' \quad (3.33)$$

[6] The exact graph was produced with the code found in appendix A.4, and the approximation was produced with the code in appendix A.5.



(a) Exact equation ***(b) Taylor approximation***

Figure 3.5: Comparison of exact and second-order approximation of time evolution of the disk from the comoving frame. $\beta = 0.75$ and $t' = 0.1$

$$v'_{yj}(R, dt') = \dot{y}'_j(R, 0) + \ddot{y}'_j(R, 0)dt' \tag{3.34}$$

We want to find the Lorentz transformation into the frame in which our reference point is stationary. The velocity of this frame has components $v'_{x0}(dt') - v'_{x0}(0) = \ddot{x}'_0(0)dt'$ in the x-direction and $v'_{y0}(dt') - v'_{y0}(0) = \ddot{y}'_0(0)dt'$ in the y-direction. These evaluate to

$$v_x = 0 \qquad\qquad v_y = \omega dt' \tag{3.35}$$

The Lorentz transformations for a boost in the y'-direction are

$$\begin{aligned}\bar{x} &= x' \\ \bar{y} &= \bar{\gamma}\left(y' - \omega t' dt'\right) \\ \bar{t} &= \bar{\gamma}\left(t' - \tfrac{y'\omega dt'}{c^2}\right)\end{aligned} \tag{3.36}$$

where $\bar{\gamma} = \frac{1}{\sqrt{1-\omega^2 dt'^2/c^2}}$. To second order in dt',

$$\bar{\gamma} \approx 1 + \frac{\omega^2 dt'^2}{2c^2} \tag{3.37}$$

Using equation 3.37 and a small value of t' (such that $t' = dt'$), the Lorentz transformations look like

$$\begin{aligned}\bar{x} &= x' \\ \bar{y} &= \left(1 + \tfrac{\omega^2 dt'^2}{2c^2}\right)\left(y' - \omega dt'^2\right) \approx y' - \omega dt'^2 + \tfrac{y'\omega^2 dt'^2}{2c^2} \\ \bar{d}t &= \left(1 + \tfrac{\omega^2 dt'^2}{2c^2}\right)\left(t' - \tfrac{y'\omega dt'}{c^2}\right) \approx dt'\left(1 - \tfrac{y'\omega}{c^2}\right)\end{aligned} \tag{3.38}$$

to second order. Using equation 3.32, the Lorentz transformations are

$$\bar{y}_j(R,t') = y'_j(R,0) + \dot{y}'_j(R,0)dt' + \frac{\ddot{y}'_j(R,0)}{2}dt'^2 - \omega dt'^2 + \frac{y'_j(R,0)\omega^2 dt'^2}{2c^2} \tag{3.39}$$

$$\bar{d}t_j(R) = dt' - \frac{y'_j(R,0)\omega dt'}{c^2} - \frac{\dot{y}'_j(R,0)\omega dt'^2}{c^2} \tag{3.40}$$

Equation 3.40 may be integrated up to some finite time to obtain an expression for the disk time.

$$\begin{aligned}\bar{t}_j(R) &= \left(1 - \frac{y'_j(R,0)\omega}{c^2}\right)t' - \frac{\dot{y}'_j(R,0)\omega}{c^2}t'dt' \\ &= \left(1 - \frac{y'_j(R,0)\omega}{c^2}\right)t'\end{aligned} \tag{3.41}$$

Note that $\bar{t}$ is not equivalent to t'. The clocks in our comoving frames apparently don't behave quite like clocks on the rotating disk; they tick more quickly. This is a problem if we are trying to build up the rotating frame by moving from instantaneous comoving frame to instantaneous comoving frame as the reference point moves. It shows that second-order adjustments in the reference point's motion have an effect on time measurement at that point, making it less likely that comoving frames can describe the disk frame.

3.6 Summary

We have examined a numeric method of picturing the rotating frame, first by studying instantaneous comoving inertial frames at the moment they are comoving with a given reference point, and then by looking at the time evolution of the disk from the vantage point of those inertial frames. From the latter, we attempted to construct a conglomerate frame made by switching quickly from comoving frame to comoving frame, and to describe the disk in that way. Unfortunately, there is some doubt that the conglomerate frame will accurately describe the disk. We found that clocks run more slowly in our constructed disk frame than in comoving frames with the same value of β. This is a second-order effect of the equations describing the points of the disk, and its existence suggests that frames with rotation or acceleration are more difficult to build up from inertial frames than we had hoped.

The numeric method does allow us insight into a related problem, however. It provides a way to study the shape of a rotating disk that is undergoing translational motion past an inertial observer, such as a wheel rolling by at a relativistic speed. We have seen the bending of the wheel's spokes away from the point of contact with the ground (the point with velocity 0), and observed that the disk's shape changes in this frame as it travels past the observation point.

3.7 Conclusions and Directions for Future Research

Arzeliès' approach to the relativistic rotating disk shows the disadvantages of starting one's examination of the system with clocks and rods. The spacetime geometry becomes variable and time is especially difficult to define. One could attempt in the future to find a definition of time for which the spacetime geometry is flat, but most of the options have been examined already and found unsatisfactory.

The numeric method shows some promise as an approach to describing the disk. It neatly avoids the problems with Arzeliès' approach, and in return asks only that we not try to define time over the entire disk at once. Modern general relativists tend to agree that time is generally only well-defined locally. Moreover, the numeric method is clear and precise every step of the way. There is no ambiguity about what definition of time to use, since the Lorentz transformation clearly defines the relationship between time in different inertial frames. Also, as shown in section 3.4, the disk has the same appearance in all comoving frames of a given reference point.

Something tricky happens when we try to make the switch from comoving frames to a continuous frame, though, and the present work only touches on this briefly. Because two "neighboring" comoving frames are moving with respect to each other, transforming from one to the other causes a clock slowdown. This seems to indicate that the time on the disk is different from the time in the comoving frames. Future research on the numeric method should examine this. There are a few possibilities that I can think of with which to begin. First, a calculation of the shape of the disk via the Taylor expansion and integration techniques of section 3.5, to second order and then to higher orders in dt', would be potentially illuminating. This will be a difficult calculation because any effect will be overshadowed by the changes of the disk's orientation in space and translational motion that we saw in figure 3.4, and those effects will be difficult to filter out of the calculations. If successful, however, one can look for any warping of the disk caused by continuous infinitesimal Lorentz transformations. For the model to hold up, warping cannot occur.

One can also do a higher-order calculation of the disk time, expanding equations 3.33 and 3.34 to second order or even higher. This will enhance our understanding of the time problem described above. It is possible that one might simply ignore the effects on time; in that case, the numeric method could be used as-is to describe the rotating disk. There is some question about the legitimacy of such a move, unfortunately, and it brings us closer to problems with time definitions.

After successfully describing the disk from one reference point by use of the numeric method, one might wish to generalize to any observer travelling on the disk. This problem may or may not be feasible. The most direct approach to it is to perform two infinitesimal Lorentz transformations instead of the one in section 3.5, and to integrate that over the path taken by the observer. This might not work, however, and I am unsure how one would proceed from there.

There is still much to be learned about the relativistic rotating disk. In this work we have looked at it from two very different perspectives, each with advantages and disadvantages, and there are other perspectives that we did not examine. My goal is to have clarified some problems typical of the system for those who are familiar with it, and made it accessible to those who are not. I hope you have found it illuminating.

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Appendix A

Mathematica Code

A.1 A Ring of Points

```
n = 50; (* Number of points *)
\[Beta] = 0.8; (* (Angular) velocity of the reference point (the radius and \
speed of light have both been set to 1) *)
Array[x, n, 0];
Array[y, n, 0];
R = 1;
w[j_] := w /.
      FindRoot[w/R == Sin[\[Beta]^2 w + 2 Pi*j/n], {w, 0.5},
        MaxIterations -> 1000];
x[j_] := Sqrt[1 - \[Beta]^2]*w[j];
y[j_] := -R*Cos[\[Beta]^2*w[j] + 2 Pi*j/n];
pairs2 = Table[{x[j], y[j]}, {j, 0, n - 1}];
ListPlot[pairs2, AspectRatio -> Automatic]
```

A.2 The Whole Disk

```
n = 50; (* Circumferential resolution *)
\[Beta] = 0.5; (* (Angular) velocity of the reference point (the radius and \
speed of light have both been set to 1) *)
Rres = 10; (* Radial resolution *)
Rinc = 1/(\[Beta]*Rres);
Array[x, n, 0];
Array[y, n, 0];
R = 0;
pairs = Table[{0, -1}, {1}];
For[ i = 0, i < Rres, i++,
  R = R + Rinc;
  w[j_] :=
```

```
    w /. FindRoot[w/R == Sin[\[Beta]^2 w + 2 Pi*j/n], {w, 0.5},
        MaxIterations -> 1000];
  x[j_] := Sqrt[1 - \[Beta]^2]*w[j];
  y[j_] := -R*Cos[\[Beta]^2*w[j] + 2 Pi*j/n];
  pairs2 = Table[{x[j], y[j]}, {j, 0, n - 1}];
  pairs = Join[pairs, pairs2];
  ]
ListPlot[pairs, AspectRatio -> Automatic,]
```

A.3 Time Evolution

```
n = 50; (* Circumferential resolution *)
\[Beta] =
    0.75; (* (Angular) velocity of reference point (R_0 and c are set to 1) *)


Rres = 10; (* Radial resolution *)
t = 0; (* Time elapsed in first comoving frame before transformation *)

(* Assorted dependent constants below *)
Rinc = 1/(\[Beta]*Rres);
gamma = 1/Sqrt[1 - \[Beta]^2];
x0 = Sqrt[1 - \[Beta]^2]*w /.
      FindRoot[w ==
          Sin[\[Beta]^2 w + \[Beta]/Sqrt[1 - \[Beta]^2]*t] - \[Beta]*t, {w,
          0.5}, MaxIterations -> 1000];
tpp = t + gamma*\[Beta]*Sin[gamma*\[Beta]^2*x0 + gamma*\[Beta]*t]; (*
  Constant time in the second comoving frame *)
gamma2 = gamma^2*(1 - \[Beta]^2*Cos[gamma*\[Beta]^2*x0 + gamma*\[Beta]*t]);
vx0 = \[Beta]*(Cos[gamma*\[Beta]^2*x0 + gamma*\[Beta]*t] -
            1)/(1 - \[Beta]^2*Cos[gamma*\[Beta]^2*x0 + gamma*\[Beta]*t]);
vy0 = \[Beta]*
      Sin[gamma*\[Beta]^2*x0 +
            gamma*\[Beta]*
              t]/(gamma*(1 - \[Beta]^2*
                  Cos[gamma*\[Beta]^2*x0 + gamma*\[Beta]*t]));
v0 = Sqrt[vx0^2 + vy0^2];

(* And now, the meat of the program *)
R = 0;
pairs = Table[{0, -1}, {1}];
For[ i = 0, i < Rres, i++,
  R = R + Rinc;
  t1[j_] :=
```

```
    t1 /. FindRoot[{t1 ==
            tpp/gamma2 + vx0*x1 -
              vy0*R*Cos[gamma*\[Beta]^2*x1 + 2*\[Pi]*j/n + \[Beta]*gamma*t1],
          x1 == (R/gamma)*
                Sin[gamma*\[Beta]^2*x1 +
                    2*\[Pi]*j/n + \[Beta]*gamma*t1] - \[Beta]*t1}, {{t1,
            t}, {x1, 0.5}}, MaxIterations -> 1000];
  x1[j_] :=
    x1 /. FindRoot[{t1 ==
            tpp/gamma2 + vx0*x1 -
              vy0*R*Cos[gamma*\[Beta]^2*x1 + 2*\[Pi]*j/n + \[Beta]*gamma*t1],
          x1 == (R/gamma)*
                Sin[gamma*\[Beta]^2*x1 +
                    2*\[Pi]*j/n + \[Beta]*gamma*t1] - \[Beta]*t1}, {{t1,
            t}, {x1, 0.5}}, MaxIterations -> 1000];
  y1[j_] := -R*Cos[gamma*\[Beta]^2*x1[j] + 2 Pi*j/n + \[Beta]*gamma*t1[j]];
  x2[j_] :=
    gamma2*((vx0^2*x1[j] + vx0*vy0*y1[j])/v0^2 - vx0*t1[j]) -
      vy0/v0^2*(vx0*y1[j] - vy0*x1[j]);
  y2[j_] :=
    gamma2*((vy0^2*y1[j] + vx0*vy0*x1[j])/v0^2 - vy0*t1[j]) +
      vx0/v0^2*(vx0*y1[j] - vy0*x1[j]);
  pairs2 = Table[{x2[j], y2[j]}, {j, 0, n - 1}];
  pairs = Join[pairs, pairs2];
  ]
ListPlot[pairs, AspectRatio -> Automatic, ImageSize -> 500]
```

A.4 Time Evolution in One Comoving Frame

```
n = 50; (* Circumferential resolution *)
\[Beta] = 0.75; (* Velocity of the comoving frame *)
Rres = 10; (* Radial resolution *)
t = 0.1; (* Time *)
Rinc = 1/(\[Beta]*Rres);
gamma = 1/Sqrt[1 - \[Beta]^2];
Array[x, n, 0];
Array[y, n, 0];
R = 0;
pairs = Table[{Null}, {0}];
For[ i = 0, i < Rres, i++,
  R = R + Rinc;
  w[j_] :=
    w /. FindRoot[
        w/R == Sin[\[Beta]^2 w +
```

```
               2 Pi*j/n + \[Beta]/Sqrt[1 - \[Beta]^2]*t] - \[Beta]*gamma*
             t/R, {w, 0.5}, MaxIterations -> 1000];
  x[j_] := Sqrt[1 - \[Beta]^2]*w[j];
  y[j_] := -R*Cos[\[Beta]^2*w[j] + 2 Pi*j/n + \[Beta]/Sqrt[1 - \[Beta]^2]*t];
  pairs2 = Table[{x[j], y[j]}, {j, 0, n - 1}];
  pairs = Join[pairs, pairs2];
  ]
ListPlot[pairs, AspectRatio -> Automatic, ImageSize -> 500]
```

A.5 Taylor Expansion of Time Evolution in One Comoving Frame

```
n = 50;(* Circumferential resolution *)
\[Beta] = 0.75; (* Velocity of the comoving frame *)
Rres = 10; (* Radial resolution *)
t = 0.1; (* Time *)
Rinc = 1/(\[Beta]*Rres);
gamma = 1/Sqrt[1 - \[Beta]^2];
Array[x, n, 0];
Array[y, n, 0];
R = 0;
pairs = Table[{Null}, {0}];
For[ i = 0, i < Rres, i++,
  R = R + Rinc;
  w[j_] :=
    w /. FindRoot[w/R == Sin[\[Beta]^2 w + 2 Pi*j/n], {w, 0.5},
        MaxIterations -> 1000];
  x[j_] := Sqrt[1 - \[Beta]^2]*w[j];
  y[j_] := -R*Cos[\[Beta]^2*w[j] + 2 Pi*j/n];
  theta0[j_] := gamma*\[Beta]^2*x[j] + 2*Pi*j/n;
  x2[j_] :=
    R*Sin[theta0[j]]/
          gamma + \[Beta]*(R*Cos[theta0[j]] - 1)/(1 -
              R*\[Beta]^2*Cos[theta0[j]])*t +
      R*\[Beta]^2*Sin[theta0[j]]*
        t^2/(2*gamma^3*(R*\[Beta]^2*Cos[theta0[j]] - 1)^3);
  y2[j_] := -R*Cos[theta0[j]] +
      R*\[Beta]*Sin[theta0[j]]*t/(gamma*(1 - R*\[Beta]^2*Cos[theta0[j]])) +
      R*\[Beta]*(R*\[Beta]^2 - Cos[theta0[j]])*
        t^2/(2*gamma^2*(R*\[Beta]^2*Cos[theta0[j]] - 1)^3);
  pairs2 = Table[{x2[j], y2[j]}, {j, 0, n - 1}];
  pairs = Join[pairs, pairs2];
  ]
ListPlot[pairs, AspectRatio -> Automatic, ImageSize -> 500]
```

Appendix B

Maxima Transcripts

B.1 Central Time

```
Maxima 5.9.1 http://maxima.sourceforge.net
Using Lisp Kyoto Common Lisp GCL 2.6.5 (aka GCL)
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
This is a development version of Maxima. The function bug_report()
provides bug reporting information.
(%i1) tsetup();
(%o1)     TSETUP()
(%i2) load(ctensr);
(%o2)  F:/PROGRA~1/MAXIMA~1.1/share/maxima/5.9.1/share/tensor/ctensr.mac
(%i3) tsetup();
Enter the dimension of the coordinate system:
4;Do you wish to change the coordinate names?
y;Enter a list containing the names of the coordinates in order
[t,r,p,z];Do you want to

1. Enter a new metric?

2. Enter a metric from a file?

3. Approximate a metric with a Taylor series?
1;
Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric  4. General
Answer 1, 2, 3 or 4 :
2;Row 1 Column 1:
-sqrt(1-w^2*r^2/c^2);Row 1 Column 2:
0;Row 1 Column 3:
w*r^2;Row 1 Column 4:
0;Row 2 Column 2:
```

```
1;Row 2 Column 3:
0;Row 2 Column 4:
0;Row 3 Column 3:
r^2;Row 3 Column 4:
0;Row 4 Column 4:
1;
Matrix entered.
Enter functional dependencies with the DEPENDS function or 'N' if none
n;Do you wish to see the metric?
y;         [            2  2               ]
           [           r  w        2       ]
           [ - SQRT(1 - -----)  0  r  w  0 ]
           [              2                ]
           [             c                 ]
           [                               ]
           [          0         1   0    0 ]
           [                               ]
           [          2             2      ]
           [       r  w         0   r    0 ]
           [                               ]
           [          0         0   0    1 ]

Do you wish to see the metric inverse?
n;(%o3)       DONE
(%i4) riemann(true);
```

(%t4) $$R_{1,2,1,2} = \text{large equation}^{1}$$

(%t5) $$R_{1,3,1,3} = \frac{2\,c^2\,r^2\,w^2\,\sqrt{c^2 - r^2\,w^2} - |c|\,r^2\,w^2}{2\,c^2\,\sqrt{c^2 - r^2\,w^2}}$$

```
(%o5)       DONE
(%i5)
```

B.2 Local Time

```
Maxima 5.9.1 http://maxima.sourceforge.net
Using Lisp Kyoto Common Lisp GCL 2.6.5 (aka GCL)
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
This is a development version of Maxima. The function bug_report()
```

[1] This term does not fit nicely on the page. Nevertheless, the code listed above will calculate it if given enough time.

```
provides bug reporting information.
(%i1) load(ctensr);
(%o1)  F:/PROGRA~1/MAXIMA~1.1/share/maxima/5.9.1/share/tensor/ctensr.mac
(%i2) tsetup();
Enter the dimension of the coordinate system:
4;Do you wish to change the coordinate names?
y;Enter a list containing the names of the coordinates in order
[t,r,p,z];Do you want to

1. Enter a new metric?

2. Enter a metric from a file?

3. Approximate a metric with a Taylor series?
1;
Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric  4. General
Answer 1, 2, 3 or 4 :
2;Row 1 Column 1:
-1;Row 1 Column 2:
0;Row 1 Column 3:
w*r^2/sqrt(1-r^2*w^2/c^2);Row 1 Column 4:
0;Row 2 Column 2:
1;Row 2 Column 3:
0;Row 2 Column 4:
0;Row 3 Column 3:
r^2;Row 3 Column 4:
0;Row 4 Column 4:
1;
Matrix entered.
Enter functional dependencies with the DEPENDS function or 'N' if none
n;Do you wish to see the metric?
y;    [                               2           ]
      [                             r  w          ]
      [   - 1               0  ---------------  0 ]
      [                                 2  2      ]
      [                               r  w        ]
      [                        SQRT(1 - -----)    ]
      [                                  2        ]
      [                                 c         ]
      [                                           ]
      [     0               1         0         0 ]
      [                                           ]
      [      2                                    ]
      [    r  w                        2          ]
      [ ---------------     0          r        0 ]
```

```
[           2  2                             ]
[          r  w                              ]
[ SQRT(1 - -----)                            ]
[            2                               ]
[           c                                ]
[                                            ]
[     0                0         0         1 ]
```

```
Do you wish to see the metric inverse?
n;(%o2)       DONE
(%i3) RIEMANN(TRUE);
```

(%t3) $$R_{1,2,1,2} = \text{large equation}$$

(%t4) $$R_{1,2,2,3} = \text{large equation}$$

(%t5) $$R_{1,3,1,3} = -\frac{c^6\, r^6\, w^6 - 4\, c^8\, r^4\, w^4 + 4\, c^{10}\, r^2\, w^2}{4\, c^4\, r^6\, w^6 - 12\, c^6\, r^4\, w^4 + 12\, c^8\, r^2\, w^2 - 4\, c^{10}}$$

(%t6) $$R_{2,3,2,3} = \text{large equation}$$

```
(%o6)       DONE
(%i6)
```

B.3 Natural Time

```
Maxima 5.9.1 http://maxima.sourceforge.net
Using Lisp Kyoto Common Lisp GCL 2.6.5 (aka GCL)
Distributed under the GNU Public License. See the file COPYING.
Dedicated to the memory of William Schelter.
This is a development version of Maxima. The function bug_report()
provides bug reporting information.
(%i1) load(ctensr);
(%o1)  F:/PROGRA~1/MAXIMA~1.1/share/maxima/5.9.1/share/tensor/ctensr.mac
(%i2) tsetup();
Enter the dimension of the coordinate system:
4;Do you wish to change the coordinate names?
y;Enter a list containing the names of the coordinates in order
[t,r,p,z];Do you want to

1. Enter a new metric?
```

```
2. Enter a metric from a file?

3. Approximate a metric with a Taylor series?
1;
Is the matrix  1. Diagonal  2. Symmetric  3. Antisymmetric  4. General
Answer 1, 2, 3 or 4 :
1;Row 1 Column 1:
-1;Row 2 Column 2:
1;Row 3 Column 3:
r^2/(1-r^2*w^2/c^2);Row 4 Column 4:
1;
Matrix entered.
Enter functional dependencies with the DEPENDS function or 'N' if none
n;Do you wish to see the metric?
y;       [ - 1  0       0      0  ]
         [                        ]
         [  0   1       0      0  ]
         [                        ]
         [               2        ]
         [              r         ]
         [  0   0   ---------  0  ]
         [                2  2    ]
         [               r  w     ]
         [          1 - -----     ]
         [                 2      ]
         [                c       ]
         [                        ]
         [  0   0       0      1  ]

Do you wish to see the metric inverse?
n;(%o2)       DONE
(%i3) RIEMANN(TRUE);
```

(%t3) $$R_{2,3,2,3} = \frac{3\,c^4\,r^2\,w^2}{r^6\,w^6 - 3\,c^2\,r^4\,w^4 + 3\,c^4\,r^2\,w^2 - c^6}$$

```
(%o3)       DONE
(%i4)
```